

Markov Concurrent Processes

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Abstract

We introduce a model for probabilistic systems with concurrency. The system is distributed over two local sites. Global trajectories of the system are composed of local trajectories glued along synchronizing points. Global trajectories are thus given as partial orders of events, and not as *paths*. As a consequence, time appears as a *dynamic partial order*, contrasting with the universal chain of integers we are used to. It is surprising to see how natural it is to adapt mathematical techniques for processes to this new conception of time.

The probabilistic model has two features: first, it is Markov, in a sense convenient for concurrent systems; and second, the local components have maximal independence, beside their synchronization constraints. We construct such systems and characterize them by finitely many real parameters, that are the analogous to the transition matrix for discrete Markov chains. This construction appears as a generalization of the “synchronization of Markov chains” developed in an earlier collaboration.

Introduction

This paper introduces a class of probabilistic processes intended to model concurrent systems, and characterizes them by a finite number of parameters. “Probabilistic process” is not to be taken here in the usual sense of a sequence of random variables, and that is certainly the most unusual part of these “processes”.

To model a system evolving through time, nothing is more natural than to consider a sequence of random variables, each one taken its values among the possible states of the system, at least when time is discrete. This approach however brings some issues in case the system is distributed over several interacting sites. Considering a new state space defined as the product of several local state spaces, and then a process on the new state space, immediately comes in mind. By this way however, the interaction of the different sites is not clearly apparent. Furthermore, and maybe more important, a unique time line evolving synchronously for all sites does not render any kind of concurrency between sites. One would have to consider additional conditions, for instance, equaling elementary sequences of transitions like $ab = ba$. One would then naturally enter the field of random walks on algebraic structures such as semigroups. Here, concurrency of parallel actions is obtained *a posteriori*, by equaling some sequences of actions, the interleaving of which is considered as irrelevant.

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The approach we consider in this paper is based on a treatment of concurrency in a more structural way. A system is distributed over several sites. On each site a local process evolves in the usual way, and is thus rendered as a sequence of random variables taking values in the local state space. The interaction of sites is taken into account by considering that local state spaces share some common values, so that local processes are forced to synchronize on these values. Local processes are otherwise *totally asynchronous*. Although *local trajectories* can be seen as sequences of local states, and thus unfolded as infinite paths in a regular tree, the *global trajectories* are not paths anymore. Each global trajectory is instead correctly represented as a *partial order* of events, resulting of the gluing of different local trajectories along synchronizing states.

This paper presents some results for such models in the case where concurrency is non trivial, but still kept to its simplest expression, namely, the case of a system with only two sites. It will be demonstrated that this case already contains interesting mathematical issues and original features that are completely absent from the traditional theory of probabilistic processes.

In the classical theory of probabilistic processes, the class of Markov processes has demonstrated its importance, both from the practical and from the theoretical viewpoints. The model is acclaimed because of the many systems it can faithfully model, by still allowing analytical treatment. For us, having defined generalized probabilistic processes for two sites, with partial orders for trajectories as a main feature, our next task will be to particularize to Markov generalized processes. Unlike in the case of usual Markov chains, describing generalized Markov processes by a complete set of parameters, analogous to the transition matrix, is a non trivial task, that will occupy us next. We will then come to a related construction proposed earlier, and that can be seen as the *synchronization of Markov chains* [3]. Finally, we will end the paper with a discussion for future work, and how our work is related to other theories, in particular with the theory of random walks in random environments. We will also discuss some strange feature and consequences of a partially ordered time. Although it might appear strange at first sight, it is actually quite intuitive that a distributed state space implies a distributed time as well. For a discussion on other related work, and in particular for a comparison with the model of *probabilistic automata*, the reader is referred to the discussion found in [3].

The paper is organized as follows. In Section 1 we introduce the general framework for the model, including a general definition of a probabilistic process with 2 components and no ordered clock. The two next sections, Sections 2 and 3, respectively deal with the Markov property and with the local independence property (LIP). Section 4 is devoted to the construction and to the characterization of Markov concurrent processes with the LIP. Finally, Section 5 discusses research perspectives on the one hand, and provides elements for an understanding of a “partially ordered time”. An Appendix quickly reviews some background on conditional expectation and conditional independence.

1 Probabilistic 2-Components Processes

1.1 General Framework

We consider a system distributed over two sites, named 1 and 2. We refer to the parts of the system related to each site as to the *local systems*, having *local states*, the evolution of which gives rise to two *local processes*. We refer to each site by the *superscript* 1 or 2: for $i = 1, 2$, we note S^i the finite set of *local states* of site i , we use x^i to denote a generic element of S^i . We use symbols such as s^i or $(x_j^i)_j$ to denote sequences of local states. Such sequences can be finite or infinite. If it is infinite, we call the sequence a **local trajectory**. We denote by Ω^i the sets of local trajectories.

The two local state sets S^1 and S^2 are intended to have a non empty intersection, otherwise the theory has little interest. We put $Q = S^1 \cap S^2$. Elements of Q are called **common states** or **shared states**.

Given a sequence $(x_j)_j$ of elements in a set S , with finitely many of infinitely many index j , and given a subset $A \subseteq S$, the **A-sequence induced by $(x_j)_j$** is defined as the sequence, finite or infinite, of elements of A encountered by the sequence $(x_j)_j$, in their order of appearance. In particular, given two local trajectories $(x_n^1)_{n \geq 0}$ and $(x_n^2)_{n \geq 0}$, we will say that they **synchronize** if the two Q -sequences they induce are *equal*. Such a pair of two synchronizing local trajectories will be called a **global trajectory**. More generally, we define a **partial trajectory** as a pair $((x_j^1)_j, (x_j^2)_j)$, where $(x_j^1)_j$ and $(x_j^2)_j$ are two finite or infinite sequences of states such that the two Q -sequences they induce coincide.

Partial trajectories are ordered component by component: if $s = (s^1, s^2)$ and $t = (t^1, t^2)$ are two partial trajectories, we define $s \leq t$ if $s^1 \leq t^1$ and $s^2 \leq t^2$, where the order on sequences of states is the usual prefix order on words. The resulting binary relation on partial trajectories is a partial order, which maximal elements are exactly the global trajectories defined earlier. The set of global trajectories is denoted by Ω .

Given any partial trajectory $s = (s^1, s^2)$, the **subtrajectories of s** are those partial trajectories t such that $t \leq s$. Observe that not any prefix t of s is a subtrajectory; since t could very well not be a partial trajectory itself.

Given a partial trajectory (s^1, s^2) , we denote by $(y_j)_j$ the Q -sequence induced by both sequences s^1 and s^2 . It can be finite or infinite, even empty. We refer to $(y_j)_j$ as to the **Q-sequence induced by (s^1, s^2)** .

The following notion will be useful: we call **global state** any pair $(x^1, x^2) \in S^1 \times S^2$. Observe that partial trajectories of the global system are *not* defined as sequences of global states.

Some typology about partial trajectories: we say that (s^1, s^2) is *finite* if both s^1 and s^2 are finite sequences of states. In that case, we define

$$\gamma(s^1, s^2) = (x^1, x^2) \in S^1 \times S^2 \tag{1}$$

as the pair of last states of the two sequences s^1 and s^2 . It is to be understood as the current global state “after” execution of the finite trajectory (s^1, s^2) , although time does not have the usual signification we are used to, in particular because there is no canonical totally ordered clock at the scale of the system.

We introduce the somewhat unusual notion of length for partial trajectories.

We denote by \mathcal{T} the set

$$\mathcal{T} = (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}).$$

The set \mathcal{T} is partially ordered component by component, with the natural order on each component. If $s = (s^1, s^2)$ is any partial trajectory, the **length of s** is defined by

$$|s| = (|s^1|, |s^2|) \in \mathcal{T}$$

where $|s^1|$ and $|s^2|$ denote the length, finite or infinite, of sequences. In a first approach, lengths can be thought of as time instants—the fact that this approach is too rough is discussed in § 5.1—it becomes then quite clear that time is only partially ordered, and not totally ordered.

There is a **concatenation** operation defined on partial trajectories. If $s = (s^1, s^2)$ and $t = (t^1, t^2)$ are two partial trajectories, and if s is a finite trajectory, then the component wise concatenation, denoted by $s \cdot t$ and defined by

$$s \cdot t = (s^1 \cdot t^1, s^2 \cdot t^2)$$

is obviously a partial trajectory. If (t^1, t^2) is furthermore a global trajectory, then $s \cdot t$ is a global trajectory as well. There is an obvious **addition** on lengths defined above, compatible with concatenation, in the sense that $|s \cdot t| = |s| + |t|$. If the left member is kept fixed, the concatenation defines an application

$$\Phi_s : t \mapsto s \cdot t, \tag{2}$$

which is a bijection from the set of global trajectories onto the set of global trajectories that have s as a prefix.

1.2 Generalized Probabilistic Processes

Although time has been abstracted from the framework, the notion of trajectory is still present; this is all we need to introduce a probabilistic layer. Recall that Ω denotes the set of global trajectories defined above. The set Ω is defined as a subset of the product $\Omega^1 \times \Omega^2$, where $\Omega^i = (S^i)^{\mathbb{N}}$ is the set of local trajectories on site i , for $i = 1, 2$. If \mathfrak{F}^i denotes the Borel σ -algebra on Ω^i , we denote by \mathfrak{F} the σ -algebra on Ω defined as the restriction of $\mathfrak{F}^1 \otimes \mathfrak{F}^2$ on subsets of Ω . It is clear that Ω itself is $\mathfrak{F}^1 \otimes \mathfrak{F}^2$ -measurable as a subset of $\Omega^1 \times \Omega^2$, and this justifies that we refer to \mathfrak{F} as to the *Borel σ -algebra* on Ω .

Definition 1.1. *A 2-components probabilistic process is given by the following data: two state sets S^1 and S^2 , usually with a nonempty intersection, and a probability measure \mathbb{P} on the induced space (Ω, \mathfrak{F}) of trajectories.*

*Let $\omega^i = (X_j^i)_{j \geq 0}$ be the sequence describing the local trajectories, for $i = 1, 2$. The **initial distribution** of a given 2-components probabilistic process is defined as the probability distribution μ on $S^1 \times S^2$ given by*

$$\forall (x^1, x^2) \in S^1 \times S^2 \quad \mu(x^1, x^2) = \mathbb{P}(X_0^1 = x^1, X_0^2 = x^2).$$

This definition is a natural extension of the general definition of a probabilistic process in discrete time over a single state set ([4]). In the following, we will refer to such “classical” processes as to *sequential probabilistic processes*.

Sequential processes contrast with the 2-components probabilistic processes, because of the *concurrency* feature of the later.

Observe that, for each finite trajectory s , the mapping Φ_s defined in Eq. (2) is not only a bijection, but is also bi-measurable. Next, a 2-components probabilistic process being given with the notations of Def. 1.1, introduce the symbol $\uparrow s$ to denote the set of global trajectories t such that $s \leq t$. Assume that $\mathbb{P}(\uparrow s) > 0$. Considering first the conditional probability $\mathbb{P}(\cdot | \uparrow s)$ on $\uparrow s$, and then its image on Ω by the inverse of Φ_s defines a probability measure on Ω , that we denote by \mathbb{P}_s , and characterized by

$$\mathbb{P}_s(\uparrow t) = \frac{1}{\mathbb{P}(\uparrow s)} \mathbb{P}(\uparrow (s \cdot t)),$$

for every finite trajectory t . Intuitively, probability \mathbb{P}_s represents the evolution of the process “after” execution of s .

So far, a general model of a distributed system over two sites with synchronization has been defined. Interaction between sites is rendered by the synchronization constraint. Except for synchronization, local components have their processes evolving asynchronously from one another. A general notion of probabilistic process on top of such systems has been introduced, to be particularized to Markov systems now.

2 The Markov Property and Some Consequences

Markov chains, and more precisely homogeneous Markov chains, as a class of sequential probabilistic processes, can be defined in different equivalent ways. One of them is the following: the evolution of the chain after a finite trajectory, seen as a function of the finite trajectory, actually only depends on the last state of the trajectory—and not on the previous values, nor on the length of the chain. This definition departs from another popular one, involving the initial distribution of the chain and a transition matrix.

For 2-components probabilistic processes, it is suitable to extend the former formulation rather than the later; since there is no obvious analogous to the transition matrix in our case. It will then be our task to discover what the analogous of the transition matrix is. However this is postponed to Section 3, where the Markov property will be mixed with yet another ingredient, closely related to concurrency. The treatment we give of the Markov property in the present section is necessary, but is rather an *adaptation* of the usual Markov property in the concurrent framework; while the core interaction of concurrency and probability is a topic orthogonal to the Markov property, and devoted to next section.

2.1 Markov 2-Components Processes

The following definition formalizes the intuitive idea “the future only depends on the present”, in our setting where no global clock is available.

Definition 2.1. A 2-components probabilistic process defined by a probability measure \mathbb{P} on (Ω, \mathfrak{F}) is said to be **Markov** if for any two finite trajectories s and t with $\mathbb{P}(\uparrow s) > 0$ and $\mathbb{P}(\uparrow t) > 0$,

$$\gamma(s) = \gamma(t) \Rightarrow \mathbb{P}_s = \mathbb{P}_t, \quad (3)$$

using the notations previously introduced with $\gamma(\bullet)$ defined in (1) and \mathbb{P}_\bullet defined in (2).

Since \mathbb{P}_s describes the probabilistic evolution of the process “after” finite trajectory s , and if $\gamma(s)$ is understood as the current global state of the system after s , the definition renders the intuitive idea of a Markov behaviour. Constructing Markov 2-components probabilistic process is a task that we postpone for now; we temporarily ask the reader to believe they exist, so that it is worth studying them.

Assume some Markov 2-components probabilistic process \mathbb{P} is defined on (Ω, \mathfrak{F}) . Then, for every global state $x = (x^1, x^2) \in S^1 \times S^2$ such that there exists a finite trajectory s with $\gamma(s) = x$ and $\mathbb{P}(\uparrow s) > 0$, the probability measure \mathbb{P}_s only depends on x . We define thus \mathbb{P}_x as the probability measure

$$\mathbb{P}_x = \mathbb{P}_s, \quad \text{for any } s \text{ such that } \mathbb{P}(\uparrow s) > 0 \text{ and } \gamma(s) = x. \quad (4)$$

We refer to probability \mathbb{P}_x as to the probability *starting from* x . A global state x such that $\gamma(s) = x$ for some finite trajectory s satisfying $\mathbb{P}(\uparrow s) > 0$ is said to be **reachable**. More generally, if μ is a probability measure on the finite set of global states, the support of which only contains reachable global states, we put

$$\mathbb{P}_\mu = \sum_x \mu(x) \mathbb{P}_x,$$

the index x ranging over the support of μ , defining thus a probability measure over (Ω, \mathfrak{F}) . In particular, if $\mu = \delta_x$ is the Dirac measure on x , then $\mathbb{P}_\mu = \mathbb{P}_x$.

2.2 Stopping Times and the (Strong) Markov Property

In classical probability theory, stopping times are one of the basic tool to study random time specifications. Loosely speaking, we define stopping times in our framework as “random lengths”. Recall from Section 1.1 that lengths of partial trajectories range over the partially ordered set $\mathcal{T} = (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$.

Definition 2.2. With Ω denoting as above the space of global trajectories, let $T : \Omega \rightarrow \mathcal{T}$ be a given mapping. For each $\omega \in \Omega$, we denote by ω_T the prefix of ω of length $T(\omega)$. We say that T is a **stopping time** if the two following properties are satisfied:

1. The prefix ω_T is a sub-trajectory of ω , for \mathbb{P} -almost every $\omega \in \Omega$;
2. For \mathbb{P} -almost every ω and ω' in Ω , we have:

$$\omega_T \leq \omega' \Rightarrow T(\omega) = T(\omega'),$$

In this case, we also define the **shift operator** θ_T associated with T as the mapping $\theta_T : \Omega \rightarrow \Omega$ such that $\omega = \omega_T \cdot \theta_T(\omega)$ for \mathbb{P} -almost every $\omega \in \Omega$. In other words, $\theta_T(\omega)$ is the tail of ω after its prefix ω_T .

The operational tool to use and study a stopping time T is really the random prefix ω_T . Moreover, defining ω_T is equivalent to defining T ; so that we might often define a stopping time by specifying directly ω_T rather than T .

Remark. If, in Def. 2.2, Ω denotes the space of infinite sequences on a set S , and if \mathcal{T} denotes the set $\mathbb{N} \cup \{\infty\}$, then the definition reduces to that of stopping times in the classical sense [4] adapted to the natural filtration. This is at least an argument in favor of Def. 2.2 as a proper generalization of the classical notion of stopping time.

Definitions 2.1 and 2.2 are variants of definitions previously introduced in [1] in the framework of probabilistic event structures. The point of view is quite different here, in particular next section cannot be developed in the general model of probabilistic event structures. Nevertheless, the Strong Markov Property shown in [1] is valid with the following formulation: for any stopping time T , and for any $h : \Omega \rightarrow \mathbb{R}$ non negative and measurable function,

$$\mathbb{E}(h \circ \theta_T | \mathfrak{F}_T) = \mathbb{E}_{\gamma(\omega_T)}(h), \quad \mathbb{P}\text{-almost surely,} \quad (5)$$

where: \mathbb{E} denotes the mathematical expectation with respect to \mathbb{P} ; \mathbb{E}_x denotes the expectation with respect to \mathbb{P}_x , defined in (4); θ_T is the shift operator associated with T , as in Def. 2.2; and finally \mathfrak{F}_T is the σ -algebra associated with T , and defined by

$$\forall A \in \mathfrak{F}, \quad A \in \mathfrak{F}_T \iff \forall \omega, \omega' \in \Omega, \quad \omega \in A, \omega_T \leq \omega' \Rightarrow \omega' \in A. \quad (6)$$

Note that the relation (5) is an equality between *random variables*, with the same formal expression than the Strong Markov found in classical references [6]. This is meant to underline the similarity with classical Markov processes, and also thought as a convenient tool to adapt proofs and results to our new context by taking benefit from the formal analogy. We illustrate the later point by proving for Markov 2-components probabilistic processes a result well-known for Markov chains, introducing by this way the notion of recurrent and transient global state. We first define return times.

Remark. For sequential processes, a *weak* Markov property is usually first stated for *constant* times; the *strong* Markov property is then stated for *stopping times*, letting the weak property appear as a particular case of the strong property. In our case, only the strong property is given. This is because constant times have no meaning for us; this point is further discussed in § 5.1.

Definition 2.3. Let $x \in S^1 \times S^2$ be a global state. For any trajectory $\omega \in \Omega$, define a set of finite prefixes of ω by $N(\omega) = \{v \leq \omega : \gamma(v) = x \text{ and } (1, 1) \leq |v|\}$. Then define the **return time to x** as the stopping time T such that

$$\forall \omega \in \Omega, \quad \omega_T = \begin{cases} \omega, & \text{if } N(\omega) = \emptyset, \\ \inf N(\omega), & \text{otherwise.} \end{cases}$$

Considering the successive returns in a given set, the question is to determine the probability of returning in it infinitely often. To formalize the problem, we introduce the successive return times to x as the following sequence $(T_n)_{n \geq 0}$ of stopping times:

$$T_0 = (0, 0), \quad T_{n+1}(\omega) = \begin{cases} \infty, & \text{if } T_n(\omega) = \infty, \\ T_n(\omega) + T \circ \theta_{T_n}(\omega), & \text{otherwise.} \end{cases}$$

In the previous expression, recall that θ_{T_n} is the shift operator associated with T_n , so that $\theta_{T_n}(\omega)$ is merely the tail of ω after the prefix ω_{T_n} . Observe that $T_1 = T$, and that $T_n(\omega)$ is the n^{th} return of ω in x . We say that a trajectory $\omega \in \Omega$ has **infinite return in x** if all return times $T_n(\omega)$ are finite, for $n \geq 1$.

Proposition 2.1. *For any reachable state $x \in S^1 \times S^2$, the set of trajectories that return infinitely often in x either has \mathbb{P}_x -probability 0 or 1. The probability is 1 if and only if $\mathbb{P}_x(T \text{ is finite}) = 1$, where T is the return time to x . Using the Markov chains terminology [6], we will say that:*

- x is **recurrent** if the \mathbb{P}_x -probability of infinitely many returns in x is 1;
- x is **transient** if the \mathbb{P}_x -probability of infinitely many returns in x is 0.

Proof of Proposition 2.1. The proof is borrowed from [6]. For any measurable subset H , we denote by $\mathbf{1}_H$ the characteristic function of H . We compute as follows, using the σ -algebra \mathfrak{F}_T defined in (6) and the usual properties of conditional expectation:

$$\begin{aligned} \mathbb{P}_x(T_n < \infty) &= \mathbb{P}_x(T_{n-1} + T \circ \theta_{T_{n-1}} < \infty) \\ &= \mathbb{E}_x(\mathbb{E}_x(\mathbf{1}_{\{T_{n-1} + T \circ \theta_{T_{n-1}} < \infty\}} | \mathfrak{F}_{T_{n-1}})) \\ &= \mathbb{E}_x(\mathbf{1}_{\{T_{n-1} < \infty\}} \mathbb{E}_x(\mathbf{1}_{\{T \circ \theta_{T_{n-1}} < \infty\}} | \mathfrak{F}_{T_{n-1}})). \end{aligned} \quad (7)$$

On the other hand, and using the Markov property as in Eq. (5), we have:

$$\mathbb{E}_x(\mathbf{1}_{\{T \circ \theta_{T_{n-1}} < \infty\}} | \mathfrak{F}_{T_{n-1}}) = \mathbb{E}_{\gamma(\omega_{T_{n-1}})}(\mathbf{1}_T < \infty) = \mathbb{P}_x(T < \infty). \quad (8)$$

Hence, putting $a = \mathbb{P}(T < \infty)$, we get from Eqs. (7) and (8):

$$\mathbb{P}_x(T_n < \infty) = a \mathbb{P}_x(T_{n-1} < \infty).$$

It follows from the Borel-Cantelli lemma that x is recurrent if $a = 1$, and transient if $a < 1$. \square

Another consequence of the Markov property will be very helpful to understand how synchronization occurs in Markov 2-components probabilistic process. For each global trajectory $\omega = (\omega^1, \omega^2)$, let $Y(\omega)$ be the Q -sequence induced by ω , defined as the common Q -sequence induced by both ω^1 and ω^2 . Then $Y(\omega)$ can be either infinite or finite, even empty.

Definition 2.4. *We say that a global trajectory **synchronizes infinitely often** if the Q -sequence it induces is infinite. If a 2-components probabilistic process has the property that, with probability 1, global trajectories synchronize infinitely often, we say that the process is **closed**; if, with probability 1, global trajectories induce an empty Q -sequence, then we say that the process is **open**.*

Proposition 2.2. *For any recurrent state x of a Markov 2-components probabilistic process, \mathbb{P}_x either defines a closed or an open process.*

Proof of Proposition 2.2. Let x be a recurrent global state. It follows from the Markov property that the finite trajectories delimited by the successive returns in x form an infinite sequence of independent and identically distributed random variables. Hence, if the \mathbb{P}_x probability of hitting Q between two returns in x is 0, then the process is open; while, by the Borel-Cantelli lemma, if the \mathbb{P}_x -probability of hitting Q is positive, then the probability of hitting Q infinitely often is 1, and thus the process is closed. \square

3 The Local Independence Property

Contrary to the Markov property, which can be expressed in both the sequential and in the concurrent cases, the probabilistic interaction of components in the present model has no analogous for sequential processes.

Just as the Markov property limits dependency between past and future, it is natural to limit dependency between the components. How tiny can dependency be kept? It is unreasonable to expect *independence* between local components since, in general, the synchronization constraint is far from being trivial—and probability cannot add independence on top of a model with intrinsic dependency. Instead, we will require *minimal dependency*, besides the synchronisation constraints. The minimal dependency property that we introduce is called the *local independence property*, abbreviated in LIP, and it intuitively renders the following behaviour: apart from synchronizing, the two components are free to evolve “on their own”.

3.1 First Formulation of the LIP

The most convenient way to formulate the property we have in mind, although not the most intuitive, is the following.

Definition 3.1. *Let Y denote as above the Q -sequence induced by a generic global trajectory.*

*We say that a 2-components probabilistic process enjoys **the local independence property (LIP)** if its two components $\omega^1 : \Omega \rightarrow \Omega^1$ and $\omega^2 : \Omega \rightarrow \Omega^2$ are independent with respect to the random variable Y .*

Intuitively, the definition says that, conditionally on each synchronization constraint Y , the two local components are independent.

Besides allowing a concise definition, treating Y as a random variable in Def. 3.1 has the advantage of not distinguishing between the cases where the synchronization sequence Y is finite or infinite. We give below however a more concrete characterization of the LIP (§ 3.2).

It is useful to examine first some particular cases of Def. 3.1, that can be seen as trivial cases. For this, we need a lemma that explores the case where relative independence of Def. 3.1 reduces to actual independence.

Lemma 3.1. *Assume that \mathbb{P} defines a Markov 2-components probabilistic process such that the two local trajectories given by $\omega \in \Omega \mapsto \omega^1$ and $\omega \in \Omega \mapsto \omega^2$ are independent. Then ω^1 and ω^2 define two independent homogeneous Markov chains with values in S^1 and S^2 respectively.*

Proof of Lemma 3.1. Since ω^1 and ω^2 are assumed to be independent, we only need to show that their laws are those of two homogeneous Markov chains. For $i = 1, 2$, let \mathbb{P}^i be the law of ω^i , characterized by the following values:

$$\mathbb{P}^i(\omega^i \geq s^i) = \mathbb{P}(\omega^i \geq s^i),$$

for s^i ranging over the set of finite local trajectories (= sequences in S^i). The ordering “ \leq ” used above is the prefix ordering on sequences. To show that \mathbb{P}^i is the law of a homogeneous Markov chain, it is enough to show that for any finite local trajectory s^i , the conditional probability $\mathbb{P}^i(s^i \cdot \bullet \mid \uparrow s^i)$ only depends on

the last state in sequence s^i . Fix $i = 1$, the case $i = 2$ is similar. For s^1 a finite sequence in S^1 , such that $\mathbb{P}(\omega^1 \geq s^1) > 0$, let x^1 be the last state of s^1 . Pick an arbitrary finite sequence s^2 in S^2 such that $\mathbb{P}(\uparrow (s^1, s^2)) > 0$; such a sequence exists since $\mathbb{P}(\omega^1 \geq s^1) > 0$. Put $s = (s^1, s^2)$, and let x^2 be the last element of s^2 . For any finite sequence σ in S^1 , we have:

$$\begin{aligned}
\mathbb{P}^1(\omega^1 \geq s^1 \cdot \sigma | \omega^1 \geq s^1) &= \frac{\mathbb{P}(\omega^1 \geq s^1 \cdot \sigma)}{\mathbb{P}(\omega^1 \geq s^1)} \\
&= \frac{\mathbb{P}(\omega^1 \geq s^1 \cdot \sigma)}{\mathbb{P}(\omega^1 \geq s^1)} \times \frac{\mathbb{P}(\omega^2 \geq s^2)}{\mathbb{P}(\omega^2 \geq s^2)} \\
&= \frac{\mathbb{P}(\omega^1 \geq s^1 \cdot \sigma, \omega^2 \geq s^2)}{\mathbb{P}(\omega^1 \geq s^1, \omega^2 \geq s^2)} && \text{by independence} \\
&= \mathbb{P}_s(\uparrow (\sigma, \epsilon)) && \text{with } \epsilon \text{ the empty sequence} \\
&= \mathbb{P}_{(x^1, x^2)}(\uparrow (\sigma, \epsilon)) && \text{since } \mathbb{P} \text{ is Markov.} \blacksquare
\end{aligned}$$

We thus obtain the following identity:

$$\mathbb{P}^1(\omega^1 \geq s^1 \cdot \sigma | \omega^1 \geq s^1) = \mathbb{P}_{(x^1, x^2)}(\uparrow (\sigma, \epsilon)). \quad (9)$$

It is obvious that the left member of (9) is independent of x^2 , last state of s^2 arbitrarily chosen. Therefore, the right member of (9) is independent of x^2 as well. It is then visible that the right member of (9), only depends on x^1 and σ . Therefore the conditional probability $\mathbb{P}^1(\omega^1 \geq s^1 \cdot \bullet | \omega^1 \geq s^1)$ only depends on x^1 , which was to be shown. \square

Case where Q is empty. Then the random variable Y is constant, equal to the empty sequence. The independence with respect to Y in Def. 3.1 reduces to usual independence: the two local trajectories ω^1 and ω^2 , seen as random variables, are independent. According to Lemma 3.1, the local trajectories are two independent Markov chains.

Case where Q has one element. Say $Q = \{a\}$. We analyse the process defined by \mathbb{P}_x , where x is any recurrent state. According to Prop. 2.2, we are in one of the following cases:

- If \mathbb{P}_x is open, then it follows from the previous case that \mathbb{P}_x defines a product of two independent Markov chains.
- If \mathbb{P}_x is closed, then the random variable Y is constant, this time equal to the infinite sequence $(aaaa\dots)$ with probability 1. Hence, as in the previous case, the independence relative to Y reduces to usual independence, and Lemma 3.1 applies: the local trajectories are also two independent Markov chains, with a as a recurrent state.

We summarize the above results in the following proposition:

Proposition 3.1. *If the common state space Q of a Markov 2-components probabilistic process \mathbb{P} , satisfying the LIP, has no element, then \mathbb{P} is the law of two independent Markov chains. If Q has only one element, the same is true for \mathbb{P}_x , where x is any recurrent state.*

It also follows from the above discussion that, in the study of Markov 2-components probabilistic processes with the LIP, we might only retain the case of *closed* processes. Indeed, since there are finitely many states, transient states are eventually left and therefore we may assume without loss of generality that the process starts from a recurrent state. From there, the process is either open or closed according to Prop. 2.2; and it reduces to a product of independent Markov chains if it is open, by the same argument as above using Lemma 3.1. Hence, the core of our theory is occupied by closed processes.

3.2 Second Formulation of the LIP

We decompose global trajectories into pieces, in such a way that the LIP is equivalent to a suitable independence between these pieces. Intuitively, the decomposition follows the alternation between shared and non-shared states.

Thanks to the remark following Prop. 3.1, we consider a *closed* Markov 2-components probabilistic process. Fix a trajectory $\omega = (\omega^1, \omega^2)$. Let Y denote the Q -sequence induced by ω ; then Y is an infinite sequence with probability 1 and we put $Y = (y_1, y_2 \dots)$. Then the local trajectories ω^1 and ω^2 are represented as an infinite concatenation

$$\omega^i = (Z_1^i \cdot Z_2^i \cdot Z_3^i \dots), \quad i = 1, 2 \quad (10)$$

where all Z_j^i have the following form

$$Z_j^i = (E_j^i \cdot y_j), \quad j = 1, 2, \dots,$$

and all E_j^i are finite sequences with values in $S^i \setminus Q$. It is clear that each ω synchronizing infinitely often has a unique such decomposition.

All the above values, namely Y , y_j , Z_j^i and E_j^i for $j \geq 1$, are defined with probability 1 if the process is closed. In other words, they define random variables. With these notations in mind, we give an alternative formulation to Def. 3.1 for closed processes.

Proposition 3.2. *A closed Markov 2-components probabilistic process has the LIP if and only if for all $j \geq 1$, the two random variables Z_j^1 and Z_j^2 are independent conditionally on the values of y_{j-1} and y_j (where y_0 is given an arbitrary, constant value).*

Remark. We have formulated the proposition for closed processes. Treating the case of general processes would lead us to distinguish between the indices j such that y_j is defined, and those for which y_j is not defined.

Proof of Proposition 3.2. Let (a) be the property that ω^1 and ω^2 are independent with respect to Y , and let (b) be the property stated in the proposition.

Proof of (a) \Rightarrow (b). Assume that property (a) is in force, it is enough to show

$$\mathbb{P}(Z_j^2 = z^2 \mid y_{j-1} = a, y_j = b, Z_j^1 = z^1) = \mathbb{P}(Z_j^2 = z^2 \mid y_{j-1} = a, y_j = b), \quad (11)$$

for all integers $j \geq 1$, for all possible values z^1, z^2 of Z_j^1 and Z_j^2 respectively and for all $a, b \in Q$. Using the Markov property, there is no loss of generality

by assuming that $j = 1$; using the notations Z^1 and Z^2 for Z_1^1 and Z_1^2 , Eq. (11) reduces to

$$\mathbb{P}(Z^2 = z^2 | y_1 = b, Z^1 = z^1) = \mathbb{P}(Z^2 = z^2 | y_1 = b). \quad (12)$$

Since $\{y_1 = b\}$ is Y -measurable, and $\{Z^1 = z^1\}$ and $\{Z^2 = z^2\}$ are respectively ω^1 and ω^2 -measurable, Eq. (12) follows from the independence of ω^1 and ω^2 with respect to Y .

Proof of (b) \Rightarrow (a). Assuming that property (b) is in force, we claim first that for all possible values $a_j \in Q$ and z_j^i with $i = 1, 2$ and $j = 1, \dots, n$, we have:

$$\begin{aligned} \mathbb{P}(Z_1^2 = z_1^2, \dots, Z_n^2 = z_n^2 | y_1 = a_1, \dots, y_n = a_n, Z_1^1 = z_1^1, \dots, Z_n^1 = z_n^1) \\ = \mathbb{P}(Z_1^2 = z_1^2, \dots, Z_n^2 = z_n^2 | y_1 = a_1, \dots, y_n = a_n). \end{aligned} \quad (13)$$

Indeed, Eq. (13) follows from property (b) used in conjunction with the Markov property and usual recursive conditioning technique. Then, by summing and conditioning over all possible values of y_j and Z_j^1 for $j = 1, \dots, m$, we get that for any integer $n \geq 1$ and for any integer $m \geq 1$ large enough:

$$\begin{aligned} \mathbb{P}(Z_1^2 = z_1^2, \dots, Z_m^2 = z_m^2 | y_1 = a_1, \dots, y_n = a_n, Z_1^1 = z_1^1, \dots, Z_n^1 = z_n^1) \\ = \mathbb{P}(Z_1^2 = z_1^2, \dots, Z_m^2 = z_m^2 | y_1 = a_1, \dots, y_n = a_n). \end{aligned}$$

Since the process is assumed to be closed, the elementary cylinders $\{Z_1^2 = z_1^2, \dots, Z_m^2 = z_m^2\}$ generate the σ -algebra generated by ω^2 . Hence we obtain, for any function $h : \Omega^2 \rightarrow \mathbb{R}$ bounded and measurable:

$$\begin{aligned} \forall n \geq 1, \quad \mathbb{E}(h(\omega^2) | y_1 = a_1, \dots, y_n = a_n, Z_1^1 = z_1^1, \dots, Z_n^1 = z_n^1) \\ = \mathbb{E}(h(\omega^2) | y_1 = a_1, \dots, y_n = a_n). \end{aligned} \quad (14)$$

By the Martingale convergence theorem, the left member of Eq. (14) converges \mathbb{P} -almost surely to $\mathbb{E}(h(\omega^2) | Y, \omega^1)$, while the right member converges to $\mathbb{E}(h(\omega^2) | Y)$. Therefore:

$$\mathbb{E}(h(\omega^2) | Y, \omega^1) = \mathbb{E}(h(\omega^2) | Y),$$

and this holds for every function $h : \Omega^2 \rightarrow \mathbb{R}$ bounded and measurable. This is equivalent to property (a). \square

4 Characterization of Markov 2-Components Probabilistic Processes with the LIP

We now seek a complete description of Markov 2-components probabilistic processes satisfying the LIP. We have defined these processes in an abstract manner, by a condition on the probability measure \mathbb{P} defined on the space Ω of global trajectories. What we are looking for now is a complete characterization of such a process by a finite number of real parameters, analogous to the description of a finite Markov chain by means of its initial distribution together with its transition matrix. Although the initial distribution has a clear analogue here, the corresponding transition matrix is not obvious, in particular since the transition matrix of a Markov chain is tightly dependent of the *totally ordered* nature of time.

4.1 Construction and Characterization

As we will see, the correct notion to quantify a Markov 2-components probabilistic process is not a single transition matrix with the LIP, but rather a *family of transition matrices*, introduced in Def. 4.1 below. Before that, a first step is to establish the two following results, valid without the LIP.

Lemma 4.1. *If Y denotes the (infinite) synchronization sequence induced by a closed Markov 2-components probabilistic process, then Y is a Markov chain on the set $Q = S^1 \cap S^2$.*

Proof of Lemma 4.1. The formulation (3) applies to the chain Y as follows: for any two finite sequences s and u in Q , the conditional probability $\mathbb{P}(Y \geq s \cdot u | Y \geq s)$ only depends on u and on the last state of s . This shows that Y is a homogeneous Markov chain. \square

Remark. If the process considered was not closed, the resulting synchronizing sequence Y would be in general a *stopped* Markov chain.

We have in view to use the decomposition stated in Eq. (10) to entirely describe the 2-components probabilistic process. Focusing on local sequences $(Z_j^i)_{j \geq 1}$, for $i = 1, 2$, our next observation is the following—keeping in mind that Z_j^i itself is defined as a *sequence* of states in S^i .

Lemma 4.2. *Consider a closed Markov 2-components probabilistic process. Then, for $i = 1, 2$ and for every $j \geq 1$, the sequence Z_j^i is a stopped Markov chain conditionally on the values of y_{j-1}, y_j (where y_0 is given a conventional, constant value).*

Proof of Lemma 4.2. Consider any reachable state $x = (x_0^1, x_0^2) \in S^1 \times S^2$, and an element $a \in Q$ such that $\mathbb{P}_x(y_1 = (a, a)) > 0$. We denote by $\omega^1 = (X_n^1)_{n \geq 0}$ the local trajectory on site 1. Consider $\tau : \Omega \rightarrow \mathbb{N}$, defined almost surely as the first hitting time of Q for the sequence $(X_n^1)_{n \geq 0}$. Denote by \mathbb{Q} the conditional probability

$$\mathbb{Q} = \mathbb{P}_x(\cdot | y_1 = (a, a)).$$

Our first claim is that the stopped process $Z_1^1 = (X_0^1, X_1^1, \dots, X_\tau^1)$ is a stopped homogeneous Markov chain under \mathbb{Q} . For this, let $n \geq 1$ be any integer and let x_1, \dots, x_n be values in $S^1 \setminus Q$, and $x_{n+1} \in S^1 \cup \{a\}$. We put

$$\delta = \mathbb{Q}(X_{n+1}^1 = x_{n+1} | X_1^1 = x_1, \dots, X_n^1 = x_n)$$

and we show that δ only depends on x_n and x_{n+1} . Indeed, we compute:

$$\begin{aligned}
\delta &= \frac{\mathbb{Q}(X_1^1 = x_1, \dots, X_n^1 = x_n, X_{n+1}^1 = x_{n+1})}{\mathbb{Q}(X_1^1 = x_1, \dots, X_n^1 = x_n)} \\
&= \frac{\mathbb{P}_x(X_1^1 = x_1, \dots, X_n^1 = x_n, X_{n+1}^1 = x_{n+1}, X_\tau^1 = a)}{\mathbb{P}_x(X_1^1 = x_1, \dots, X_n^1 = x_n, X_\tau^1 = a)} \\
&= \frac{\mathbb{P}_x(\omega^1 \geq (x_0^1 x_1^1 \dots x_{n+1}^1), X_\tau^1 = a)}{\mathbb{P}_x(\omega^1 \geq (x_0^1 x_1^1 \dots x_n^1), X_\tau^1 = a)} \\
&= \frac{\mathbb{P}_x(\omega \geq (x_0^1 \dots x_{n+1}^1, x_0^2), X_\tau^1 = a \mid \omega \geq (x_0^1 \dots x_n^1, x_0^2))}{\mathbb{P}_x(X_\tau^1 = a \mid \omega \geq (x_0^1 \dots x_n^1, x_0^2))} \\
&= \frac{\mathbb{P}_{(x_n, x_0^2)}(X_1^1 = x_{n+1}, X_\tau^1 = a)}{\mathbb{P}_{(x_n, x_0^2)}(X_\tau^1 = a)} = \mathbb{P}_{(x_n, x_0^2)}(X_1^1 = x_{n+1} \mid X_\tau^1 = a).
\end{aligned}$$

It is now visible on the above expression that δ only depends on x_n and x_{n+1} , and not on x_1, \dots, x_{n-1} , showing our first claim.

Since the first claim is true for any initial state x , it is actually true for any initial distribution, and thus it is true for \mathbb{P} instead of \mathbb{P}_x ; and this is the statement of the lemma for $j = 1$, since conditioning with respect to the conventional value y_0 is equivalent to not conditioning, and thus to considering probability $\mathbb{P}(\cdot \mid y_1)$ directly. The statement of the lemma then follows for all $j \geq 1$ using the Markov property. \square

As a consequence, one can consider the following construction for a 2-components probabilistic processes: first construct the sequence $Y = (y_j)_{j \geq 1}$, which is a Markov chain by Lemma 4.1. Then, for each $j \geq 1$ and $i = 1, 2$, replace y_j by a stopped Markov chain Z_j^i that ends with y_j , whose transition matrix only depends on the value y_j and whose initial distribution is the Dirac distribution on y_{j-1} (excepted for $j = 0$, where the initial distribution is arbitrary). By the LIP, for a given integer $j \geq 1$, the two local sequences E_j^1 and E_j^2 are independent with respect to y_j . The 2-components probabilistic process is now entirely constructed. This description is formalized in Theorem 4.1 below. It is convenient to adopt the following definitions:

Definition 4.1. *Let two finite sets S^1 and S^2 be given, with $Q = S^1 \cap S^2$. An **adapted family of transition matrices** is given by two families $(R_y^i)_{y \in Q}$, one for each $i = 1, 2$, such that:*

1. *The R_y^i 's are transition matrices on $S^i \setminus Q \cup \{y\}$;*
2. *Any Markov chain with R_y^i as transition matrix has probability 1 to reach y .* \blacksquare

With this definition at hand, the existence and uniqueness result states as follows.

Theorem 4.1. *Any closed Markov 2-components probabilistic process with the LIP induces the following elements, that entirely characterize the probability measure \mathbb{P} defined on the space (Ω, \mathfrak{F}) of global trajectories:*

1. *An initial distribution μ on the space of global states, given as in Def. 1.1.*

2. A transition matrix R on Q , defined as the transition matrix of the induced Markov chain $Y = (y_j)_{j \geq 1}$.
3. An adapted family of transition matrices $(R_y^i)_{y \in Q}$, for $i = 1, 2$. For each $i = 1, 2$ and $y \in Q$, and for any integer $j \geq 1$, R_y^i is the transition matrix of the chain Z_j^i conditionally on $y_j = y$.

Conversely, given:

1. A probability distribution μ on $S^1 \times S^2$;
2. A transition matrix R on the set Q ;
3. An adapted family of transition matrices $(R_y^i)_{y \in Q}$, $i = 1, 2$;

there exists a unique closed Markov 2-components probabilistic process inducing such elements. In other words, there is a unique probability measure \mathbb{P} on the space Ω of global trajectories, with μ as initial distribution, with R as transition matrix for the induced Markov chain Y , and such that for each $j \geq 1$ and $i = 1, 2$, the sequence Z_j^i is, conditionally on $y_j = y$, a stopped Markov chain with R_y^i as transition matrix and stopped when y is reached.

Remark. An interpretation of Theorem 4.1 is that, starting from a Markov chain Y , replacing each state y of Y by a pair of stopped Markov chains yields a process that is still Markov, in the 2-components sense. In other words, the private parts do not disturb the global process.

Proof of Theorem 4.1. We quickly review the routine probability arguments, and we give more details on the new part of the theorem.

The LIP guarantees that, given the two laws of E_j^1 and E_j^2 , conditionally on y_j , the law of the pair (E_j^1, E_j^2) conditionally on y_j is known, as the product of the two marginal laws. Hence, applying the Kolmogorov theorem (see, e.g., [6, Th. 2.8 p. 16]) with state space the infinite countable set of pairs (z^1, z^2) , where z^i is any finite sequence in $S^i \setminus Q$, except for the last element that belongs to Q , the uniqueness property shows that the data 1-2-3 in the statement of the theorem are enough to uniquely characterize a given 2-components probabilistic process with the LIP.

Conversely, if the data 1-2-3 is given, using this time the existence part of the Kolmogorov theorem, we construct a closed 2-components probabilistic process, with a probability measure \mathbb{P} inducing the given elements. Hence, beside the routine arguments that apply with no mathematical difficulty in our case, what is really needed to be shown is that the so-obtained process satisfies the Markov property; since, by construction and in conjunction with Prop. 3.2, it is clear that the LIP is satisfied by the process.

We have to show that, for any finite trajectory t such that $\mathbb{P}(\uparrow t) > 0$, the probability \mathbb{P}_t on $\gamma(t)$. It is clear that this is the case if $\gamma(t)$ has the form $\gamma(t) = (a, a)$ for some element $a \in Q$. Hence, up to a shift in t , we may assume without loss of generality that, if $t = (t^1, t^2)$, then t^1 and t^2 are sequences in $S^1 \setminus Q$ and $S^2 \setminus Q$ respectively, except maybe for their first element that may belong to Q .

Assuming that t has this particular form, we need to show that, for any finite trajectory σ , the quantity $\mathbb{P}_t(\uparrow \sigma)$ only depends on $\gamma(t)$ and σ . Our first

claim is that this is the case if σ has the form $\sigma = (\sigma^1, \sigma^2)$, where σ^1 and σ^2 are sequences in $S^1 \setminus Q$ and $S^2 \setminus Q$ respectively, except for their last element which is a common element $a \in Q$. We have:

$$\uparrow(t \cdot \sigma) = \{Z_1^1 = t^1 \cdot \sigma^1, Z_1^2 = t^2 \cdot \sigma^2\}.$$

Considering an element $y \in Q$, let \mathbb{Q}^1 and \mathbb{Q}^2 denote the probability laws for Markov chains associated with transition matrices R_y^1 and R_y^2 , respectively, and starting from the same elements in S^1 and S^2 respectively than t . We also denote, for $i = 1, 2$, by \mathbb{Q}_b^i the law of the Markov chain with the same transition matrix, and starting from state $b \in S^i$. We compute:

$$\begin{aligned} \mathbb{P}_t(\uparrow \sigma | Y_1 = y) &= \frac{\mathbb{P}(\uparrow(t \cdot \sigma) | Y_1 = y)}{\mathbb{P}(\uparrow t | Y_1 = y)} \\ &= \frac{\mathbb{Q}^1(\uparrow(t^1 \sigma^1))}{\mathbb{Q}^1(\uparrow t^1)} \times \frac{\mathbb{Q}^2(\uparrow(t^2 \sigma^2))}{\mathbb{Q}^2(\uparrow t^2)} \quad \text{by independence w.r.t. } Y_1 \\ &= \mathbb{Q}_{\gamma(t^1)}^1(\uparrow \sigma^1) \times \mathbb{Q}_{\gamma(t^2)}^2(\uparrow \sigma^2), \end{aligned}$$

where $\gamma(t^1)$ and $\gamma(t^2)$ denote the last states of t^1 and t^2 respectively. It is clear on the last term that $\mathbb{P}_t(\uparrow \sigma | Y_1 = y)$ only depends on $\sigma = (\sigma^1, \sigma^2)$, $\gamma(t)(\gamma(t^1), \gamma(t^2))$, and on y . Therefore the probability $\mathbb{P}_t(\uparrow \sigma) = \sum_y \mathbb{P}_t(y_1 = y) \mathbb{P}_t(\uparrow \sigma | Y_1 = y)$ only depends on σ and γ . This shows our first claim.

We deduce from this that $\mathbb{P}_t(\uparrow \sigma)$ only depends on $\gamma(t)$ and σ , not only if σ has the above form, but also if $\sigma = (\sigma^1, \sigma^2)$ is composed of sequences σ^1 and σ^2 that only contain elements of $S^1 \setminus Q$ and $S^2 \setminus Q$, respectively. For, for such a finite trajectory σ , we have:

$$\mathbb{P}_t(\uparrow \sigma) = \sum_{\sigma'} \mathbb{P}_t(\uparrow \sigma'),$$

where σ' ranges over those finite trajectories such that $\sigma \leq \sigma'$, and that are composed of private elements exclusively, but for their last element which belongs to Q . By the previous case, and thanks to the above expression, we conclude that $\mathbb{P}_t(\uparrow \sigma)$ only depends on $\gamma(t)$ and σ , in this case as well.

We then extend the result to any σ since, by construction of \mathbb{P} and as we observed earlier, as soon as a synchronizing element $(a, a) \in Q \times Q$ occurs in a finite trajectory σ , the probability of the tail of σ after (a, a) only depends on a . Finally, we conclude that $\mathbb{P}_t(\bullet)$ only depends on $\gamma(t)$, showing that the 2-components probabilistic process is Markov. \square

4.2 An Example: Synchronization of Markov Chains

This work was initiated by the study of an example, the so-called synchronization of Markov chains, which now appears as a particular case of Markov 2-components probabilistic processes with the LIP. Consider two ergodic Markov chains $(X_n^i)_{n \geq 0}$, $i = 1, 2$, with state spaces S^1 and S^2 , with a nonempty intersection $Q = S^1 \cap S^2$. Excepted in very particular cases, the event that two local trajectories are two synchronizing sequences has zero probability in the product space, spanned by the pairs of trajectories. Hence, “brute conditioning” on the synchronization constraint is not operational for the construction of a 2-components probabilistic process.

We proposed in [3] the following construction: first stop both chains on their reaching time of Q , let τ^1 and τ^2 be the reaching times, and condition the pair of processes under their product probability on the event $\{X_{\tau^1}^1 = X_{\tau^2}^2\}$. Under quite general conditions, this event has positive probability in the product space. Then shift the process and repeat the construction by conditioning, infinitely many times. This amounts to the construction of Theorem 4.1, in a particular case. It is now a consequence of Theorem 4.1 that this construction yields a Markov 2-components probabilistic process with the LIP. Furthermore, it is now easy to recognize that not any Markov 2-components probabilistic process with the LIP can be obtained as the synchronization of two Markov chains.

5 Conclusion and Discussion

We have introduced a model a probabilistic process allowing concurrency of local components. We have distinguished two properties: the Markov property, that extends the Markov property in the case of usual, sequential processes, and the local independence property. The later property is specific to our model, since it is a condition on the relative independence of local components, and is thus not applicable for sequential processes, without concurrency. The model we consider has the same basic ingredients than any probabilistic model: a space of trajectories, on top of which we construct some probability measure with particular properties. However the meaning of “trajectory” is quite different in our case: instead of considering sequences of successive states, a trajectory consists of a partial order of local states. Hence, insisting on a state space that takes into account the *distributed* character of the system induces a distributed property for time as well. We detail this point below.

5.1 Time in 2-Component Probabilistic Processes

It has been noted in § 2.2 that the only Markov property available in our context is a Strong Markov property, that is to say, a property formulated with respect to stopping times. In other words, a weak Markov property formulated with *constant* times is unavailable. We argue in fact that the notion of *constant time* itself is unavailable.

In sequential processes, all trajectories are infinite sequences. Seen as partial orders, they are thus all isomorphic with each other. There is therefore an abstract “skeleton”, namely the ordered set \mathbb{N} of integers, that is a universal model for every trajectory. Considering a “constant time” is nothing but designating a place on this universal scale; and then stopping any given trajectory on the place of the trajectory that corresponds to the given time on the universal scale.

Now in a context with concurrency, although we have introduced the notion of length as a pair of integers, it cannot replace the notion of constant time, basically because there is no universal skeleton for all trajectories. For instance, consider the length $(4, 5)$; now, extracting from a given trajectory ω the unique prefix of ω of length $(4, 5)$ will not produce a subtrajectory of ω in general, because local components are asynchronous. It has thus no sense to talk about an instant that would be indexed by $(4, 5)$.

Therefore, the natural solution to obtain “time instants” in general is to use stopping times, that are instants of times that depend on the given trajectory.

This emphasizes the *dynamics of time for concurrency*.

Another topic related to time and concurrency has counter-intuitive consequences. In view of Th. 4.1, one can be surprised by the construction suggested. For, if one only observes one component, it seems that the current evolution depends on the future; since the current evolution law is determined by next synchronisation state. In particular, a local component is by no mean a Markov chain itself. This strange effect has its roots in concurrency; it only appears when looking at one particular component. This disturbing effect of dependency with respect to future disappears when considering the global process, where the concurrent Markov property is in force.

5.2 Two Research Directions

There are two obvious directions for future work: one concerns the extension of this work to an arbitrary number of sites; the other one concerns the study of the asymptotic behaviour of 2-components probabilistic processes, or of more general multi-component processes. We will quickly review some of the difficulties that we face for these two challenges.

The extension to an arbitrary number of sites should start from the same principles than we did for two sites. With two sites only, synchronizations occur in a totally ordered manner: this is the chain Y of this paper. The novelty with more sites is that synchronization events themselves will be concurrent. If each site has private and public states, one can actually make abstraction of private states; for, as the present paper has shown, the interleaving of private sequences between synchronization on public states does not disturb the global process, and keeps all required properties such as the Markov property. Hence, focusing on the public part, one realizes that the probabilistic behaviour of synchronization is another topic, where concurrency and probabilistic independence may not be related as we did for private states by conditioning with respect to the appropriate variable. In other words, the combination of synchronization and concurrency brings another challenge, that we avoided by considering only two sites in this paper.

The study of asymptotic behaviours, for two or more sites, also brings new challenges. First, a notion of convergence must be explored for concurrent processes, since no ordered time is available. In the, related but different, study of probabilistic event structures [2], we considered appropriate *sequences of stopping times* as replacements for “ $n \rightarrow \infty$ ”. This could be explored here as well. The difficulty, and also the mathematical interest, lies in the structure of each individual component. As we observed earlier, it is not Markov. A prospective research direction to relate them to known theories is to explore the theory of random walks in random environments, where typically transition matrices are first chosen at random, then used as “normal” transition matrices. Intuitively, each local site does evolve in a *random environment with feedback*: the transition matrices that govern its evolution depend on the environment (*i.e.*, the other component), but each component has some effect on the environment as well. The theory of random walks in random environments has been studied since the 1970’s, and is still a very active field of research. Establishing a clear bridge with the theory of random walks in random environments is one of the most promising research direction for the study of concurrent probabilistic processes.

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Appendix: Background on Conditional Independence and Conditional Expectation

In this appendix we recall some basic definitions on conditional independence and conditional expectation. This is *not* a short presentation of these notions; in particular, no intuition is given. For longer explanations, see classical references on probability theory (Doob's *Stochastic processes*, Breiman's *Probability* or Billingsley's *Probability and Measure*, for instance).

A probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is given throughout the Appendix.

1 Background on Conditional Expectation

Let $X : \Omega \rightarrow \mathbb{R}$ be a bounded random variable, and let \mathfrak{G} be a sub- σ -algebra of \mathfrak{F} . The *conditional expectation of X with respect to \mathfrak{G}* is the random variable Y , uniquely defined \mathbb{P} -almost surely, such that:

1. Y is \mathfrak{G} -measurable; and
2. For every bounded and \mathfrak{G} -measurable variable $\phi : \Omega \rightarrow \mathbb{R}$, we have $\mathbb{E}(X\phi) = \mathbb{E}(Y\phi)$.

The conditional expectation of X with respect to \mathfrak{G} is denoted $\mathbb{E}(X|\mathfrak{G})$.

If \mathfrak{G} is the σ -algebra generated by a random variable Z , we use the notation

$$\mathbb{E}(X|\mathfrak{G}) = \mathbb{E}(X|Z).$$

2 Background on Conditional Independence

Let X_1 and X_2 be two random variables, with values in any measurable space, and let Y be a random variable defined on $(\Omega, \mathfrak{F}, \mathbb{P})$. We say that X_1 and X_2 are *independent with respect to Y* if, for any two functions $h_1 : \Omega \rightarrow \mathbb{R}$ and $h_2 : \Omega \rightarrow \mathbb{R}$ such that h_i is bounded and measurable with respect to X_i , for $i = 1, 2$, we have:

$$\mathbb{E}(h_1 h_2 | Y) = \mathbb{E}(h_1 | Y) \mathbb{E}(h_2 | Y), \quad \mathbb{P}\text{-a.s.}$$

Equivalently, X_1 and X_2 are independent with respect to Y if and only if, for any function $h_2 : \Omega \rightarrow \mathbb{R}$ bounded and X_2 -measurable, we have

$$\mathbb{E}(h_2 | X_1, Y) = \mathbb{E}(h_2 | Y),$$

where $\mathbb{E}(h_2 | X_1, Y)$ denotes the conditional expectation of h_2 with respect to the σ -algebra generated by Y and X_1 .

References

- [1] S. Abbes. The (true) concurrent Markov property and some applications to Markov nets. In G. Ciardo and P. Darondeau, editors, *International Conference on Theory and Applications of Petri Nets (ICATPN 05)*, volume 3536 of *LNCS*, pages 70–89, 2005.
- [2] S. Abbes and A. Benveniste. Probabilistic true-concurrency models: Markov nets and a law of large numbers. *Theoretical Computer Science*, 390(2-3):129–170, 2008.
- [3] S. Abbes and A. Benveniste. The truly concurrent synchronization of Markov chains. Technical report, PPS preprints, 2009.
- [4] J.L. Doob. *Stochastic Processes*. John Wiley, 1953.
- [5] J. Neveu. *Mathematical Foundations of the Calculus of Probability*. Holden-Day, 1965.
- [6] D. Revuz. *Markov Chains*. North Holland, 1975.