Determinism in the one-way model

Vincent Danos (CNRS, Université Paris 7)
Elham Kashefi (IQC - University of Waterloo)*

June 6, 2005

Abstract

We introduce a *flow condition* on one-way measurement patterns which guarantees globally deterministic behaviour. Dependent Pauli corrections are derived for all such patterns, which 1) equalise all computation branches, and 2) only depend on the underlying entanglement graph and its choice of inputs and outputs.

The class of patterns having flow is stable under composition and tensorisation, and has unitary embeddings as realisations. The restricted class of patterns having both flow and reverse flow, supports an operation of adjunction, and has all and only unitaries as realisations.

Keywords: measurement based quantum computing.

1 Introduction

The recent one-way quantum computing model [1, 2, 3] has already drawn considerable attention, mainly because it suggests different physical realisations of quantum computing [4, 5, 6, 7, 8, 9]. Computation consists of a first phase of preparation and entanglement, followed by 1-qubit measurements and a final round of corrections. Making measurements an integral part of computation will in general induce non-deterministic behaviours. To counter this, both measurements and corrections are allowed to depend on the outcomes of previous measurements. This mechanism of feed-forwarding classical observations is known to be a necessary requirement for the model to be universal [10]. Whether and how a given pattern can be controlled so as to obtain a globally deterministic behaviour is the question we address in this paper.

Specifically, we introduce a condition on one-way measurement patterns which guarantees a strong form of deterministic behaviour, and bears only on the underlying structure of their entanglement. This condition singles out a

*Corresponding author: ekashefi@iqc.ca. Postal address: Institute for Quantum Computing, University of Waterloo 200 University Ave. W., Waterloo, ON, Canada, N2L 3G1. Phone: (519) 888-4567 X4021. Fax: (519) 888-7610.

class of patterns, said to *have flow*, which is stable under composition and tensorisation (parallel composition) and is large enough to realise all unitaries. At the time of writing this paper, we do not know of any pattern realising a unitary operator which does not have flow, and the flow condition may well turn out to be also necessary.

Be that as it may, patterns with flow have interesting additional properties. First, these are uniformly deterministic, in the sense that no matter which are the measurements angles, the obtained set of corrections, which depends only on the underlying geometry, will make the global behaviour deterministic. Second, all computation branches have equal probabilities, which means in particular these probabilites are independent of the inputs, and as a consequence, one can show that all such patterns implement unitary embeddings. Third, a more restricted class of patterns having both flow and reverse flow supports an operation of adjunction, corresponding to time-reversal of unitary operations. This smaller class implements all and only unitary transformations.

Acknowledgements. The authors wish to thank Ignacio Cirac, Damian Markham, Keiji Matsumoto, Harold Ollivier and Marcus Silva for useful comments and discussions.

2 Measurement patterns

We briefly recall the definition of measurement patterns and various notions of determinism. Computations in a pattern involve a combination of 1-qubit preparations N^i_α, 2-qubit entanglement operators $E_{ij} := \wedge Z_{ij}$, 1-qubit measurements M^α_i, and 1-qubit Pauli corrections X_i, Z_i, where i, j represent the qubits on which each of these operations apply, and α is a parameter in $[0, 2\pi]$.

Let $|\pm_\alpha\rangle$ stand for $\frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\alpha}|1\rangle)$.

Preparation N^i_α prepares qubit i in state $|+_\alpha\rangle_i$. Measurement M^α_i is defined by orthogonal projections $\langle\pm_\alpha|_i$, applied at qubit i, with the convention that $\langle+_\alpha|_i$ corresponds to the outcome 0, while $\langle-_\alpha|_i$ corresponds to 1.

Since qubits are measured at most once in a pattern, we may represent unambiguously the outcome of the measurement done at qubit j by s_j. Dependent corrections, used to control non-determinism, will be written $X^{s_j}_i$ and $Z^{s_j}_i$, with $X^0_i = Z^0_i = I$, $X^1_i = X_i$, and $Z^1_i = Z_i$.

A *measurement pattern*, or simply a pattern, is defined by the choice of V a finite set of qubits, two possibly overlapping subsets I and O determining the pattern inputs and outputs, and a finite sequence of commands acting on V.

Such a pattern is said to be *runnable* if it satisfies the following:
—(D0) no command depends on an outcome not yet measured;
—(D1) no command acts on a qubit already measured or not yet prepared;
—(D2) a qubit i is measured (prepared) if and only if i is not an output (input).

Write $\mathfrak{H}_I$ ($\mathfrak{H}_O$) for the Hilbert space spanned by the inputs (outputs). The run of a runnable pattern consists simply in executing each command in sequence. If n is the number of measurements (which by (D2) is also the number of non outputs) then the run may follow 2^n different branches. Each branch is associated with a unique binary string $\mathbf{s}$ of length n, representing the classical outcomes of the measurements along that branch, and a unique *branch map* $A_\mathbf{s}$ representing the linear transformation from $\mathfrak{H}_I$ to $\mathfrak{H}_O$ along that branch.

Branch maps decompose as $A_\mathbf{s} = C_\mathbf{s}\Pi_\mathbf{s}U$, where:
— $C_\mathbf{s}$ is unitary map over $\mathfrak{H}_O$ collecting all corrections on outputs,
— $\Pi_\mathbf{s}$ is a projection from $\mathfrak{H}_V$ to $\mathfrak{H}_O$ representing the particular measurements performed along the branch,
— U is a unitary embedding from $\mathfrak{H}_I$ to $\mathfrak{H}_V$ collecting the branch preparations, and entanglements.
Therefore, $\sum_\mathbf{s} A_\mathbf{s}^\dagger A_\mathbf{s} = \sum_\mathbf{s} U^\dagger \Pi_\mathbf{s} U = I$, and $T(\rho) := \sum_\mathbf{s} A_\mathbf{s}\rho A_\mathbf{s}^\dagger$ is a trace-preserving cp-map, explicitely given as a Kraus decomposition.

One says that the pattern *realises* T.

A pattern is said to be *deterministic* if it realises a cp-map that sends pure states to pure states. This is equivalent to saying that branch maps are proportional, that is to say, for all $q \in \mathfrak{H}_I$ and all $\mathbf{s}_1$, $\mathbf{s}_2 \in \mathbb{Z}_2^n$, $A_{\mathbf{s}_1}(q)$ and $A_{\mathbf{s}_2}(q)$ differ only up to a scalar. A pattern is said to be *strongly deterministic* when branch maps are equal, *i.e.*, for all $\mathbf{s}_1$, $\mathbf{s}_2 \in \mathbb{Z}_2^n$, $A_{\mathbf{s}_1} = A_{\mathbf{s}_2}$. A pattern is said to be *uniformly deterministic* if it is deterministic for all values of its measurement angles.

If a pattern is strongly deterministic, then $T(\rho) = A\rho A^\dagger$ with $A := 2^{n/2}A_\mathbf{s}$, and A must be a unitary embedding, because $\sum_\mathbf{s} A_\mathbf{s}^\star A_\mathbf{s} = A^\dagger A = I$. In such cases, one says that the pattern realises the unitary embedding A.

Example. Not all deterministic patterns are uniformly or strongly so. To see this, choose as command sequence $X_1^{s_2}M_2^0E_{12}N_2^0$, with $V = \{1, 2\}$, and $I = O = \{1\}$. The two branch maps are given by $A_0 = |0\rangle\langle 0|$, and $A_1 = |0\rangle\langle 1|$, so they are proportional, but distinct, and the pattern is deterministic, but not strongly so. The associated cp-map $T(|\psi\rangle\langle\psi|) = \langle\psi,\psi\rangle|0\rangle\langle 0|$ does not correspond to a unitary transformation. This pattern is not uniformly deterministic either, since $\alpha = 0$ is the only angle value for M_2^α which makes it deterministic.

3 Geometries and flows

A *geometry* (G, I, O) consists of a undirected graph G together with two subsets of nodes I and O, called inputs and outputs. We write V for the set of nodes in G, I^c, and O^c for the complements of I and O in V, $G(i)$ for the set of neighbours of i in G, and $E_G := \prod_{(i,j)\in G} E_{ij}$ for the global entanglement operator associated to G.[1]

One may think of a geometry as the beginning of the definition of a pattern, where one has already decided how many qubits will be used (V), how they will be entangled (E_G), and which will be inputs and which outputs (I and O). To complete the definition a pattern it remains to decide which angles will be used to prepare (measure) qubits in I^c (O^c), and most importantly, if one is interested

[1]Note that all E_{ij} commute so the order in this product is irrelevant.

in determinism; which dependent corrections will be used. Conversely, any pattern has a unique underlying geometry, obtained by forgetting measurements and corrections.

For instance, the geometry associated to the example above is the graph G with nodes $\{1,2\}$, inputs and outputs $\{1\}$, and $E_G = E_{12}$. To complete the definition, one has to choose the angles of the measurement and preparations done at qubit 2, and define the dependent corrections.

We give now a condition bearing on geometries under which one can construct a set of dependent corrections such that the obtained pattern is strongly and uniformly deterministic.

Definition 1 *A geometry (G, I, O) has* flow *if there exists a map $f : O^c \to I^c$ and a partial order $>$ over V such that for all $i \in O^c$:*
— (F0) $(i, f(i)) \in G$;
— (F1) $f(i) > i$;
— (F2) for all $k \in G(f(i)) \smallsetminus \{i\}$, $k > i$.

The coarsest order $>$ such that (F1) and (F2) holds will be called the *dependency order* induced by f, and the length of the longest increasing sequence in $>$ will be called the *depth* of f.

Figure 1 shows a geometry together with a flow function represented as arrows from O^c (solid circles) to I^c (non rectangled nodes). The associated dependency order is given by $1 < \{2,3,6,5\}$, $2 < \{4,6,7\}$, $4 < 7$, and $3 < 5$. In general flows may or may not exist, and are not unique either.

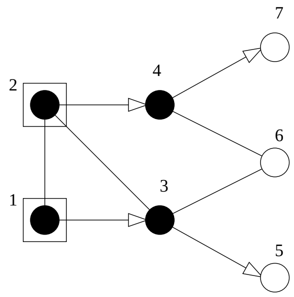

Figure 1: A geometry that has flow.

Theorem 1 *Suppose the geometry (G, I, O) has flow f, then the pattern:*

$$\mathfrak{P}_{f,G,\vec{\alpha}} \quad := \quad \prod^{>}_{i \in O^c} (X^{s_i}_{f(i)} \prod_{k \in G(f(i)) \smallsetminus \{i\}} Z^{s_i}_k M^{\alpha_i}_i) E_G N^0_{I^c}$$

where the product follows the dependency order $>$ of f, is runnable, uniformly and strongly deterministic, and realises the unitary embedding:

$$U_{G,I,O,\vec{\alpha}} \quad := \quad (\prod_{i \in O^c} \sqrt{2} \langle +_{\alpha_i} |_i)\, E_G N^0_{I^c}$$

Proof. The proof is based on the following equations:

$$
\begin{array}{rcll}
\langle +_\alpha|_i & = & M_i^\alpha Z_i^{s_i} & (1)\\
Z_i^s E_{ij} & = & X_j^{s_i} E_{ij} X_j^s & (2)\\
X_i^s E_{ij} & = & E_{ij} Z_j^s X_i^s & (3)\\
Z_i^s E_{ij} & = & E_{ij} Z_j^s & (4)\\
X_i^s N_i^0 & = & N_i^0 & (5)
\end{array}
$$

Equation (1) amounts to saying that $Z_i|\pm_\alpha\rangle_i = |\mp_\alpha\rangle_i$ (this property is uniquely defining Z by the way). Equations (2), (3), and (4) come from the fact that $\wedge Z$ is in the normaliser of the Pauli group, and are easy to verify. Equation (5) is obvious.

By (1):

$$\prod_{i\in O^c}\langle +_\alpha|_i E_G N_{I^c}^0 \quad =_{(1)} \quad (\prod_{i\in O^c} M_i^{\alpha_i} Z_i^{s_i}) E_G N_{I^c}^0$$

so the right hand side is clearly a deterministic pattern, but just as clearly it violates condition (D0), since $Z_i^{s_i}$ depends on a measurement which has not been done yet. At that point, entanglement comes to rescue. Write $G(f(i))^c$ for the graph obtained by removing $G(f(i))$ from G. One has:

$$
\begin{array}{ll}
Z_i^{s_i} E_G N_{I^c}^0 & =\\
Z_i^{s_i} E_{G(i)} E_{G(i)^c} N_{I^c}^0 & =\\
Z_i^{s_i} E_{if(i)} (\prod_{k\in G(f(i))\smallsetminus\{i\}}) E_{f(i)k} E_{G(f(i))^c} N_{I^c}^0 & =_{(2)}\\
X_{f(i)}^{s_i} E_{if(i)} X_{f(i)}^{s_i} (\prod_{k\in G(f(i))\smallsetminus\{i\}} E_{f(i)k}) E_{G(f(i))^c} N_{I^c}^0 & =_{(3)}\\
X_{f(i)}^{s_i} E_{G(f(i))} (\prod_{k\in G(f(i))\smallsetminus\{i\}} Z_k^{s_i}) X_{f(i)}^{s_i} E_{G(f(i))^c} N_{I^c}^0 & =_{(4)}\\
X_{f(i)}^{s_i} (\prod_{k\in G(f(i))\smallsetminus\{i\}} Z_k^{s_i}) E_G X_{f(i)}^{s_i} N_{I^c}^0 & =_{(5)}\\
X_{f(i)}^{s_i} (\prod_{k\in G(f(i))\smallsetminus\{i\}} Z_k^{s_i}) E_G N_{I^c}^0 &
\end{array}
$$

Condition (F0) is used in the third step. From this computation follows that:

$$\prod_{i\in O^c}\langle +_\alpha|_i E_G N_{I^c}^0 \quad =_{(1)} \quad (\prod_{i\in O^c} X_{f(i)}^{s_i} (\prod_{k\in G(f(i))\smallsetminus\{i\}} Z_k^{s_i}) M_i^{\alpha_i}) E_G N_{I^c}^0$$

By conditions (F1) and (F2) the obtained pattern is runnable, since the product can always be ordered according to $>$. Moreover, by the last equation, all branch maps are equal, and therefore the pattern is strongly deterministic. Finally, since the proof uses nowhere the particular values of the measurement angles α_i, it is also uniformly so. □

Note that the unitary embedding associated to $\mathcal{P}_{f,G}$ (we drop the angles $\vec{\alpha}$ in the following) does not depend on f. Yet, the choice of f determines the structure of the corrections used by the pattern, and has therefore an influence on its depth complexity, which is the depth of f. Another thing worth noticing, is that using the G-stabiliser [11, 12] at i, defined as $K_{G(i)} := X_i(\prod_{j\in G(i)} Z_j)$, $\mathfrak{P}_{f,G}$ can be equivalently written as:

$$\mathfrak{P}_{f,G} \quad = \quad \prod_{i\in O^c}^{>} (M_i^{\alpha_i} K_{G(f(i))}^{s_i} Z_i^{s_i}) E_G N_{I^c}^0$$

and the proof above can be reread in terms of stabilisers.

3.1 X and Y measurements

Not all geometries have flow, the example presented in Section 2 does not for instance. Another example is given Figure 2. Sometimes, one can still obtain deterministic patterns as we will see now.

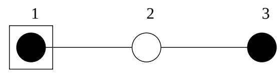

Figure 2: A geometry with no flow.

Condition (F1) forbids $f(i) = i$, yet, in the special case where qubit i is measured with angle $\frac{\pi}{2}$ (Y measurement), choosing $f(i)$ to be i will work. The reason for this is:

$$M_i^{\frac{\pi}{2}} X_i^s = M_i^{\frac{\pi}{2}} Z_i^s$$

This obtains an extended flow for the geometry above as one can see in Figure 3. However the obtained pattern is deterministic only if qubit 3 is measured at angle $\frac{\pi}{2}$, and is therefore not uniformly deterministic.

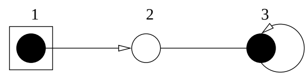

Figure 3: A geometry with a looped flow.

Another special case is when $f(i)$ is a measurement with angle 0 (X measurement). Again the requirement that $f(i) > i$ can be dropped because:

$$M_i^0 X_i^s = M_i^0$$

3.2 General preparations

As yet, our flow theorem is only valid when preparations are all of the form N_0^i, since (5) is valid only for them. Define $X_i^\alpha = Z_i^\alpha X_i Z_i^{-\alpha}$, with Z_i^α the phase operator with angle α applied at i. One has $Z_i = Z_i^\pi$. To handle general phase preparations one only needs the analog of equations (2), (3) and (5):

$$Z_i^s E_{ij} = (X_i^\alpha)^s E_{ij} (X_i^\alpha)^s \quad (6)$$

$$(X_i^\alpha)^s E_{ij} = E_{ij} Z_j^s (X_i^\alpha)^s \quad (7)$$

$$(X_j^\alpha)^s N_i^\alpha = N_i^\alpha \quad (8)$$

and the flow theorem works as before. Note that we had to extend the set of corrections to include X_i^α. This extension will prove handy and actually natural, in the next and last section where we develop the algebra of patterns with flow.

4 Algebraic structure

Say a geometry (G, I, O) has *bi-flow*, if both (G, I, O) and the dual geometry (G, O, I) have flow. Say a pattern has flow (bi-flow) if its underlying geometry does. The class of patterns with flows (bi-flows) is closed under composition and tensorisation. It is also universal, in the sense that all unitaries can be realised within this class. This follows from the existence of a set of generating patterns having bi-flow [13].

Figure 4 shows the geometry corresponding to a pattern realising $\wedge U$, for U an arbitrary 1-qubit unitary, and obtained by combining these generators [13].

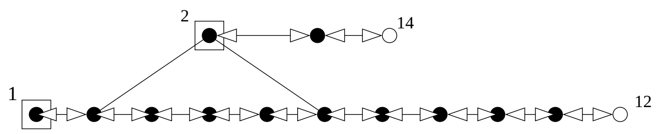

Figure 4: A geometry with bi-flow.

Patterns with bi-flows realise unitary operators. Indeed, by (F2), a flow f is one-one. Therefore the orbits $f^n(i)$ for $i \in I$ define an injection from I into O. In the case of a bi-flow, I and O are therefore in bijection, and since one knows already that patterns with flows realise unitary embeddings, it follows that patterns with bi-flow implement unitaries.

Interestingly, one can define directly the adjoint of a pattern in the subcategory of patterns with bi-flows. Specifically, given f a flow for (G, I, O), and angles $\{\alpha_i; i \in I^c\}$ for preparations, and $\{\beta_j; j \in O^c\}$ for measurements, we write $\mathfrak{P}_{f,G,\vec{\alpha},\vec{\beta}}$ for the pattern obtained as in the extension to general preparations of theorem above. Suppose a dual flow g is given on (G, I, O), one can define:

$$\mathfrak{P}^{\dagger}_{f,G,\vec{\alpha},\vec{\beta}} \quad := \quad \mathfrak{P}_{g,G,\vec{\beta},\vec{\alpha}}$$

There are two things to note here: first, for this definition to make sense, one needs to have general preparations as in the preceding section; second, this adjunction operation depends on the choice of a reverse flow g. It is easy to see that $P^{\dagger}_{f,G,\alpha,\beta}$ and $P_{f,G,\beta,\alpha}$ realise adjoint unitaries.

An example is the pattern $\mathfrak{H} := X_2^{s_1} M_1^0 E_{12} N_2^0$ with $I = \{1\}$ and $O = \{2\}$. It has a unique bi-flow, and is self-adjoint in the sense that $\mathfrak{H}^{\dagger} = \mathfrak{H}$, therefore it must realise a self-adjoint operator, and indeed it realises the Hadamard transformation.

5 Conclusion

Whereas the one-way model had been mostly thought of in relation with the traditional circuit model, we have proposed here a flow condition, which is clearly divorced from the circuit model, and guarantees the existence of a set of

Pauli corrections obtaining a (strongly and uniformly) deterministic behaviour. In essence, while dealing with patterns with flow, one can wholly forget about corrections, and think of measurements as being simply projections. If one is ready to lose uniform determinism, this condition can be somewhat extended when dealing with Pauli measurements. It may be however that strong and uniform determinism is an interesting property, when it comes to fault-tolerant computing in the one-way model.

Another point worth making is that the notion of flow gives a better understanding of why X^α, Z corrections and N^α preparations are needed. From the point of view of our determinism theorem, they represent a natural and universal way to control the non deterministic evolutions induced by 1-qubit xy measurements on a graph state.

Finally and although no counterexample is known, and the obtained class of patterns is universal, it remains to be seen whether this condition is also necessary.

References

[1] R. Raussendorf and H.-J. Briegel. A one-way quantum computer. *Physical Review Letters*, 86(5188), 2001.

[2] R. Raussendorf and H.-J. Briegel. Computational model underlying the one-way quantum computer. *QIC*, 2(6), 2002.

[3] R. Raussendorf, D. E. Browne, and H.-J. Briegel. Measurement-based quantum computation on cluster states. *Phys. Rev. A*, 68(022312), 2003.

[4] M. A. Nielsen. Optical quantum computation using cluster states. quant-ph/0402005, 2004.

[5] M. A. Nielsen and C. M. Dawson. Fault-tolerant quantum computation with cluster states. quant-ph/0405134, 2004.

[6] D. E. Browne and T. Rudolph. Efficient linear optical quantum computation. quant-ph/0405157, 2004.

[7] S.R. Clark, C. Moura Alves, and D. Jaksch. Efficient generation of graph states for quantum computation. *New J. Phys.*, 7(124), 2005. quant-ph/0406150.

[8] S. D. Barett and P. Kok. Efficient high-fidelity quantum computation using matter qubits and linear optics. quant-ph/0408040, 2004.

[9] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger. Experimental one-way quantum computing. *Nature*, 434:169–176, March 2005.

[10] V. Danos, E. Kashefi, and P. Panangaden. The measurement calculus. quant-ph/0412135, 2004.

[11] M.A. Nielsen and I.L. Chuang. *Quantum Computation and Quantum Information*. CUP, Cambridge, 2000.

[12] M. Hein, J. Eisert, and H.-J. Briegel. Multi-party entanglement in graph states. *Phys. Rev. A*, 69:62311, 2004.

[13] V. Danos, E. Kashefi, and P. Panangaden. Robust and parsimonious realisations of unitaries in the one-way model. quant-ph/0411071, 2004.