LKQ and **LKT**: Sequent calculi for second order logic based upon dual linear decompositions of classical implication

by

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As is well known, one can recover intuitionistic logic within linear logic. Indeed, linear logic found its origin in a semantical decomposition of intuitionistic type constructors corresponding, in the sense of the 'Curry-Howard-de Bruijn isomorphism' (Howard(1980)), to the intuitionistic connectives (see Girard et al.(1988) for details). (A such decomposition in fact already appears in the simple set-theoretical models for the untyped lambda calculus.) Conversely this decomposition gives rise to Girard's embedding of intuitionistic into linear logic, which, for (the $\rightarrow, \forall_2$-fragment of) second order propositional intuitionistic logic, a.k.a.[1] system $\mathcal{F}$, is inductively defined as follows.

For atomic p let $p^\star := p$; then put

$$\begin{aligned} (A \rightarrow B)^\star &:= \; !A^\star \multimap B^\star \\ (\forall \alpha A)^\star &:= \; \forall \alpha A^\star. \end{aligned}$$

This mapping of intuitionistic to linear formulas is an example of what we call a(n) *(inductive) modal translation*: the translation of a given formula is obtainable by inductively prefixing the formula's subformulas by *modalities*, i.e. (possibly empty) strings of exponentials '!', '?'.[2]

Observe that for all such modalities μ both $!A \Rightarrow \mu A$ and $\mu A \Rightarrow ?A$ are derivable, whatever A: starting from an axiom $A \Rightarrow A$ and an application of L!, we can without restriction apply R? and R! to obtain μA in the first, and by the unrestricted possibility of using L! and L? starting from an axiom followed by an application of R? in the second case. Otherwise said: in

[1]I.e. *'also known as'*

[2]Note that a modal translation $(\cdot)^\checkmark$ will be *compatible with substitution* (i.e. for all A, B the formulas $(A[B/\alpha])^\checkmark$ and $A^\checkmark[B^\checkmark/\alpha]$ are identical) if and only if it is the identity on atoms.

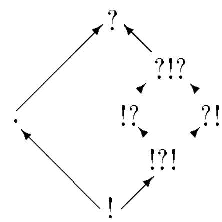

Figure 1: The lattice of linear modalities

the partial ordering on modalities induced by linear derivability ($\mu \preceq \nu$ iff $\vdash \mu A \Rightarrow \nu A$ is derivable in linear logic, for any A) we have that "!" is minimal, "?" is maximal.

If we consider the equivalence relation induced by this ordering, we find *seven* equivalence classes: calling $\cdot, ?, !, !?, ?!, !?!$ and $?!?$ (where "$\cdot$" stands for the void modality) *basic* modalities, one easily shows (e.g. using the *idempotency* of these basic modalities) that for any modality μ there is a *unique* basic μ_0 such that $CLL \vdash \mu A \Longleftrightarrow \mu_0 A$ for all A. So,

> *modulo provable linear equivalence, there are precisely seven modalities in linear logic.*

(Consequently, in *intuitionistic* linear logic, there is modulo linear equivalence just *one* non-trivial modality: "!".) Basic modalities are related as in figure 1, where an arrow from μ to ν indicates that $\mu \prec \nu$ (see Joinet(1993)).

As an easy corollary we then find that *all* modalities are idempotent: $CLL \vdash \mu A \Longleftrightarrow \mu\mu A$ for all μ, A.

Girard's embedding is both sound- and faithful, i.e. if , $\Rightarrow A$ is derivable in system $\mathcal{F}$, then so is !, $^* \Rightarrow A^*$ in classical linear logic; conversely, if CLL derives !, $^* \Rightarrow A^*$, then $\mathcal{F}$ will prove , $\Rightarrow A$ (see Schellinx(1994)).

Moreover, if in a linear derivation π of !, $^* \Rightarrow A^*$ we replace the linear connectives by their non-linear analogues, and simply forget about the exponentials, delete resulting repetitions of sequents, then what we find will in general be an intuitionistic derivation of , $\Rightarrow A$. We make the proviso "in general", for example because of the devious behaviour of the linear constant 0 (cf. Schellinx(1991)), which is linearly equivalent to $\forall\alpha.\alpha$. Anyway, it is a pretty trivial remark that this *collapsing* of a linear derivation π results in a derivation in *classical logic*, which we will refer to as π's *skeleton*. We express this as follows:

> *the* skeleton $sk(\pi)$ *of a linear derivation* π *is a derivation in (intuitionistic or) classical logic.*

Despite its obviousness, we consider this observation to be essential, not in the least because we will show that it has a converse:

CLAIM. *Any derivation in intuitionistic or classical sequent calculus occurs as the skeleton of a derivation in linear logic.*

This provides an interesting argument supporting linear logic's claim to being a *refinement* of intuitionistic and classical sequent calculus. Note e.g. that it enables us to obtain eliminability of cut from classical and intuitionistic derivations as a corollary to the linear cut elimination theorem.

DEFINITION. A *decoration* of a (classical, intuitionistic) derivation π is a linear derivation $\partial(\pi)$ such that $sk(\partial(\pi)) = \pi$; by a decoration-strategy for a given (sequent)calculus we mean a uniform procedure (algorithm) that outputs a decoration for any given derivation in the calculus. □

How to prove our claim? Well, one simply shows, given any intuitionistic or classical derivation π, how to produce a decoration $\partial(\pi)$. One possibility is to start from π and 'linearize' it by tracing the effects of occurrences of structural rules in π. We looked at this option in detail for intuitionistic implicational logic in Danos et al.(1993b) (see also Joinet(1993)).

A second possibility is to try to transform a given derivation π into a linear derivation $\partial(\pi)$, by inductively applying a modal translation to the sequents occurring in π.

DEFINITION. Let a modal translation $(\cdot)^{\checkmark}$ and modalities μ, ν be given. We say that the triple $\langle (\cdot)^{\checkmark}, \mu, \nu \rangle$ defines an *inductive decoration strategy* for a sequent calculus $\mathcal{S}$ if

1/ for all $\mathcal{S}$-axioms , $\Rightarrow \Delta$ it holds that $\mu, {}^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$ is a CLL-axiom or obtainable from such solely by means of zero or more applications of exponential contextual and/or dereliction rules;

Identity axiom and cut rule:

$$\text{Ax}\quad A \Rightarrow A \qquad\qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Logical rules:

$$\text{L}{\multimap}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', A \multimap B \Rightarrow \Delta, \Delta'} \qquad\qquad \text{R}{\multimap}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}$$

Rules for the second order quantifier (β not free in Γ, Δ):

$$\text{L}\forall_2\ \frac{\Gamma, A[X/\alpha] \Rightarrow \Delta}{\Gamma, \forall\alpha\, A \Rightarrow \Delta} \qquad\qquad \text{R}\forall_2\ \frac{\Gamma \Rightarrow \Delta, A[\beta/\alpha]}{\Gamma \Rightarrow \Delta, \forall\alpha\, A}$$

Exponential structural rules:

$$\text{W!}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad \text{W?}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?A, \Delta} \qquad \text{C!}\ \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad \text{C?}\ \frac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

Exponential contextual rules:

$$\text{L?}\ \frac{!\Gamma, A \Rightarrow ?\Delta}{!\Gamma, ?A \Rightarrow ?\Delta} \qquad\qquad \text{R!}\ \frac{!\Gamma \Rightarrow A, ?\Delta}{!\Gamma \Rightarrow !A, ?\Delta}$$

Exponential dereliction rules:

$$\text{R?}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow ?A, \Delta} \qquad\qquad \text{L!}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$

Table 1: Linear logic, the $\{!, ?, \multimap, \forall_2\}$-fragment

2/ for all $\mathcal{S}$-rules with conclusion $\ , \Rightarrow \Delta$ and premiss(es) $\ ,_i \Rightarrow \Delta_i$ we can derive $\mu, {}^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$ in linear logic from $\mu, {}^{\checkmark}_i \Rightarrow \nu\Delta^{\checkmark}_i$ by an application of the corresponding CLL-rule preceded and/or followed by zero or more applications of exponential contextual and/or dereliction rules. □

Obviously, by definition, if $\langle(\cdot)^{\checkmark}, \mu, \nu\rangle$ is an inductive decoration strategy for a calculus $\mathcal{S}$, then, given an $\mathcal{S}$-derivation π of a sequent $\ , \Rightarrow \Delta$, we can apply the translation $(\cdot)^{\checkmark}$ inductively to π and derive $\mu, {}^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$ by means of a linear derivation $\pi^{\checkmark}$ which is a decoration of the original one.

In the present paper we will be mainly interested in the $\{\rightarrow, \forall_2\}$-fragments of second order propositional classical and intuitionistic logic. To be precise, we consider the corresponding fragments of sequent calculi for these logics. For brevity's sake we will simply refer to the fragments as CL, respectively IL. The calculus for CL is described in table 2. The calculus for IL is that for CL, subjected to the usual intuitionistic restriction of the succedents to singletons. The corresponding fragment of CLL is given as table 1. (Note that extending these fragments with rules for a *first order* universal quantifier is completely straightforward, and all results stated in what follows hold for these extensions. In proofs and definitions the case of the first order quantifier is completely analogous to that of the second order one.)

As we observed in Danos et al.(1993b), Girard's translation does not extend to sequent-calculus derivations: $\langle(\cdot)^*, !, \cdot\rangle$ does not define an inductive decoration strategy for IL. Using this translation, the inductive transformation of proofs will introduce cuts at several points. To be precise (in the case of system $\mathcal{F}$), each time we encounter an application of L$\rightarrow$, we apply a cut with the canonical derivation of $!(!A^* \multimap B^*) \Rightarrow !A^* \multimap !B^*$; and each time we encounter an application of L$\forall$, we cut with the canonical derivation of $!\forall\alpha A^* \Rightarrow \forall\alpha !A$. We call these the *'correction cuts'*.

However, note that the inductive application of Girard's translation to IL-derivations in fact indicates a modified (and, with respect to the number of shrieks introduced, less economical) translation $(\cdot)^{\circledast}$ that *does* define an inductive decoration strategy $\langle(\cdot)^{\circledast}, !, \cdot\rangle$ (and, of course, also is a sound- and faithful embedding of system $\mathcal{F}$ into CLL, which can be shown e.g. using the fact that $!A^* \Longleftrightarrow !A^{\circledast}$ is linearly provable). It is inductively given as follows. For atomic p let $p^{\circledast} := p$; then put

$$\begin{array}{rcl}(A \rightarrow B)^{\circledast} & := & !A^{\circledast} \multimap !B^{\circledast} \\ (\forall\alpha A)^{\circledast} & := & \forall\alpha !A^{\circledast}.\end{array}$$

Identity axiom and cut rule:

$$\text{Ax}\quad A \Rightarrow A \qquad\qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Logical rules:

$$\text{L}{\rightarrow}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', A \rightarrow B \Rightarrow \Delta, \Delta'} \qquad\qquad \text{R}{\rightarrow}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

Rules for the second order quantifier (β not free in Γ, Δ):

$$\text{L}\forall_2\ \frac{\Gamma, A[X/\alpha] \Rightarrow \Delta}{\Gamma, \forall\alpha\, A \Rightarrow \Delta} \qquad\qquad \text{R}\forall_2\ \frac{\Gamma \Rightarrow \Delta, A[\beta/\alpha]}{\Gamma \Rightarrow \Delta, \forall\alpha\, A}$$

Structural rules:

$$\text{WL}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \text{WR}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \qquad \text{CL}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \text{CR}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

Table 2: CL, the $\{\rightarrow, \forall_2\}$-fragment

Consequently we established the first, intuitionistic, half of our claim.

But, as a matter of fact, *no* strategy can lead to decorations of IL-derivations that are always *subGirardian*, i.e. do not shriek (sub)formulas that are not banged in the $(\cdot)^*$-translation. This is shown by the following example, where each decoration of the skeleton will necessarily contain this, *minimal*, decoration.

$$\frac{C \Rightarrow C \qquad \dfrac{A \Rightarrow A}{!B, A \Rightarrow A}}{C, C \multimap !B, A \Rightarrow A}$$

The exclamation mark appearing in front of B is forced by the use of the structural rule of weakening. Deleting it results in a non-linear derivation.

The 'root of all evil' apparently is that intuitionistic sequent calculus allows applications of e.g. the rules L$\rightarrow$, L$\forall_2$ in case the active formula in the (right) premiss has been subjected to structural manipulation.

The *correctness* of the $(\cdot)^*$-translation shows that we can do without that property: the collection of derivations that do *not* use it is complete for

Identity axiom:

$$A; \;\Rightarrow A$$

Logical rules:

$$\text{L}\!\rightarrow \frac{;\,,\;\Rightarrow A \qquad B;\,,\,' \Rightarrow C}{A \rightarrow B;\,,\,,\,,\,' \Rightarrow C} \qquad \text{R}\!\rightarrow \frac{\Pi;\,,\,,A \Rightarrow B}{\Pi;\,,\;\Rightarrow A \rightarrow B}$$

Rules for the second order quantifier (β not free in , , Π):

$$\text{L}\forall_2 \frac{A[X/\alpha];\,,\;\Rightarrow B}{\forall\alpha\, A;\,,\;\Rightarrow B} \qquad \text{R}\forall_2 \frac{\Pi;\,,\;\Rightarrow A[\beta/\alpha]}{\Pi;\,,\;\Rightarrow \forall\alpha\, A}$$

Structural rules:

$$\text{WL} \frac{\Pi;\,,\;\Rightarrow A}{\Pi;\,,\,,B \Rightarrow A} \qquad \text{CL} \frac{\Pi;\,,\,,B,B \Rightarrow A}{\Pi;\,,\,,B \Rightarrow A} \qquad \text{D} \frac{B;\,,\;\Rightarrow A}{;B,\,,\;\Rightarrow A}$$

Table 3: ILU, the cut-free fragment.

intuitionistic logic. Indeed, this is an immediate corollary to the subformula-property and the fact that the skeleton of a cut free intuitionistic linear derivation of $!,^{*} \Rightarrow A^{*}$ is an IL-derivation.

This suggests a formulation of intuitionistic sequent calculus in which the use of these rules on such, non-linear, formulas is forbidden, and for which as a consequence Girard's translation should be a decoration-strategy. Such a formulation can be found by a rather straightforward abstraction of the structure of linear derivations of sequents of the form $!,^{*} \Rightarrow A^{*}$ (table 3). [3]

In a sequent $\Pi;\,,\;\Rightarrow A$ the symbol Π denotes a multiset containing *at most* one (the *head-*)formula whose occurrence in a sequent is distinguished by means of the ";". In the linear interpretation it corresponds to a formula that is not (yet) shrieked. The structural rule D is the equivalent of L!, the linear dereliction rule.

Included we find the *neutral* fragment of intuitionistic implicational logic as it appears in Girard's system of Unified Logic (LU, Girard(1993)). For this reason we refer to the above calculus as ILU.

[3] Note that the instances of rules that we will get rid of have no direct equivalent in the natural deduction formulation of intuitionistic logic. Therefore this modified sequent calculus will be closer to natural deduction and the simply typed λ-calculus than the standard formulation. (The reader will find that the 'natural' way to interpret a natural deduction derivation in sequent calculus is as an ILU-derivation!)

$\Pi; ,\ \Rightarrow A$ is derivable in ILU if and only if $\Pi^*, !, ^* \Rightarrow A^*$ is derivable in the $\{!, \multimap, \forall_2^*\}$-fragment of linear logic, where $\forall_2^*$ indicates abstraction limited to formulas of the form X^* (observe that if a sequent $\Pi^*, !, ^* \Rightarrow !\Sigma^*, \Delta^*$ is derivable in this fragment, then $|\,\Pi \cup \Sigma\,| \leq 1$ and $|\,\Sigma \cup \Delta\,| = 1$). Moreover, by construction, Girard's translation $(\cdot)^*$ determines an inductive decoration strategy (in the sense of the above definition adapted to ILU-sequents in the obvious[4] way) for ILU-derivations π.

Thus we found a first example of what we might launch as a slogan but in fact is a

FACT. *Linear logic suggests restrictions on derivations in its underlying calculi, restrictions leading to subsets of the collection of these proofs that nevertheless are* complete.

It is not difficult to show that the collection of $(\cdot)^*$-decorated ILU-sequents is closed under cut (see Danos et al.(1993b)), from which it follows that ILU is closed under the rules

$$head - cut \quad \frac{\Pi; ,_1 \Rightarrow A \qquad A; ,_2 \Rightarrow B}{\Pi; ,_1, ,_2 \Rightarrow B}$$

$$mid - cut \quad \frac{; ,_1 \Rightarrow A \quad \Pi; A, ,_2 \Rightarrow C}{\Pi; ,_1, ,_2 \Rightarrow C}$$

where the cut elimination procedure for ILU is the obvious analogue (the 'reflection') of the linear procedure, whence, using the terminology introduced in Danos et al.(1993c), the $(\cdot)^*$-decorations of ILU-derivations are *strong* decorations. As, moreover, ILU is complete for provability in intuitionistic logic, in fact what we obtained is a proof system for intuitionistic logic which is a proper fragment of CLL.

PROPOSITION. *If π is a derivation in $\{!, \multimap, \forall_2^*\}$ of $\Pi^*, !, ^* \Rightarrow !\Sigma^*, \Delta^*$ in which all cutformulas are of the form A^* or $!A^*$, and all identity axioms of the form $A^* \Rightarrow A^*$, then $sk(\pi)$ is an ILU-derivation of $\Pi; ,\ \Rightarrow \Sigma \cup \Delta$.* $\square$

Thus ILU inherits the computational properties of CLL.

[4] One merely replaces '$\mathcal{S}$' by 'ILU', ', $\Rightarrow \Delta$' by '$\Pi; ,\ \Rightarrow \Delta$', and '$\mu, ^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$' by $\Pi^{\checkmark}; \mu, ^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$'.

How about classical logic? Let us try to define a modal translation $(\cdot)^{\checkmark}$ of *classical* logic that, like the $(\cdot)^{\circledast}$-translation for intuitionistic logic, can be extended to an inductive decoration strategy for CL. In order to do so, we have to interpret sequents , $\Rightarrow \Delta$ as $\mu, {}^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$, where μ, ν are modalities. Then observe that, in order to satisfy condition 2 in the definition of decoration strategy,

1. in case of the structural rules we need that $\mu \equiv\, !\mu'$ and $\nu \equiv\, ?\nu'$, for modalities μ', ν';

2. in case of an application of cut, we have to be able to 'unify' the decorations $\mu A^{\checkmark}$ and $\nu A^{\checkmark}$ of the cut formula by some series of applications of dereliction- and/or promotion-rules. Clearly this can be done if and only if either μ is a suffix of ν or ν is a suffix of μ.

We will call a pair of modalities (μ, ν) satisfying these two conditions *adequate*.

PROPOSITION. *Let (μ, ν) be a pair of modalities. There exists a modal translation $(\cdot)^{\checkmark}$ such that $\langle(\cdot)^{\checkmark}, \mu, \nu\rangle$ is an inductive decoration strategy for CL if and only if (μ, ν) is adequate.*

PROOF: That adequacy is a necessary condition has already been shown. It is also sufficient: given an adequate pair (μ, ν) define $p^{\circledcirc} := p$ for p atomic; then take

$$
\begin{aligned}
(A \rightarrow B)^{\circledcirc} &:= max(\mu,\nu)A^{\circledcirc} \multimap max(\mu,\nu)B^{\circledcirc} \\
(\forall \alpha A)^{\circledcirc} &:= \forall \alpha\, max(\mu,\nu)A^{\circledcirc},
\end{aligned}
$$

where $max(\mu, \nu)$ denotes the longest of the two modalities. It is not difficult to verify that $\langle(\cdot)^{\circledcirc}, \mu, \nu\rangle$ is an inductive decoration strategy for CL. □

The proposition proves the second, classical, half of our claim, as obviously there exist adequate pairs of modalities.

The modal translations corresponding to the *two* simplest possible adequate pairs, namely (!, ?!) and (!?, ?), will be called the q-, respectively the t-translation. In fact, q and t are, in a way, the *unique* inductive decoration strategies for CL: in an inductive decoration strategy $\langle(\cdot)^{\circledcirc}, \mu, \nu\rangle$ as in the proof of the proposition, by adequacy, either (1) $\mu \equiv\, !\alpha?\beta$ and $\nu \equiv\, ?\beta$, or (2) $\mu \equiv\, !\beta$ and $\nu \equiv\, ?\alpha!\beta$, for modalities α, β; using the terminology and techniques introduced in Danos et al.(1993c) one then shows that in $\pi^{\circledcirc}$ exponentials in the classes induced by α and β are always superfluous, and can be stripped, hence resulting in either π^t (case 1) or π^q (case 2). (Details are in Schellinx(1994).)

Girard's embedding $(\cdot)^*$ can be seen as an optimization of the decorating embedding $(\cdot)^{\circledast}$. It appears that similar optimizations are possible in the classical case, for the q- as well as for the t-translation. To see this, note that the following are linearly derivable, for any A, B:

$$\begin{array}{rcl} !(!A \multimap ?!B) & \Rightarrow & ?!A \multimap ?!B \\ !?(!?A \multimap ?B) & \Rightarrow & !?A \multimap !?B \\ !?\forall\alpha ?A & \Rightarrow & \forall\alpha !?A. \end{array}$$

These suggest the following translations:

- the Q-translation, which maps atoms to atoms, then

$$\begin{array}{rcl} (A \rightarrow B)^Q & := & !A^Q \multimap ?!B^Q \\ (\forall\alpha A)^Q & := & \forall\alpha ?!A^Q; \end{array}$$

- the T-translation, which maps atoms to atoms, then

$$\begin{array}{rcl} (A \rightarrow B)^T & := & !?A^T \multimap ?B^T \\ (\forall\alpha A)^T & := & \forall\alpha ?A^T. \end{array}$$

Using the canonical derivations of the sequents above to 'inject' correction-cuts at the appropriate places[5] in q- respectively t-decorated CL-derivations, we find that both Q and T are sound- and faithful embeddings of CL into linear logic:

$$CLL \vdash !,^Q \Rightarrow ?!\Delta^Q \quad \text{iff} \quad CL \vdash , \Rightarrow \Delta \quad \text{iff} \quad CLL \vdash !?,^T \Rightarrow ?\Delta^T.$$

And as for Girard's embedding in the case of intuitionistic logic, the existence of these translations suggests restrictions on CL-derivations, defining a subcollection that is complete for provability.

As a matter of fact, starting from a q- or t-decorated CL-derivation π and introducing correction cuts at the appropriate places, *elimination* of precisely these cuts will result in a CL-derivation π' satisfying these restrictions. To take an example, the T-translation tells us that the succedent

[5] Being instances of L$\rightarrow$ for q, instances of L$\rightarrow$, L$\forall_2$ for t.

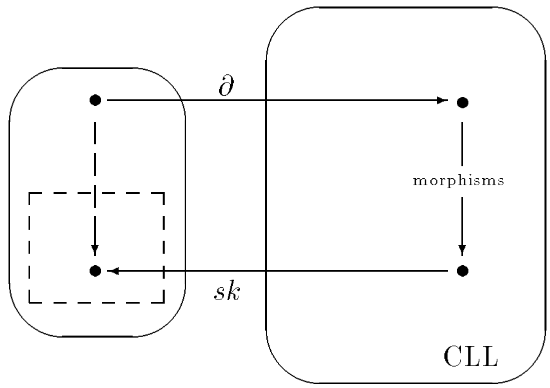

Figure 2: Transformation of proofs by means of constrictive morphisms

in an implication need never be subject to structural manipulations to the left of the entailment sign. One *constructs* a CL-derivation satisfying this restriction for a given occurrence of an implication by eliminating the correction cut with the canonical derivation of $!?(!?A \multimap ?B) \Rightarrow !?A \multimap !?B$, and taking the skeleton. We therefore speak of *'constrictive morphisms'*.

The general pattern of this proof transformation is schematized in figure 2. In case the sequent calculus of departure is IL, we start by applying inductively the $(\cdot)^*$-translation, add correction cuts whenever necessary, and then eliminate these. The skeleton of the reduct is an ILU-derivation (see Schellinx(1994) for more details).

We call the calculi corresponding to the Q- and the T-translation respectively LKQ and LKT. One obtains LKQ-, respectively LKT-derivations of a given sequent by eliminating the appropriate correction cuts applied to the q-, respectively the t-decoration of a CL-derivation of the sequent. The formulation of the rules in the calculi is found by abstraction of the structure of linear derivations of the form $!,^{Q} \Rightarrow ?!\Delta^{Q}$ respectively $!?,^{T} \Rightarrow ?\Delta^{T}$.

The calculus LKQ (table 4) has sequents $, \Rightarrow \Delta; \Pi$, where, as in ILU, the symbol Π denotes a multi-set containing *at most* one, the *'queue'* (whence Q) or *tail*-formula whose occurrence in the succedent of a sequent is distinguished by means of the ";". In the linear interpretation it corresponds to a formula that has not (yet) been questioned. Again we find a dereliction rule D, this time the equivalent of the linear R?-rule.

Identity axiom:

$$A \Rightarrow ; A$$

Logical rules:

$$\text{L}\!\rightarrow \frac{,\ \Rightarrow \Delta; A \qquad B, ,\ ' \Rightarrow \Delta';}{,\ ,\ ,\ ', A \rightarrow B \Rightarrow \Delta, \Delta';} \qquad \text{R}\!\rightarrow \frac{,\ , A \Rightarrow \Delta, B;}{,\ \Rightarrow \Delta; A \rightarrow B}$$

Rules for the second order quantifier (β not free in , , Δ):

$$\text{L}\forall_2 \frac{,\ , A[X/\alpha] \Rightarrow \Delta;}{,\ , \forall\alpha\, A \Rightarrow \Delta;} \qquad \text{R}\forall_2 \frac{,\ \Rightarrow \Delta, A[\beta/\alpha];}{,\ \Rightarrow \Delta; \forall\alpha\, A}$$

Structural rules:

$$\text{D} \frac{,\ \Rightarrow \Delta; A}{,\ \Rightarrow \Delta, A;}$$

$$\text{LW} \frac{,\ \Rightarrow \Delta; \Pi}{,\ , A \Rightarrow \Delta; \Pi} \qquad \text{RW} \frac{,\ \Rightarrow \Delta; \Pi}{,\ \Rightarrow A, \Delta; \Pi} \qquad \text{LC} \frac{,\ , A, A \Rightarrow \Delta; \Pi}{,\ , A \Rightarrow \Delta; \Pi} \qquad \text{RC} \frac{,\ \Rightarrow A, A, \Delta; \Pi}{,\ \Rightarrow A, \Delta; \Pi}$$

Cut rules:

$$\text{tail} \frac{,\ \Rightarrow \Delta; A \qquad A, ,\ ' \Rightarrow \Delta'; \Pi}{,\ ,\ ,\ ' \Rightarrow \Delta, \Delta'; \Pi} \qquad \text{mid} \frac{,\ \Rightarrow \Delta, A; \Pi \qquad A, ,\ ' \Rightarrow \Delta';}{,\ ,\ ,\ ' \Rightarrow \Delta, \Delta'; \Pi}$$

Table 4: The calculus LKQ

Derivability of , $\Rightarrow \Delta; \Pi$ in LKQ corresponds precisely to linear derivability of $!, ^Q \Rightarrow ?!\Delta^Q; \Pi^Q$, and $\langle (\cdot)^Q, !, ?! \rangle$ is an inductive decoration strategy for LKQ.

The calculus LKT (table 5) appears as the classical equivalent of ILU (as this intuitionistic calculus is obtained from LKT by the usual intuitionistic restriction of the succedents to singletons; moreover, the $(\cdot)^*$-translation is obtained by deleting all occurrences of '?' in the T-translation). Here we find sequents $\Pi; , \Rightarrow \Delta$, with Π containing at most one distinguished formula, the '*tête*' (whence T) or *head*-formula. As in ILU, it corresponds to a formula that has not yet been subjected to non-linear manipulations on the left. Included we find the *negative* fragment of classical implicational logic as it appears in LU (Girard(1993)).

Derivability of $\Pi; , \Rightarrow \Delta$ in LKT corresponds precisely to linear derivability of $\Pi^T; !?, ^T \Rightarrow ?\Delta^T$ in the $\{!, ?, \multimap, \forall_2^T\}$-fragment of linear logic, where $\forall_2^T$ indicates abstraction restricted to formulas of the form X^T (observe that

Identity axiom:

$$A; \Rightarrow A$$

Logical rules:

$$\text{L}\!\rightarrow \frac{;\, ,\ \Rightarrow \Delta, A \qquad B; ,\,' \Rightarrow \Delta'}{A \rightarrow B; ,\, ,\, ,\,' \Rightarrow \Delta, \Delta'} \qquad \text{R}\!\rightarrow \frac{\Pi; ,\, , A \Rightarrow \Delta, B}{\Pi; ,\ \Rightarrow \Delta, A \rightarrow B}$$

Rules for the second order quantifier (β not free in Π, , , Δ):

$$\text{L}\forall_2 \frac{A[X/\alpha]; ,\ \Rightarrow \Delta}{\forall\alpha\, A; ,\ \Rightarrow \Delta} \qquad \text{R}\forall_2 \frac{\Pi; ,\ \Rightarrow \Delta, A[\beta/\alpha]}{\Pi; ,\ \Rightarrow \Delta, \forall\alpha\, A}$$

Structural rules:

$$\text{D} \frac{A; ,\ \Rightarrow \Delta}{; A, ,\ \Rightarrow \Delta}$$

$$\text{LW} \frac{\Pi; ,\ \Rightarrow \Delta}{\Pi; ,\, , A \Rightarrow \Delta} \qquad \text{RW} \frac{\Pi; ,\ \Rightarrow \Delta}{\Pi; ,\ \Rightarrow A, \Delta} \qquad \text{LC} \frac{\Pi; ,\, , A, A \Rightarrow \Delta}{\Pi; ,\, , A \Rightarrow \Delta} \qquad \text{RC} \frac{\Pi; ,\ \Rightarrow A, A, \Delta}{\Pi; ,\ \Rightarrow A, \Delta}$$

Cut rules:

$$\text{head} \frac{\Pi; ,\ \Rightarrow \Delta, A \qquad A; ,\,' \Rightarrow \Delta'}{\Pi; ,\, ,\, ,\,' \Rightarrow \Delta, \Delta'} \qquad \text{mid} \frac{;\, ,\ \Rightarrow \Delta, A \qquad \Pi; A, ,\,' \Rightarrow \Delta'}{\Pi; ,\, ,\, ,\,' \Rightarrow \Delta, \Delta'}$$

Table 5: The calculus LKT

if a sequent $\Gamma_1^T, ?\Gamma_2^T, !?\Gamma_3^T \Rightarrow !?\Delta_3^T, ?\Delta_2^T, \Delta_1^T$ is derivable in this fragment, then $|\Gamma_1 \cup \Gamma_2 \cup \Delta_3| \leq 1$), and $\langle (\cdot)^T, !?, ?\rangle$ is an inductive decoration strategy for LKT.

The calculus LKT, like ILU, like LKQ, inherits the computational properties of linear logic: the T-decorations of LKT-derivations are *strong* decorations. Hence we found a proofsystem for *classical* logic as a proper fragment of CLL:

PROPOSITION. *If π is a derivation in $\{!, ?, \multimap \forall_2^T\}$ of a sequent*

$$\Gamma_1^T, ?\Gamma_2^T, !?\Gamma_3^T \Rightarrow !?\Delta_3^T, ?\Delta_2^T, \Delta_1^T$$

in which all cutformulas are of the form $A^T, ?A^T$ or $!?A^T$, and all axioms are of the form $A^T \Rightarrow A^T$, then $sk(\pi)$ is an LKT-derivation of

$$\Gamma_1 \cup \Gamma_2; \Gamma_3 \Rightarrow \Delta_3, \Delta_2, \Delta_1. \qquad \square$$

The distinction Q/T reflects the dichotomy *positive/negative* introduced in Girard(1991), even though identifying LKQ as a proper fragment of LU seems not to be possible. Maybe somewhat surprising is that we obtained two constructive calculi complete for classical logic each of which *stays* at a different 'side of the mirror'.

One final remark: the calculus LKT is closely related to the system of classical natural deduction and its term calculus $\lambda\mu$ as introduced and studied by Parigot(1992), a relation that we might state as

$$\frac{ILU}{\lambda_2} = \frac{LKT}{\lambda\mu},$$

and to which we will come back in later work (Danos et al.(1993a).

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