

# Computational isomorphisms in classical logic

Vincent Danos\*, Jean-Baptiste Joinet† & Harold Schellinx‡  
Équipe de Logique Mathématique, Université Paris VII  
Mathematisch Instituut, Universiteit Utrecht

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## Abstract

All standard ‘linear’ boolean equations are shown to be computationally realized within a suitable classical sequent calculus  $\mathbf{LK}_p^n$ .

Specifically,  $\mathbf{LK}_p^n$  can be equipped with a cut-elimination compatible equivalence on derivations based upon reversibility properties of logical rules. So that any pair of derivations, without structural rules, of  $F \Rightarrow G$  and  $G \Rightarrow F$ , where  $F, G$  are first-order formulas ‘without any qualities’, defines a computational isomorphism.

**Keywords:** proof theory - linear logic - classical logic

## 1 Introduction

### 1.1 A patch of paradise to be broadened

In recent work [1] devoted to the proof theory of classical logic, we embarked on the project of overcoming the obstacles that prevent cut from being a *decent* binary operation on the set of classical sequent derivations. To clarify what we mean by *decency*, let us have a look at the world of simply typed  $\lambda$ -calculus, which, seen from a *normalization-as-computation* point of view, is something close to a patch of paradise.

Among the ingredients of ‘computational decency’ there, we not only encounter (1) a framework to represent proofs (intuitionistic natural deduc-

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\*danos@logique.jussieu.fr; URA 753

†joinet@logique.jussieu.fr; Université Paris I

‡schellin@math.ruu.nl; research part of the project ‘The Geometry of Logic’, financed by the Netherlands Organization for Scientific Research (NWO).

tion, **IND**) and (2) a *noetherian* and *confluent* cut-elimination scheme ( $\beta$ -reduction), but also (3) a quotient of the space of proofs (the  $\eta$ -quotient) where computational isomorphisms are realized.

As an example of a computational isomorphism, which gives a good impression of how members of this *intuitionistic triple*  $(\mathbf{IND}, \beta, \overset{\eta}{\approx})$  cooperate, consider  $f$ 's 'twister'  $T[f] = \lambda y \lambda x ((f)x)y$ . Clearly:

$$f : A \rightarrow (B \rightarrow C) \vdash T[T[f]] : A \rightarrow (B \rightarrow C)$$

Note that the term  $T[T[f]]$  has the effect of switching the order of the arguments of the function  $f$ , and then switching them back again. Does such a double switching have an effect in terms of computations? Of course one would like the answer to be a firm "no!". Otherwise said - we'll be more precise later - we want this kind of "commutativity" to be a *computational isomorphism*. By two  $\beta$ -reductions,  $T[T[f]] = \lambda y \lambda x ((\lambda y \lambda x ((f)x)y)x)y$  becomes  $\lambda y \lambda x ((f)y)x$ . So in order for this double switching to be an 'action without content or meaning', we need to identify the terms  $\lambda y \lambda x ((f)y)x$  and  $f$ . But that gap between terms is exactly the one that is closed by  $\eta$ -equivalence!

## 1.2 The classical triple

In [1] we constructed a *classical triple*:  $(\mathbf{LK}_p^\eta, tq, \overset{s}{\approx})$ . It is an extension of the intuitionistic triple, because the standard embedding of natural deduction into sequent calculus actually sends  $\beta$ -equivalent derivations to  $tq$ -and-strongly equivalent ones, and  $\eta$ -equivalent ones to strongly equivalent ones:

$$(\mathbf{IND}, \beta, \overset{\eta}{\approx}) \subset (\mathbf{LK}_p^\eta, tq, \overset{s}{\approx}).$$

To build this classical triple, we start from a very general calculus for classical logic, baptized  $\mathbf{LK}^{tq}$  (which includes logical rules in 'all styles': *multiplicative, additive*), and equip it with a normalization scheme ( $tq$ ) which asks of each cut formula a "colour",  $t$  or  $q$ , to decide which sub-proof is to be moved first. This quite general scheme is shown to be noetherian and confluent using embeddings of classical logic into linear logic. (We will review some of the main notions introduced in [1] in section 2.1).

Just as asking that the above "commutativity" be a computational isomorphism *forces*  $\eta$ -equivalence on **IND**-derivations, asking the boolean equivalences we consider to be computational isomorphisms *forces strong equivalence* on  $\mathbf{LK}^{tq}$ -derivations. Strongly equivalent proofs differ only with respect to reversal of ... reversible logical rules. However, in  $\mathbf{LK}^{tq}$  pure,  $tq$ -reduction *breaks*  $\overset{s}{\approx}$ -classes! The quotient induced by  $\overset{s}{\approx}$  consequentially

is degenerated: all derivations having the same conclusion are identified. For  $\overset{s}{\approx}$  to become compatible with the  $tq$  scheme, we need to (1) narrow the space of  $\mathbf{LK}^{tq}$  proofs (the resulting fragment we call  $\mathbf{LK}^\eta$ ) and (2) restrict the normalization-space of  $\mathbf{LK}^\eta$  by *polarizing* derivations, i.e. by subordinating “colours” (hence normalization steps) to reversibility properties of connectives.

Both  $\mathbf{LK}^\eta$  and  $\mathbf{LK}_p^\eta$  are complete with respect to classical provability and closed under  $tq$ -normalization; and, by design,  $\mathbf{LK}_p^\eta$  realizes “linear” boolean equivalences as computational isomorphisms.

### 1.3 Relationships between $\mathbf{LK}_p^\eta$ and $\mathbf{LC}$

Up to the stoup/no stoup formulation of the syntax,  $\mathbf{LC}$ , Girard’s calculus for classical logic [2], simply is a fragment of  $\mathbf{LK}_p^\eta$  where one imposes a co-ordination between styles and colours, and second, our strong equivalence is a syntactic materialization of the identifications achieved by Girard’s denotational semantics for  $\mathbf{LC}$ , which by the way works for the whole of  $\mathbf{LK}_p^\eta$ .

### 1.4 An abstract criterion for isomorphisms

Let us now fix a precise definition of *computational isomorphism*. And for that, let us concentrate on sequent calculus, where composition appears via an explicit rule, the cut-rule. Given a sequent calculus  $\mathbf{L}$  with a confluent and noetherian normalization scheme, for any proofs  $\pi$  and  $\pi'$  in  $\mathbf{L}$  of  $\Gamma \Rightarrow \Delta, F$  and  $F, \Gamma' \Rightarrow \Delta'$  respectively, we can define  $\pi \odot_F \pi'$  to be *the* normal form of the derivation obtained by cutting  $\pi$  and  $\pi'$  on  $F$ . Let  $id_X$  denote the axiom  $X \Rightarrow X$  which we suppose is a unit w.r.t.  $\odot_X$ .

Let now  $\approx$  be an equivalence relation on  $\mathbf{L}$ -proofs, such that any two equivalent proofs  $\pi$  and  $\pi'$  in  $\mathbf{L}$  have equivalent normal forms (in which case we say the equivalence is *compatible* with the normalization scheme).

**Definition 1** *A pair of  $\mathbf{L}$ -derivations  $\phi$  and  $\chi$  of  $F \Rightarrow G$  and  $G \Rightarrow F$  define a computational isomorphism between  $F$  and  $G$  with respect to  $\approx$ , if  $\phi \odot_G \chi \approx id_F$  and  $\chi \odot_F \phi \approx id_G$ .*

The aim of the present paper is to provide a sufficient condition for a pair of derivations of  $F \Rightarrow G$  and  $G \Rightarrow F$  to define a computational isomorphism in  $\mathbf{LK}_p^\eta$  with respect to  $\overset{s}{\approx}$ . Actually, the pairs which are caught by our condition guarantee synonymy, i.e. *computational interchangeability* (see theorem 35), a property which is stronger than the one fixed in definition 1.

Our criterion, which replaces empirical checkings of the kind we saw in the twister example given before, is quite general. Let us say a formula  $F$  is

‘without any qualities’ when all relation symbols in  $F$  are distinct. Then: in  $\mathbf{LK}_p^\eta$ , any pair of derivations, without structural rules, of  $F \Rightarrow G$  and  $G \Rightarrow F$ , where  $F, G$  are first-order formulas without any qualities, define a computational isomorphism with respect to strong equivalence.

The only difficulty in the proof is to show (theorem 28) that for such formulas  $F$ , an  $\mathbf{LK}_p^\eta$ -derivation, with no structural rules, of the sequent  $F \Rightarrow F$  always is strongly equivalent to  $id_F$ .

Linear derivations, which can be considered as **MALL** derivations (**LL** derivations without exponentials), seem to play a distinguished rôle in the search for the algebraic structure behind classical logic considered as a computational system.

## 2 Review of basic notions

In this section we quickly review some of the basic notions and a few results from [1].

### 2.1 $\mathbf{LK}^{tq}$

The specific calculus **LK** we will consider and in which we formally distinguish between connectives with *multiplicative* and with *additive* introduction rules (cf. [1], [6]), can be found in the appendix.

Recall that there are two major sources of indeterminism in Gentzen’s original ‘cut-pushing’ elimination procedure for classical sequent calculus: a *structural* one (should one permute upwards to the left or to the right?) and a *logical* one (related to the order of the two cuts obtained in performing some key-steps). In order to be able to arrive at a *deterministic* procedure, in [1] we introduced two extensions, one of the language, the other of the calculus:

1. Each formula comes equipped with a mapping of the set of its subformulas into a ‘colour space’  $\{t, q\}$ . (When necessary, we will make explicit the colour  $\epsilon$  of the formula itself by means of a superscript:  $A^\epsilon$ .) The rules now are supposed to preserve colours, i.e. colours should respect identity classes of formulas in a proof (cf. [1]).
2. The multiplicative unary rules come in two types, prescribing whether in the ‘key-step(s)’ of cut-elimination in which they are involved, the cut on the left subformula comes before that on the right one, or conversely: we speak of the *orientation* of the unary multiplicative rules.

The system thus defined will be referred to as (all-style)  $\mathbf{LK}^{tq}$ .

The following conventions are used in distinguishing between the occurrences of formulas in a given logical rule, e.g.  $L\rightarrow$ :

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2}$$

The formula  $A \rightarrow B$  is called the *main* formula of the rule with main connective  $\rightarrow$ ; the occurrences  $A$  and  $B$  in the premisses will be referred to as the *active* formulas; all other occurrences are said to be *passive*. These are the *contextformulas* and together they constitute the *context*. In the special case of a cut, active formula occurrences are also referred to as *cutformulas*.

In case of an identity axiom we say that *both* formula-occurrences are main.

*We restrict the use of the term ‘main’ to logical rules and identity axioms.* In case of structural rules, if necessary, we speak of the *weakened*, respectively *contracted* formula.

In an occurrence in a proof  $\pi$  of a rule having as conclusion a sequent  $\sigma$  and sequent(s)  $\sigma'$  (and  $\sigma''$ ) as premise(s),  $\sigma$  is called the *successor-sequent* of  $\sigma'$  (and  $\sigma''$ ) in  $\pi$ , and  $\sigma'$  (and  $\sigma''$ ) are *ancestor-sequents* of  $\sigma$ . Similarly we will speak of the successor in  $\sigma$  of a formula occurrence in  $\sigma', \sigma''$  and of an ancestor in  $\sigma', \sigma''$  of a formula occurrence in  $\sigma$ . Using the terminology of [1]: formula occurrences  $A$  in  $\sigma$  and  $\sigma', \sigma''$  are successor-ancestor related iff both occurrences are in the same *identity class*. E.g. in

$$\frac{A \Rightarrow C \quad A \Rightarrow D}{A \Rightarrow C \wedge D}$$

the occurrences of  $A$  in the premisses are both *ancestors* of the occurrence of  $A$  in the conclusion, and the occurrence of  $A$  in the conclusion is a successor of each of the occurrences of  $A$  in the premisses; the occurrence  $C \wedge D$  does not have an ancestor.

A *block* (of sequents) in  $\pi$  is a sequence of sequents  $\sigma_0, \sigma_1, \dots, \sigma_n$  in  $\pi$  such that  $\sigma_{i+1}$  is the successor of  $\sigma_i$ . Note that a block of sequents unambiguously defines a block of (successive occurrences of) *rules* in  $\pi$ .

It is useful to think of a derivation in sequent calculus as a tree (the ‘proof tree’). Each leaf (each *axiom*) uniquely defines a branch leading *downwards* to the tree’s root (the derivation’s *final sequent*, the conclusion). Along such a branch we can follow the occurrences of a formula  $A$  from its introduction (in an axiom, a logical rule or a weakening) to its submergence (when it is active in a logical rule) or disappearance (when it is active in a cut). See figure 1. ‘Meanwhile’, in between, in general many things will happen to *other* formula-occurrences (occurrences not in the identity class of  $A$ ). We

will call this ‘meanwhile’ the *main-active* or *m-a-interspace*<sup>1</sup> of the occurrence. If *nothing* happens in between, we say that the m-a-interspace is *flat*: the formula is ‘born’, and immediately put to work.

Formally we define the m-a-interspace as follows.

Given an occurrence of a formula  $A$  in some sequent  $\sigma$  in a proof  $\pi$ , such that  $A$  is ‘*newly born*’ (i.e. is either main, or has just been weakened), we inductively define a sequence of occurrences  $A_0, \dots, A_n$  of  $A$ , together with a block  $\mathbb{B}_A$  of sequents  $\sigma_0, \dots, \sigma_n$ , by

1.  $\sigma_0 := \sigma, A_0 := A$ ;
2. if  $\sigma_i \in \mathbb{B}_A$ ,  $A_i$  is not active, and  $\sigma_{i+1}$  is the successor of  $\sigma_i$ , then  $\sigma_{i+1} \in \mathbb{B}_A$ , and  $A_{i+1}$  is the successor of  $A_i$  in  $\sigma_{i+1}$ ;
3. that’s all.

Hence  $\mathbb{B}_A$  will contain  $\sigma$ ,  $\sigma$ ’s successor, et cetera, down to either  $\pi$ ’s concluding sequent, or a sequent in which  $A$  is *active*.  $\mathbb{B}_A$  of course is  $A$ ’s *main-active interspace*.

With each *active* occurrence of a formula  $A$ , we can associate the finite set  $\{B_{A_1}, \dots, B_{A_n}\}$  of *m-a-interspaces*, where each  $A_i$  corresponds to a leaf of  $A$ ’s tree of ancestors.

Whereas the m-a-interspace is obtained by looking *downward* along the proof-tree, conversely, given an occurrence of a formula  $B$  in a sequent derivation  $\pi$ , we can look upward. There we see *the tree of  $B$ ’s ancestors in  $\pi$* , i.e. the tree-like structure obtained by following *upwards* in  $\pi$  all formula occurrences in the identity class of our initial  $B$ , up to the introductions of  $B$  by axiom, logical or weakening rule (these form the ancestor-tree’s leaves). See figure 1. The reader who wishes to do so, will easily provide the (long and boring) inductive definition.

## 2.2 Cut-elimination: the *tq*-protocol

An occurrence of a coloured formula on the left-hand side (resp. right-hand side) of the entailment-sign in a sequent is said to be *attractive* if its colour is  $t$  (resp.  $q$ ): the terminology is introduced to remind us that the subproof of the sequent containing the non-attractive active cutformula ‘has to move first’. We will often use the following alternative iconic notation for  $A^t$  (resp.  $A^q$ ), namely  $\overrightarrow{A}$  (resp.  $\overleftarrow{A}$ ).

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<sup>1</sup>Note the slight abuse of terminology: a ‘main-active interspace’ might actually be a ‘*weakening-active interspace*’.

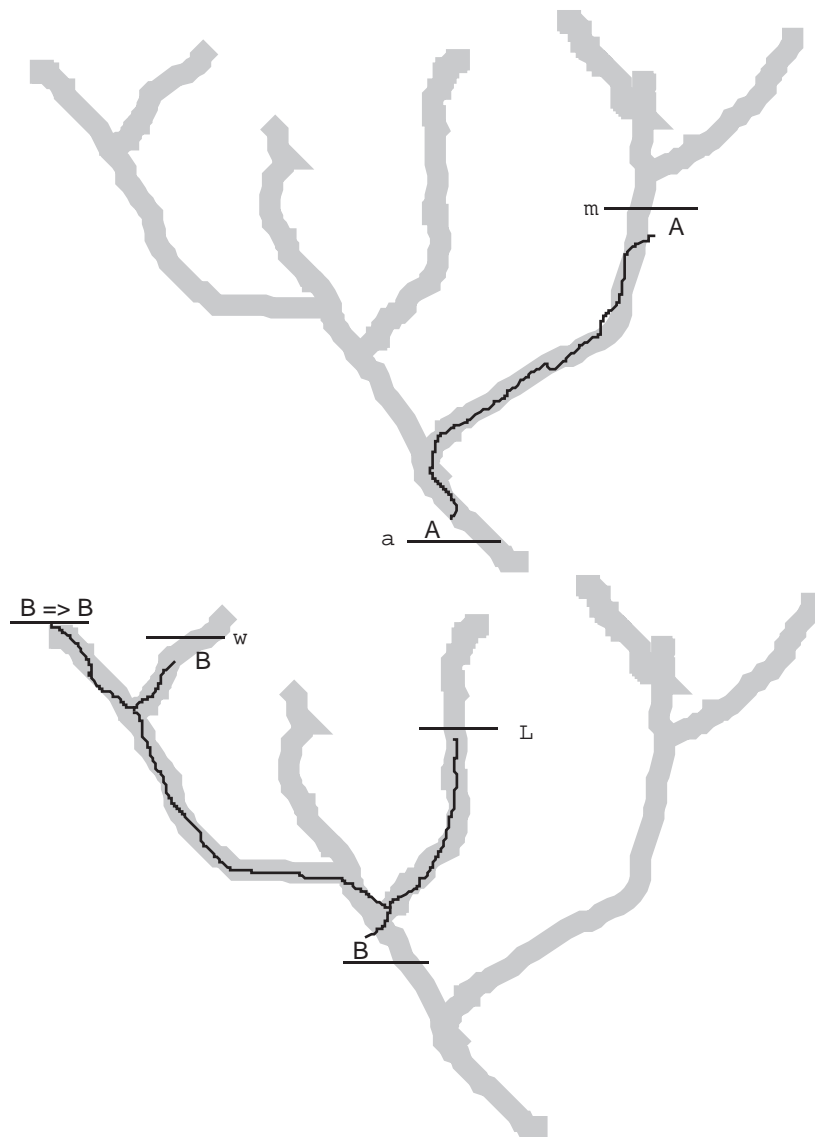


Figure 1: A proof tree  $\pi$  in which we traced (above) the m-a-interspace of a formula occurrence  $A$  and (below) the tree of ancestors of a formula occurrence  $B$ .

In each instance of a cut rule in an  $\mathbf{LK}^{tq}$ -proof, the cutformula will be coloured either  $t$  or  $q$ . Thus, using our iconic notation, cuts are of one of the two following forms:<sup>2</sup>

$$\frac{\overset{\rightarrow}{\Rightarrow A} \quad \overset{\rightarrow}{A \Rightarrow}}{\Rightarrow} \quad \text{and} \quad \frac{\overset{\leftarrow}{\Rightarrow A} \quad \overset{\leftarrow}{A \Rightarrow}}{\Rightarrow}$$

Let us call the subderivation containing the *attractive* occurrence of the cutformula the *attracting* subderivation.

**Definition 2 (tq-protocol)** *Reduction according to tq-protocol proceeds via two possible types of steps, ‘structural’ ones, S1 and S2, and ‘logical’ ones, L (‘key-steps’):*

— *An L-step applies when both cutformulas are main in a logical rule. L-steps have to be specified for each one of the connectives and, in case of the binary ones, for each of the possible combinations of a left and a right introduction rule. (We obtain as descendants one or two cuts on the immediate subformula(s) of the cutformula. In case of two descendants, the order in which these cuts are applied is determined by the orientation of the unary rule.)*

— *In case no L-step is applicable, necessarily an S-step applies, which consists in ‘transporting’ one of the cut’s subderivations up the tree of the cutformula’s ancestors in the other one, duplicating it and contracting the context whenever passing an instance of contraction (or via the context of a binary additive rule); this process ends when reaching instances of introduction in an axiom, in which case the resulting ‘axiom-cuts’ are reduced immediately, when reaching an introduction by weakening, which are replaced by weakenings of the context formulas, or when reaching instances of introduction of the main connective of the cutformula.*

*Of course, now one needs to know which of the two subderivations has to move. This is decided by asking whether or not the attractive cutformula is main in a logical rule. If the answer is “yes!”, we transport the attracting subderivation (S2); if it is “no!”, we transport the other one (S1).*

*And that’s it.*

Note that neither the choice of orientations nor that of colours has to do with imposing a reduction *strategy*. We do not select redexes, but rather the way we reduce them.

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<sup>2</sup>In depicting proof figures, we will often indicate the context only when this is relevant to the argument.

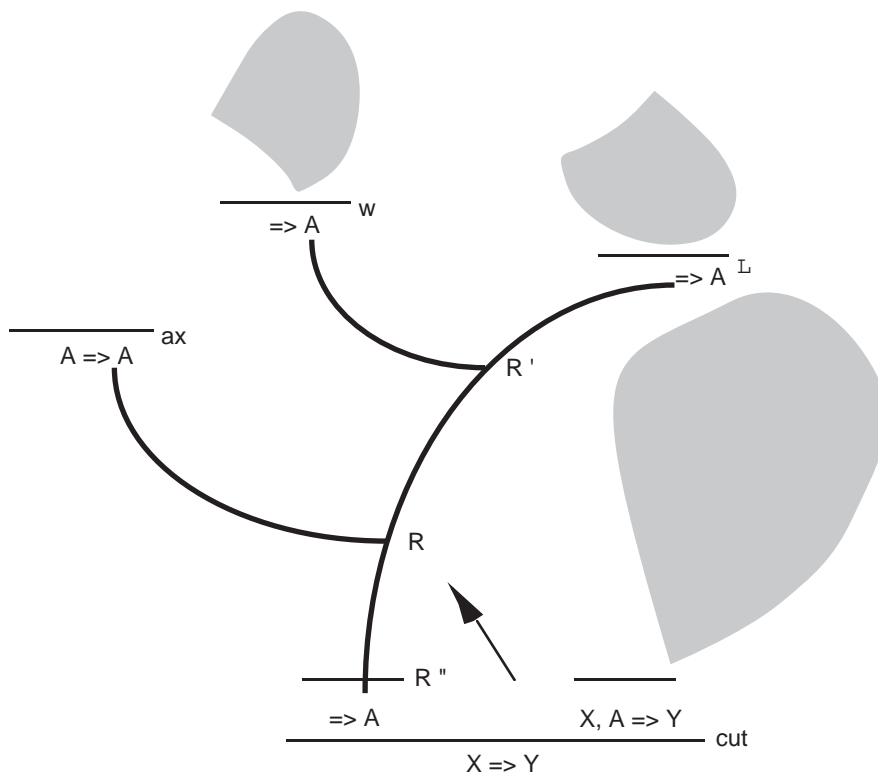


Figure 2: A structural cut, either of type S1 or of type S2, depending on whether the occurrence of the cut-formula in the left premiss of the cut is attractive or not. If it is, the reduction is of type S1. If it isn't, then the occurrence of the cut-formula in the right premiss of the cut is main in a logical rule, and the reduction is of type S2.

Figures 2 and 3 show, schematically, the ‘movement’ involved in performing a structural reduction step.

The crucial difference between  $tq$ -reduction and the standard definitions of cut-elimination steps in sequent calculus is in the definition of the *structural steps*, where the *complete* tree of ancestors of one of the occurrences of the cutformula  $A$  is involved: we raise (a copy of) the transported subderivation right up to the leaves, disregarding all ‘events’ within the m-a-interspaces that make up the tree, to each of the spots where an ancestor of the cutformula was introduced. Then we push the contextformulas of the final sequent of the transported subderivation ‘back down’ along the tree. At branchings originally due to explicit contractions on  $A$ , the contractions now are inherited by these contextformulas.

### 2.3 Reversibility: $LK^\eta$

Let  $\sigma, \sigma_i$ ’s be variables over  $\{l, r\}$ . In what follows we will sometimes use the following convenient notation:  $A_1^{\sigma_1}, \dots, A_n^{\sigma_n}$  will denote the sequent  $\Gamma \Rightarrow \Delta$  where  $\Gamma$  is the submultiset of  $A_1^{\sigma_1}, \dots, A_n^{\sigma_n}$  containing  $A_i^{\sigma_i}$ ’s such that  $\sigma_i = l$  and  $\Delta$  the complementary submultiset. We use a ‘bar’ to indicate transposition within  $\{l, r\}$  i.e.  $\bar{l} := r, \bar{r} := l$ . Similarly we will write  $\Gamma, A^\sigma$ .

**Definition 3 (reversibility) (R1)** *A rule is said to be reversible iff from its conclusion one can derive its premiss(es).*

**(R2)** *A (non-atomic) formula  $A^\sigma$  is called reversible iff, whenever a sequent  $\Gamma, A^\sigma$  is derivable, we can ask for  $A^\sigma$  to be main, without loss of provability (i.e. there exists a derivation of  $\Gamma, A^\sigma$  whose last rule introduces  $A^\sigma$ ’s main connective).*

In  $LK$  these notions coincide: a formula  $A^\sigma$  is reversible if and only if its introduction rule on the  $\sigma$ -hand side is reversible, and the reversible rules are exactly the unary multiplicative and binary additive rules, both negation rules, left existential and right universal quantifier rules: one can always permute them below any other rule, that is, except when the reversible formula is active there.

In  $LK^{tq}$  all of this continues to hold:

**Proposition 4 (reversible rules)** *Binary additive rules, unary multiplicative rules, left existential and right universal quantifier rules, as well as both rules for negation, are reversible. The other ones are not.*

**Definition 5 ( $LK^\eta$ )** *A proof of  $LK^{tq}$  is an  $LK^\eta$ -proof iff every attractive (occurrence of a) formula active in an irreversible or a negation rule is main.*

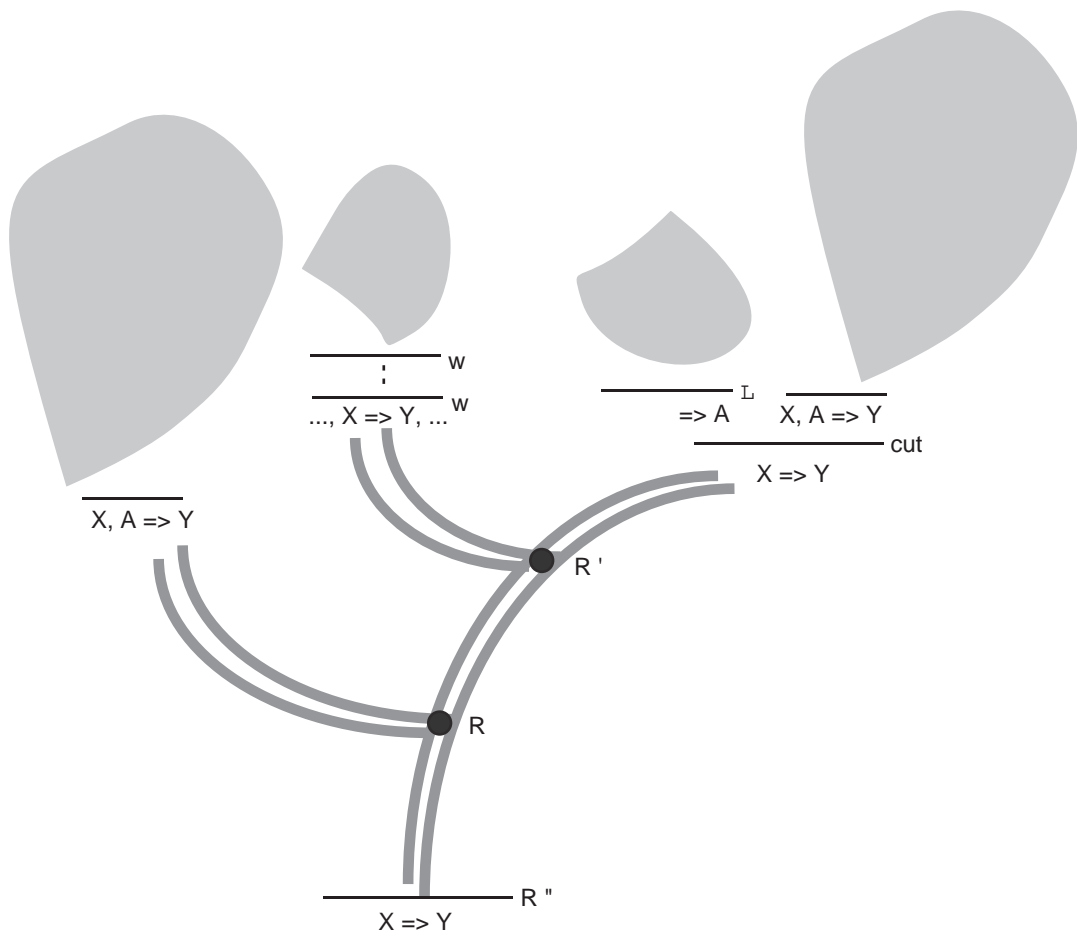


Figure 3: The result of performing the one-step structural  $tq$ -reduction induced by the structural cut in figure 2

This constraint can be rephrased as follows: the  $m$ - $a$ -interspace associated to an attractive formula occurrence active in an irreversible or a negation rule is flat.

**Theorem 6**  $\mathbf{LK}^n$  is stable under  $tq$ -normalization and all sequents provable in  $\mathbf{LK}^{tq}$  are provable in  $\mathbf{LK}^n$  (see [1]).

## 2.4 Weak equivalence, strong equivalence

A well-known, sometimes execrated, property of sequent calculus, is the high number of possible *variations* on a given derivation that is obtainable by simply permuting the order of applications of rules. An interesting and rewarding line of research is that into the question: which sort of permutations induce an equivalence relation on the set of proofs of a given sequent that is *compatible* with our reduction, meaning that if  $\pi'$  is obtained by a number of certain permutations of applications of rules in  $\pi$ , then the *normal form* of  $\pi'$  can be obtained by a number of similar permutations from the *normal form* of  $\pi$ . In [1] we consider two types of rule permutation-induced equivalence relations on  $\mathbf{LK}^{tq}$ -proofs: *weak* equivalence (induced by permutations of occurrences of structural rules) and *strong* equivalence (induced by weak equivalence and permutations of occurrences of reversible rules).

Two proofs are said to be *weakly equivalent* ( $\overset{w}{\approx}$ ) if they differ only up to contracted weakenings, re-arrangement of multiple contractions on the same formula (associativity, commutativity) and permutations of structural rules as, e.g. for a contraction and a binary multiplicative rule:

$$\frac{\frac{C, C \Rightarrow A}{C \Rightarrow A} \quad B \Rightarrow}{A \rightarrow B, C \Rightarrow} \quad \overset{w}{\approx} \quad \frac{C, C \Rightarrow A \quad B \Rightarrow}{A \rightarrow B, C, C \Rightarrow} \quad \frac{A \rightarrow B, C, C \Rightarrow}{A \rightarrow B, C \Rightarrow}$$

or a contraction and a binary additive rule:

$$\frac{\frac{C, C \Rightarrow A}{C \Rightarrow A} \quad \frac{B, C, C \Rightarrow}{B, C \Rightarrow}}{A \rightarrow B, C \Rightarrow} \quad \overset{w}{\approx} \quad \frac{C, C \Rightarrow A \quad B, C, C \Rightarrow}{A \rightarrow B, C, C \Rightarrow} \quad \frac{A \rightarrow B, C, C \Rightarrow}{A \rightarrow B, C \Rightarrow}$$

**Proposition 7 (weak equivalence is  $tq$ -compatible)** ([1]) *If two  $\mathbf{LK}^{tq}$ -proofs are weakly equivalent, then so are their normal forms.*  $\square$

Intuitively, proposition 7 tells us, among other things, that we may consider occurrences of structural rules in  $\mathbf{LK}^{tq}$ -derivations as being *not localized*.

The notion of *strong equivalence* of **LK**-derivations comes from reversibility properties of logical operators. Two **LK**-derivations  $\pi$  and  $\pi'$  are said strongly equivalent ( $\overset{s}{\approx}$ ), if they are weakly equivalent up to permutations of reversible logical rules and canonical expansions of identity-axioms.

Intuitively, strong equivalence can be thought of as the equivalence relation induced by the ‘continuous’ process of ‘opening’ and ‘closing’ in a proof, occurrences of formulas that have a reversible main connective: if you think of the main-active interspaces of such formulas as *zippers* in the proof, then ‘opening’ the formula (permuting the reversible rule downwards) *unzips* the proof, ‘closing’ it (permuting the rule upwards, which is not always possible) *zips* it. Figure 4 illustrates the zipping process:

$$\begin{array}{ccc}
\frac{\frac{A \Rightarrow A}{A, B \Rightarrow A} \quad \frac{B \Rightarrow B}{A, B \Rightarrow B}}{A \wedge_m B \Rightarrow A} & & \frac{A \Rightarrow A}{A, B \Rightarrow A} \quad \frac{B \Rightarrow B}{A, B \Rightarrow B} \\
\frac{A \wedge_m B, A \wedge_m B \Rightarrow A \wedge_m B}{A \wedge_m B \Rightarrow A \wedge_m B} & \overset{s}{\approx} & \frac{A, A, B, B \Rightarrow A \wedge_m B}{A, B \Rightarrow A \wedge_m B} \\
& & \frac{A, B \Rightarrow A \wedge_m B}{A \wedge_m B \Rightarrow A \wedge_m B} \\
& & \frac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \wedge_m B} \\
& \overset{w}{\approx} & \frac{A \wedge_m B \Rightarrow A \wedge_m B}{A \wedge_m B \Rightarrow A \wedge_m B} \\
& & \frac{A \wedge_m B \Rightarrow A \wedge_m B}{A \wedge_m B \Rightarrow A \wedge_m B} \\
& \overset{s}{\approx} & 
\end{array}$$

Figure 4: Strongly equivalent proofs

**Definition 8 (polarized proofs)** An  $\mathbf{LK}^{tq}$ -proof is said to be polarized if a non-atomic formula occurrence is reversible iff it is non-attractive.

**Definition 9 ( $\mathbf{LK}_p^\eta$ )** An  $\mathbf{LK}^\eta$ -proof is in  $\mathbf{LK}_p^\eta$  iff it is polarized.

Note that, as  $\mathbf{LK}^\eta$  is complete for classical provability, so is  $\mathbf{LK}_p^\eta \subset \mathbf{LK}^\eta$ .

Only within  $\mathbf{LK}_p^\eta$ , as was proved in [1], is strong equivalence ‘computationally meaningful’:

**Proposition 10** If two  $\mathbf{LK}_p^\eta$  proofs are strongly equivalent, then so are their normal forms.

## 2.5 Archetypes, linear derivations, and other characters

We consider a first-order language for classical logic built from a set of variables  $x_1, x_2, \dots$ , a set of  $n$ -ary function symbols  $f_1, f_2, \dots$ , a set of  $n$ -ary relation symbols  $R_1, R_2, \dots$  (where  $n = 0, 1, \dots$  and each function and

relation symbol is supposed to come with a fixed arity; 0-ary relation symbols are sometimes referred to as propositional variables; for each  $n$  the set of  $n$ -ary function, relation symbols is supposed to be infinite), negation  $\neg$ , quantifiers  $\forall, \exists$  and binary connectives  $\wedge_a, \vee_a, \overset{a}{\rightarrow}, \wedge_m, \vee_m, \overset{m}{\rightarrow}$  (the *additive* and *multiplicative* versions of the connectives well-known from classical propositional logic).

Define as usual the set of *terms* inductively by: all variables are terms, and whenever  $t_1, \dots, t_n$  are terms and  $f$  is an  $n$ -ary function symbol, then  $f(t_1, \dots, t_n)$  is a term; and the set of formulas by: if  $R$  is an  $n$ -ary relation symbol and  $t_1, \dots, t_n$  are terms, then  $R(t_1, \dots, t_n)$  is a(n atomic) formula and, whenever  $F, G$  are formulas and  $x$  a variable, then  $\neg F, \mathcal{Q}x F, F \circ G$  are formulas (where  $\mathcal{Q}$  ranges over quantifiers and  $\circ$  over binary connectives).

**Definition 11** *A first-order formula  $F$  is an archetype iff all relation symbols occurring in  $F$  are distinct, and, whenever  $\mathcal{Q}xG$  is a subformula of  $F$ , then  $x$  is a free variable of  $G$  (i.e., there is no vacuous binding).*

For example,  $(\forall x R_1(x)) \vee_a R_2(y, z)$  is an archetype, but both  $\forall x R(z)$  and  $R(f(t, t')) \wedge_a (\exists x R(x, z))$  are not.

**Definition 12** *A linear derivation of  $F \Rightarrow F$  for some first-order formula  $F$  (notation:  $\tau_F$ ) is a cut-free derivation of  $F \Rightarrow F$  in the ‘all-style’ sequent calculus **LK**, that does not make use of structural rules.*

Clearly, given some  $F$ , we can in general not expect  $\tau_F$  to be unique. Indeed, if  $F$  is not atomic, then obvious distinct examples of  $\tau_F$  are the proof consisting in nothing but the identity axiom  $F \Rightarrow F$  (the *trivial*  $\tau_F$ , written as  $id_F$ ), and iterations of the derivation called  $\eta_F$  in [1]:

**Definition 13** *If  $F_i$  is (are) the immediate subformula(s) of  $F$ , an iterated  $\eta$ -proof of  $F \Rightarrow F$  (notation:  $\eta_F$ ), consists in axiom(s)  $F_i \Rightarrow F_i$  and/or iterated  $\eta$ -proofs of  $F_i \Rightarrow F_i$ , followed by precisely one instance of each of the logical rules introducing  $F$ ’s main connective.*

### 3 Linear derivations of archetypical identities are units

In what follows we will characterize the derivations  $\tau_F$ , and show that, in case  $F$  is an archetype, *any*  $\tau_F$  necessarily ends in an application of a reversible rule. Also, every  $\tau_F$ , by permutations of instances of reversible rules, can be transformed in an iterated  $\eta$ -proof of  $F \Rightarrow F$ . As a consequence we

get that: for archetypes  $F$ , any linear derivation of an identity  $F \Rightarrow F$  is strongly equivalent to  $id_F$ , a result which we then relativize to  $\mathbf{LK}_p^\eta$ .

We start by assuming the archetype to be *propositional*, and then will use the characterization of  $\tau_F$  for propositional archetypes  $F$  in order to extend the characterization to first-order archetypes.

### 3.1 The propositional case

In the proof of the propositional case several times the following simple property is invoked, which states that whenever a sequent  $\Gamma \Rightarrow \Delta$  is provable in non-exponential propositional *linear* logic without constants **MALL** (or equivalently **LK** without structural rules), in a cut-free derivation, because of the absence of weakening, every formula  $X$  in  $\Gamma \cup \Delta$  can be ‘traced upwards’ to *at least one* identity axiom (otherwise said: every formula  $X$  has *at least one* atomic subformula whose tree of ancestors has a leaf in an identity axiom).

**Lemma 14** *Let  $\pi$  be a normal **MALL**-derivation (without constants) of  $X^\sigma, \Delta^{\sigma_i}$ . Then there is at least one atomic subformula  $p$  of  $X$  that occurs positively (negatively) in  $X$  and negatively (positively) in  $X \cup \Delta$ . Hence if a sequent  $\Gamma$  is provable, then any formula in the multi-set  $\Gamma$  contains at least one atom  $p$  that occurs more than once in  $\Gamma$ .*

**proof:** By induction on the length of cut free **MALL**-derivations.  $\square$

Any propositional formula  $F$  is of the form  $\neg^m F'$ , where  $\neg^m$  denotes  $m \geq 0$  negation signs and  $F'$  is either atomic or of the form  $F_1 \circ F_2$  for some binary connective  $\circ$ .

**Lemma 15** *Let  $F$  be a propositional archetype. Then any non-trivial  $\tau_F$  ends in an application of the reversible rule introducing  $F$ 's main connective.*

**proof:** If  $F$ 's main connective is a negation, then this clearly holds, as both introduction rules are reversible.

Otherwise  $F$  is of the form  $F_1 \circ F_2$ , where  $\circ$  is a binary connective. Let us suppose that  $\tau_F$  ends with an instance of the *irreversible* introduction rule.

In case  $\circ$  is multiplicative,  $\tau_F$ , written schematically, ends as follows:

$$\frac{\begin{array}{c} \vdots \\ F_1 \circ F_2, F_i \end{array} \quad \begin{array}{c} \vdots \\ F_j \end{array}}{F_1 \circ F_2, F_1 \circ F_2}$$

As  $F_j$  is an archetype, its derivability contradicts lemma 14.

In case  $\circ$  is additive,  $\tau_F$  ends as in:

$$\frac{\begin{array}{c} \vdots \\ F_1 \circ F_2, F_i \end{array}}{F_1 \circ F_2, F_1 \circ F_2}$$

Because of the fact that  $F_1 \circ F_2$  is an archetype, all atoms in  $F_j$  ( $j \neq i$ ) occur precisely once in the sequent  $F_1 \circ F_2, F_i$ . Now follow some branch containing ancestors of  $F_1 \circ F_2$  upward in the proof tree. Such a branch necessarily ends in an instance of the reversible introduction rule of  $\circ$ :

$$\frac{\begin{array}{c} \vdots \\ F_1, \Delta \end{array} \quad \begin{array}{c} \vdots \\ F_2, \Delta \end{array}}{F_1 \circ F_2, \Delta}$$

where  $\Delta$  consists entirely of subformulas of  $F_i$ . However, for  $j \neq i$  derivability of  $F_j, \Delta$  contradicts lemma 14, as  $F_j$  is an archetype and has no atoms in common with  $\Delta$ .  $\boxtimes$

By the above lemma we know that, for  $F$  a non-atomic propositional archetype, a non-trivial  $\tau_F$  necessarily ends in an application of the reversible rule introducing  $F$ 's main connective.

But we also know what are the lowest occurrences of irreversible rules:

**Lemma 16** *Let  $F \equiv \neg^m(F_1 \circ F_2)$  be an archetype. All lowest occurrences of irreversible rules in  $\tau_F$  introduce the principal connective of  $F_1 \circ F_2$ . Moreover, all passive formulas occurring in a premise of such a rule are subformulas of the active formula.*

**proof:** Let us follow some branch of the proof tree from the root up to a first occurrence of an irreversible rule.

If  $F_1 \circ F_2$  is main formula in the rule, then a slight modification of the argument of the proof of lemma 15 shows that its conclusion can not be of the form  $\neg^k(F_1 \circ F_2), F_1 \circ F_2$ , for no  $k \geq 0$ . Hence it is of the form  $\Gamma, F_1 \circ F_2$ , with all formulas in  $\Gamma$  subformulas of  $F_1, F_2$ . As to the premiss(es), of course a passive occurrence of a subformula of  $F_i$  in a premiss where  $F_j$  is active contradicts lemma 14, unless  $i = j$ . This proves the second half of our claim.

For the first half, suppose  $F_1 \circ F_2$  is *not* main formula in the rule. Then the conclusion of the rule is a sequent of the form  $\Delta, X_1 \bullet X_2, \neg^k(F_1 \circ F_2)$ , where  $k \geq 0$  and  $X_1 \bullet X_2$  is main in the irreversible rule. Clearly  $X_1 \bullet X_2$

and all formulas in  $\Delta$  are subformulas of either  $F_1$  or  $F_2$ , and all atoms in  $\Delta$  and  $X_1 \bullet X_2$  are distinct. If the rule applied is *multiplicative* we find the following sub-derivation in  $\tau_F$ :

$$\frac{\begin{array}{c} \vdots \\ \Delta_1, X_1 \end{array} \quad \begin{array}{c} \vdots \\ \Delta_2, X_2 \end{array}}{\neg^k(F_1 \circ F_2), \Delta, X_1 \bullet X_2}$$

As the rule is *multiplicative*,  $\neg^k(F_1 \circ F_2)$  will occur either in  $\Delta_1$  or in  $\Delta_2$ ; hence derivability of the other premiss contradicts lemma 14.

If the rule applied is *additive* we find the following sub-derivation in  $\tau_F$ :

$$\frac{\begin{array}{c} \vdots \\ \neg^k(F_1 \circ F_2), \Delta, X_i \end{array}}{\neg^k(F_1 \circ F_2), \Delta, X_1 \bullet X_2}$$

where  $\Delta$  and  $X_i$ , of course, again are subformulas of  $F_1, F_2$ .

Now take any branch in the tree of ancestors of the occurrence of  $X_1 \bullet X_2$  in  $\neg^k(F_1 \circ F_2)$ . Following it upwards, we necessarily arrive at an introduction of  $X_1 \bullet X_2$ . An introduction is either in an identity-axiom, a reversible rule having  $X_1 \bullet X_2$  as a main formula, or an additive irreversible rule having a super-formula of  $X_1 \bullet X_2$  as main formula.

The first case is obviously excluded.

For the second case we reason as in the proof of lemma 15: such an introduction has the form

$$\frac{\begin{array}{c} \vdots \\ X_1, \Gamma \end{array} \quad \begin{array}{c} \vdots \\ X_2, \Gamma \end{array}}{X_1 \bullet X_2, \Gamma}$$

where  $X_j$ , for  $j \neq i$ , is easily seen to have no atoms in common with  $\Gamma$ , contradicting lemma 14.

For the final case: if  $X_1 \bullet X_2$  is introduced in an additive irreversible rule, then it occurs in  $F_i$  in a subformula of the form  $(\dots(X_1 \bullet X_2)\dots)@Z$  or  $Z@(\dots(X_1 \bullet X_2)\dots)$  (where @ denotes an additive connective). Hence, w.l.o.g., the introduction has the form

$$\frac{\begin{array}{c} \vdots \\ Z, \Gamma \end{array}}{(\dots(X_1 \bullet X_2)\dots)@Z, \Gamma}$$



We already saw that any  $\tau_{-G}$  can be transformed in a derivation that is locally  $\eta$ . The following lemma shows that this can always be done, for whatever archetype  $F$ .

**Lemma 18 (zipping lemma)** *Let  $F \equiv F_1 \circ F_2$  be a propositional archetype. Then for any  $\tau_F$  there exists a permutation of its closing rules with the irreversible bar such that the resulting derivation is locally  $\eta$ .*

**proof:** By induction on the number of closing rules in  $\tau_F$ . If there is precisely *one* closing rule, then our derivation already is locally  $\eta$ , hence the ‘empty’ permutation will do. Otherwise: it is easy to see that a multiplicative reversible rule can be ‘pushed over’ the irreversible bar. Let us show that the same holds true for an additive reversible rule.

Suppose that  $\circ$  is multiplicative. We have, w.l.o.g. (the number of closing rules is bigger than one, hence  $A \bullet B$  necessarily is subformula of either  $F_1$  or  $F_2$ ), the following sub-derivation:

$$\frac{\frac{\frac{\pi_{11} \quad \vdots \quad F_1, \Gamma_1}{F_1 \circ F_2, \Gamma, A} \quad \frac{\pi_{12} \quad \vdots \quad F_2, \Gamma_2, A}{F_2, \Gamma_2, A}}{F_1 \circ F_2, \Gamma, A} \quad \frac{\frac{\pi_{21} \quad \vdots \quad F_1, \Gamma_3}{F_1, \Gamma_3} \quad \frac{\pi_{22} \quad \vdots \quad F_2, \Gamma_4, B}{F_2, \Gamma_4, B}}{F_1 \circ F_2, \Gamma, B}}{F_1 \circ F_2, \Gamma, A \bullet B}$$

But clearly this implies that  $\Gamma_1 = \Gamma_3$  and  $\Gamma_2 = \Gamma_4$ . Hence we can push the reversible rule ‘over the bar’:

$$\frac{\frac{\pi_{j1} \quad \vdots \quad F_1, \Gamma_1}{F_1, \Gamma_1} \quad \frac{\frac{\pi_{12} \quad \vdots \quad F_2, \Gamma_2, A}{F_2, \Gamma_2, A} \quad \frac{\pi_{22} \quad \vdots \quad F_2, \Gamma_4, B}{F_2, \Gamma_4, B}}{F_2, \Gamma_2, A \bullet B}}{F_1 \circ F_2, \Gamma, A \bullet B}$$

In case  $\circ$  is additive the argument is similar, and left to the reader.  $\boxtimes$

The following theorem then is an immediate corollary:

**Theorem 19** *Any linear derivation of  $F \Rightarrow F$ , with  $F$  an archetype, is strongly equivalent to  $id_F$ .*

**proof:** By induction on the complexity of  $F$ , using (the zipping) lemma 18. As reversal of  $F \Rightarrow F$  can introduce structural rules, the converse does not hold: there are non-linear derivations strongly equivalent to the identity axiom.  $\boxtimes$

Let  $\tau'_F$  denote any derivation obtained from a  $\tau_F$  by removing zero or more of its closing rules. Also the following proposition is a corollary to the above.

**Proposition 20** *Let  $F$  be an archetype. Then any sub-derivation of  $\tau_F$  is of the form  $\tau'_G$  for some subformula  $G$  of  $F$ ; moreover, any sequent in  $\tau_F$  is of the form  $\Gamma, \neg^n H$ , where  $H$  is either atomic or has an irreversible main connective, and all formulas in  $\Gamma$  are subformulas of  $\neg^n H$ .*

Let us mention another corollary, which is often used in the proofs of the lemmas in the next section.

**Lemma 21** *Suppose  $\tau'_F$  derives  $\Gamma, F$ . Then, for every atomic formula  $p$  occurring in  $\Gamma$  there is an axiom  $A \Rightarrow A$  in  $\tau'_F$ , such that  $A$  contains  $p$ .*

**proof:** By induction on the complexity of  $F$ : if  $F$  is atomic, then  $\tau'_F$  is the axiom  $p \Rightarrow p$ ; otherwise, by proposition 20, following upwards a branch of ancestors of  $p$  in  $\tau'_F$  will lead us to the conclusion of some  $\tau'_G$  for a proper subformula  $G$  of  $F$ , hence containing an axiom as wanted by induction hypothesis.  $\square$

### 3.2 Extension to first-order

In order to extend the above characterization to the first-order case it suffices to extend lemmas 15, 16 and 18 to first-order archetypes. We will make use of the fact that first-order formulas have an obvious underlying *propositional* structure.

**Definition 22** *We inductively define a mapping  $(\cdot)^b$  ('flat') from first-order formulas to propositional formulas by:*

$$\begin{aligned} R_i(t_1, \dots, t_n)^b &:= r_i \\ (\neg F)^b &:= \neg F^b \\ (Qx F)^b &:= F^b \\ (F_1 \circ F_2)^b &:= F_1^b \circ F_2^b \end{aligned}$$

where  $r_i$  is a (new) propositional variable, and call  $F^b$  the propositional collapse of  $F$ . (Cf. [5], chapter 9.)

The propositional collapse of first-order formulas extends in an obvious way to first-order proofs; if  $\pi$  is a first-order proof of  $\Gamma$ , then, (modulo possible

repetitions of sequents due to erasing quantifiers)  $\pi^b$  is propositional proof of  $\Gamma^b$ .

We are going to use the following trivial property of the  $(\cdot)^b$ -mapping:

**Lemma 23** *Let  $F$  be a first-order archetype and  $\tau_F$  a linear proof of  $F \Rightarrow F$ . Then  $F^b$  is a propositional archetype and  $(\tau_F)^b$  a linear proof  $\tau_{F^b}$  of  $F^b \Rightarrow F^b$ .*

Now, using the results in the propositional case, one shows that lemmas 15 and 16 continue to hold in the first-order case.

**Lemma 24** *Let  $F$  be a first-order archetype. Then any non-trivial  $\tau_F$  ends in an application of the reversible rule introducing  $F$ 's main connective.*

**proof:** In case  $F$ 's main connective is not a quantifier, the argument is identical to that given in the proof of lemma 15. Therefore, let us assume that  $F \equiv \mathcal{Q}xG$ . In case  $\tau_F$  does not end in an instance of the reversible quantifier rule, it has the following form:

$$\frac{\begin{array}{c} \vdots \\ G[t/x], (\mathcal{Q}xG)^\sigma \end{array}}{(\mathcal{Q}xG)^\sigma, (\mathcal{Q}xG)^\sigma}$$

Let  $R_1[t/x], \dots, R_n[t/x]$  be the atomic subformulas of  $G[t/x]$ , containing the term  $t$ . Now follow some branch upward in the proof tree. Suppose at some point  $(\mathcal{Q}xG)^\sigma$  is 'split off' in a binary irreversible rule:

$$\frac{\begin{array}{c} \vdots \\ \Gamma_1[t/x] \end{array} \quad \begin{array}{c} \vdots \\ \Gamma_2[t/x], (\mathcal{Q}xG)^\sigma \end{array}}{\Gamma[t/x], (\mathcal{Q}xG)^\sigma}$$

But then  $(\tau_F)^b$  contains only isolated atoms in the sequent  $(\Gamma_1[t/x])^b$ , which consists solely in distinct subformulas of the archetype  $(G[t/x])^b$ . This contradicts lemma 14.

Hence every branch upward from  $G[t/x], (\mathcal{Q}xG)^\sigma$  will pass through an occurrence of the reversible rule introducing  $(\mathcal{Q}xG)^\sigma$ :

$$\frac{\begin{array}{c} \vdots \\ \Delta, G[y/x] \end{array}}{\Delta, (\mathcal{Q}xG)^\sigma}$$

with  $y$  not free in  $\Delta, G(x)$ : our  $\tau_F$  has a ‘reversible bar’, say.

Now take some instance of a rule in this reversible bar such that  $\Delta$  contains an ancestor  $R[t/x]$  of a predicate ‘bound’ by  $Qx$ . By lemma 21 the corresponding sub-derivation in the propositional collapse of our first order generic proof contains an identity-axiom containing  $r$ . In the first-order proof such an axiom, by the subformula-property, is either of the form  $\dots R_i[t/x]\dots \Rightarrow \dots R_i[t/x]\dots$  or  $\dots R_i[y/x]\dots \Rightarrow \dots R_i[y/x]\dots$

But of course that is not possible, as it would imply  $t \equiv y$ , thus violating the variable condition.  $\boxtimes$

**Lemma 25** *Let  $F \equiv \neg^m G$  (with  $G$  non-atomic, not starting with a negation) be a first-order archetype. All lowest occurrences of irreversible rules in  $\tau_F$  introduce the principal connective or quantifier of  $G$ . Moreover, all passive formulas occurring in a premise of such a rule are subformulas of the active formula.*

**proof:** The (somewhat lengthy) proof uses the same type of argument as the proof of lemma 24, and is left mostly to the reader. Let us just show that all passive formulas occurring in the premiss of a lowest instance of an irreversible quantifier-rule introducing the principal quantifier of  $F \equiv QxG$  in  $\tau_F$  are subformulas of the active formula.

Such an instance is separated by a certain number of closing rules from the concluding inference:

$$\begin{array}{c} \tau' \\ \vdots \\ \frac{\Gamma, G[t/x]}{\Gamma, QxG} \\ \vdots \\ \frac{G[y/x], QxG}{QxG, QxG} \end{array}$$

In case there is an  $R[y/x]$  ‘bound’ by the quantifier present in  $\Gamma$ , then, by lemma 21 we know that  $(\tau')^b$  contains an axiom  $A(r) \Rightarrow A(r)$ ; hence in  $\tau'$  we need to have  $t \equiv y$ , so indeed all formulas in  $\Gamma$  are subformulas of  $G[t/x]$ .  $\boxtimes$

Similarly, with due care as to the possibility of, when necessary, renaming variables and terms, one may verify that also lemma 18 continues to hold:

**Lemma 26** *Let  $F$  be a first-order archetype. Then for any  $\tau_F$  there exists a permutation of its closing rules with the irreversible bar such that the resulting derivation is locally  $\eta$ .*

**proof:** Again we leave most of the verification to the reader. Let us only show that in a  $\tau_F$  we may push a reversible quantifier rule up over a lowest irreversible one, for, say,  $F \equiv \forall aG$ .

$$\begin{array}{c} \tau' \\ \vdots \\ \frac{\Gamma, A[y/x], G[t/a]}{\Gamma, A[y/x], \forall aG} \\ \frac{\Gamma, \mathcal{Q}xA, \forall aG}{\Gamma, \mathcal{Q}xA, \forall aG} \\ \vdots \\ \frac{G[z/a], \forall aG}{\forall aG, \forall aG} \end{array}$$

Now it *might* be the case that  $t \equiv y$ , which would prohibit the permutation of rules we want to perform; however, in that case, due to the variable condition, the predicates ‘bound’ by  $\forall a$  can not be present in  $A(x), \Gamma$ . But then *all* their occurrences in  $G[t/a]$  have to be introduced in  $\tau'$  by additive weakening (i.e. unary irreversible rules). Hence  $t$  can be taken different from  $y$ , w.o.l.g.  $\boxtimes$

We therefore find:

**Theorem 27** *Theorem 19 and proposition 20 hold for all first-order archetypes.*

## 4 Classical isomorphisms

### 4.1 Back to $\mathbf{LK}_p^\eta$

Linear derivations of archetypical identities, hence, are strongly equivalent to identity axioms; shown while pretending to be ‘colour-blind’, this property of course continues to hold in  $\mathbf{LK}^{tq}$  for coloured archetypes.

Note that if  $\pi$  is a derivation in  $\mathbf{LK}^\eta$ , or  $\mathbf{LK}_p^\eta$ , then *zipping* it can always be done *within*  $\mathbf{LK}^\eta$ , or  $\mathbf{LK}_p^\eta$ . Hence the last theorem can be relativized to  $\mathbf{LK}^\eta$  or  $\mathbf{LK}_p^\eta$ , thus:

**Theorem 28** *Any linear  $\mathbf{LK}_p^\eta$  derivation of  $F \Rightarrow F$ , with  $F$  an archetype, is strongly equivalent to  $id_F$ .*

(Observe that the converse is false : figure 4 shows a non linear derivation strongly equivalent to an  $id_F$ ). In  $\mathbf{LK}_p^\eta$  linear derivations of  $F \Rightarrow F$ , for polarized first-order archetypes, are, of a strikingly simple form. E.g., the structure of the fully expanded  $\tau_F$  (all occurring identity axioms are atomic) in  $\mathbf{LK}_p^\eta$  is the following:

(i) do *all* possible reversible rules, starting from the reversible rule introducing  $F$ 's main connective (be careful: only *one* of the negation-rules is reversible in  $\mathbf{LK}_p^\eta$  and while in  $\mathbf{LK}^{tq}$  derivations the negation-rules are in some sense ‘roaming free’, in  $\mathbf{LK}_p^\eta$  they are strictly localized), until you are left with only atomic formulas or formulas with an irreversible main connective; (ii) then decompose the ‘irreversible’  $F$ , up to the ‘duals’ of the formulas left in (i).

After step (ii) all leaves are of the form  $F_i \Rightarrow F_i$ , and the process starts over again.

The result of step (i) is unique up to possible permutations of ‘independent’ reversible rules, but this is the *only* degree of freedom.

## 4.2 The criterion: linearity - The harvesting: classical isomorphisms.

The following theorem gives a sufficient condition for the existence of a computational isomorphism between  $F$  and  $G$ :

**Theorem 29** *Suppose  $\phi$  and  $\chi$  are linear  $\mathbf{LK}_p^\eta$ -derivations of  $F \Rightarrow G$  and  $G \Rightarrow F$  respectively, where  $F$  and  $G$  are first-order archetypes. Then  $\phi$  and  $\chi$  define a computational isomorphism between  $F$  and  $G$  w.r.t.  $\overset{s}{\approx}$ .*

**proof:** As being linear is stable under  $tq$ -reduction, we find  $\phi \odot_G \chi = \tau_F \overset{s}{\approx} id_F$  and  $\chi \odot_F \phi = \tau_G \overset{s}{\approx} id_G$ . Hence  $\phi$  and  $\chi$  define a computational isomorphism.  $\square$

As a by-product of the above analysis we recover most linear boolean equivalences : commutativity and associativity of conjunction and disjunction, involutivity of negation, de Morgan laws, etc.

However observe that we cannot *always* use the condition of theorem 29 above to ‘catch’ isomorphisms. An example is given by the distributivity  $A \wedge_m (B \vee_a C) \Leftrightarrow (A \wedge_m B) \vee_a (A \wedge_m C)$ , for which there is a computational isomorphism; but of course the formulas are not archetypical.

### 4.3 Linear isomorphisms

**Definition 30** An isomorphism  $(\phi, \chi)$  is linear whenever  $\phi$  and  $\chi$  are (strongly equivalent to) linear derivations.

The following proposition expresses a necessary condition for the existence of linear isomorphisms between  $F^\epsilon$  and  $G^{\epsilon'}$  (where the superscripts indicate the colour of the formulas, cf. [1]), in case  $F^\epsilon, G^{\epsilon'}$  are first-order archetypes.

**Proposition 31** Let  $F^\epsilon, G^{\epsilon'}$  be first-order archetypes. If  $\epsilon \neq \epsilon'$ , then there are no linear isomorphisms between  $F^\epsilon$  and  $G^{\epsilon'}$ .

**proof:** Suppose  $\phi$  and  $\chi$  are  $\mathbf{LK}_p^\eta$  derivations of the sequents  $F \Rightarrow G$  and  $G \Rightarrow F$  respectively, where  $F$  and  $G$  are first-order archetypes of opposite polarities, then they must be both attractive in one of the sequents, say  $F \Rightarrow G$ , and both non-attractive in the other. Because of the absence of structural rules, one of  $F$  and  $G$ , say  $G$ , has to be logically main in  $\phi$ 's last rule. Then  $\chi \odot_F \phi$  must end in this same rule since  $F$  is attractive in  $\phi$ , so that the structural step will carry  $\chi$  above  $G$ 's last rule. But then  $\chi \odot_F \phi$  can't be a  $\tau_G$ : its last rule is an irreversible one, which contradicts lemma 24.  $\square$

Let us observe that even with  $\epsilon \neq \epsilon'$ , archetypal  $F^\epsilon, G^{\epsilon'}$  could be isomorphic. Consider for instance the two following style-switching  $\mathbf{LK}_p^\eta$  derivations:

$$\begin{array}{ccc}
 \phi & & \psi \\
 \\
 \frac{\frac{A^t \Rightarrow A^t}{A^t, B^t \Rightarrow A^t}}{(A^t \wedge_m B^t)^q \Rightarrow A^t} & \frac{\frac{B^t \Rightarrow B^t}{A^t, B^t \Rightarrow B^t}}{(A^t \wedge_m B^t)^q \Rightarrow B^t} & \frac{\frac{\frac{A^t \Rightarrow A^t}{(A^t \wedge_a B^t)^t \Rightarrow A^t} \quad \frac{B^t \Rightarrow B^t}{(A^t \wedge_a B^t)^t \Rightarrow B^t}}{(A^t \wedge_a B^t)^t, (A^t \wedge_a B^t)^t \Rightarrow A^t \wedge_m B^t}}{\frac{(A^t \wedge_a B^t)^t \Rightarrow (A^t \wedge_m B^t)^q}{(A^t \wedge_a B^t)^t \Rightarrow (A^t \wedge_m B^t)^q}}
 \end{array}$$

Figure 6: Style-switchers  $\phi$  and  $\psi$

An easy computation shows that both  $\phi \odot \psi$  (which is any of the strongly equivalent derivations depicted in figure 4) and  $\psi \odot \phi$  are strongly equivalent, respectively, to the corresponding identity axiom. The style-switchers  $\phi$  and  $\psi$  being non linear derivations, this however does not contradict Proposition 31.

Now here is an example of a ‘good taste’ corollary to our approach, namely the *unicity* of the archetypal, linear computational isomorphisms in  $\mathbf{LK}_p^\eta$  caught by means of our criterion:

**Theorem 32** *Let  $F, G$  be archetypes,  $\phi$  and  $\chi$  linear  $\mathbf{LK}_p^\eta$  derivations of  $F \Rightarrow G$  and  $G \Rightarrow F$ . Any linear  $\mathbf{LK}_p^\eta$  derivation  $\phi'$  of  $F \Rightarrow G$  is strongly equivalent to  $\phi$ .*

**proof:** Using proposition 31 and lemma 4 of [1] we have  $\phi \odot_G (\chi \odot_F \phi') = (\phi \odot_G \chi) \odot_F \phi'$ , that is these two cuts commute. By linearity of  $\phi'$  and theorem 28,  $\chi \odot_F \phi' \stackrel{s}{\approx} id_G$  and since  $\phi \odot_G \chi \stackrel{s}{\approx} id_F$ ,  $\phi \stackrel{s}{\approx} \phi'$ .  $\square$

Conversely, the necessary condition expressed by proposition 31, shows that a certain number of equivalences can't be recovered at the computational level: style-switchings, prenexifications and some distributivities, etc.

The reader will readily convince himself that the composition of proofs in classical logic is *not* associative, not even in  $\mathbf{LK}_p^\eta$ . As a result, two formulas may very well be ‘isomorphic’ (in the sense of definition 1) while being far from ‘synonymous’ (using the P-denotational semantics of [11], it is possible to define a restriction of  $\mathbf{LK}_p^\eta$ , endowed however with a partial reduction, which avoids the non-associativity of cuts, and for which the notion of ‘isomorphism’ used in the present paper thus implies ‘synonymy’; see [12]). Indeed: it will in general be the case that identifying a proof  $\pi$  and the proof obtained from  $\pi$  by replacing one of the two ‘isomorphic’ formulas by the other, and then changing them back again - by means of two cuts with the proofs defining the isomorphism - collapses our universe (in the sense that it will force the identification of *all* derivations of the sequent under consideration).

However, first order archetypes that are *linearly* isomorphic, always are ‘synonymous’:

**Lemma 33** *Let  $\pi$  be an  $\mathbf{LK}_p^\eta$  derivation,  $F_0$  a positive occurrence of  $F$  in  $\pi$ , and  $F_1, \dots, F_{n+1}$  occurrences of  $F$  in  $\pi$  that are highest in some downward closed subtree (a ‘truncation’) of the tree of  $F_0$ 's ancestors in  $\pi$ . Let  $\delta$  be a proof of  $F, \Gamma \Rightarrow \Delta$ , and  $\pi'$  (resp.  $\pi''$ ) the proof obtained from  $\pi$  by cutting each  $F_{i+1}$  (resp.  $F_0$ ) with  $\delta$ . Provided in  $\delta$  each main-active interspace having the specified occurrence of  $F$  as root (i.e. each block of rules following an introduction of  $F$  in  $\delta$ ) contains at most reversible logical rules introducing subformulas of  $\Gamma$  and  $\Delta$ , the proofs  $\pi''$  and  $\pi'$  have strongly equivalent normal forms.*

**proof:** Whenever  $F_0$  is attractive, then reducing the cut on  $F_0$  in  $\pi''$ , or reducing the cuts on the  $F_{i+1}$ 's in  $\pi'$ , leads to the same proof. In case  $F_0$  is not attractive, one verifies by induction on the depth of the reversible layer, that the proof obtained from  $\pi''$  by reversing the subformulas of  $\Gamma, \Delta$  that are introduced in the ‘reversible layer’, reduces to the same proof as the one

obtained when one first reverses in  $\pi'$  those subformulas of  $\Gamma, \Delta$ , and then reduces the cuts on the  $F_{i+1}$ 's.  $\boxtimes$

**Lemma 34** *Let  $F, G$  be first-order archetypes,  $\phi$  and  $\chi$  linear cut-free  $\mathbf{LK}_p^\eta$  derivations of  $F \Rightarrow G$  and  $G \Rightarrow F$ . Suppose that  $F$  is reversible in  $\chi$  and that the last rule of  $\chi$  is not an introduction rule for  $G$ . Then the lowest part of  $\chi$  is of the following form: an ‘irreversible bar’ of rules introducing  $G$  (cf. figure 5), followed by a non void sequence of reversible rules “closing”  $F$ .*

**proof:** By theorem 29 and proposition 31,  $G$  is attractive in  $\chi$ . Thus the evaluation of  $\phi \odot_G \chi$  begins with an ascent of  $\phi$  in  $\chi$  until the introduction rules for  $G$  are met; now, the sequence of rules ‘traversed’ by  $\phi$ , will remain throughout the cut-elimination process: they are thus the last rules in  $\phi \odot_G \chi$ , which is a  $\tau_G$ . Hence, by our analysis of the structure of  $\tau$ 's, they all are logical reversible ‘closing rules’ for  $F$ .  $\boxtimes$

**Theorem 35** *Let  $F, G$  be archetypes,  $\phi$  and  $\chi$  linear  $\mathbf{LK}_p^\eta$  derivations of  $F \Rightarrow G$  and  $G \Rightarrow F$ . Let  $\pi$  be an  $\mathbf{LK}_p^\eta$  derivation,  $F_0$  an occurrence of  $F$  in  $\pi$ , and  $F_1, \dots, F_{n+1}$  occurrences of  $F$  in  $\pi$  corresponding to a truncation of the tree of  $F_0$ 's ancestors in  $\pi$ . Let  $\pi'$  be the proof obtained from  $\pi$  by first cutting each  $F_i$  ( $1 \leq i \leq n+1$ ) with  $\phi$  (or  $\chi$ ), then performing, but now on  $G$ , the rules originally applied on descendants of the  $F_i$ , and finally cutting the occurrence of  $G$  that replaces  $F$ , with  $\chi$  (or  $\phi$ ). Then the proofs  $\pi'$  and  $\pi$  have strongly equivalent normal forms.*

**proof:** By symmetry it suffices to consider the case where  $F_0$  is, say, positive in  $\pi$ . Consider the proof  $\pi''$  obtained from  $\pi$  by cutting  $F_0$  with  $\phi$ , and then, immediately below, the formula  $G$  with  $\chi$ . By proposition 31 we know that  $F$  and  $G$  have the same polarity, so that whatever the polarity of  $F$ , the ‘layer’ of rules following the introductions of  $F$  in  $\phi$ , by lemma 34, contains at most reversible rules introducing subformulas of  $G$ . Hence  $\phi$  satisfies the condition put on  $\delta$  in lemma 33, which thus applies:  $\pi'$  and  $\pi''$  have strongly equivalent normal forms. Now observe that, by proposition 31 and lemma 4 of [1], the two cuts introduced in  $\pi''$  commute. So by theorem 29,  $\pi$  and  $\pi''$  (hence  $\pi'$  as well) have strongly equivalent normal form.  $\boxtimes$

Granted that the maximization of isomorphisms reduces the ‘noise’ of the syntax, that is the amount of syntactic details which blur the actual computational phenomenon, then our classical triple should be a good calculus in which to examine the computational content of classical proofs.



### 5.3 Guess

Now for a plausible guess: 1) the equivalence relation generated by  $R$  is compatible with  $\lambda\mu$ -normalization and 2) our main result still holds, that is, two linear  $\lambda\mu$ -terms proving an equivalence between archetypes compose in both directions to a unit in the quotient.

Reversal, which is just  $\eta$ -expansion in the intuitionistic case, was already (independently) considered by Parigot as a preliminary transformation in the problem of reading back  $\lambda\mu$ -integers; also Herbelin in [4, 3] dealt with fully reversed terms (which is possible only in the absence of second order quantification) in his game-theoretic interpretation of  $\lambda\mu$ ; finally Ong, in [7], proposes a  $\mu\zeta$ -rule, which might define the same equivalence relation as ours, and proves its soundness by model-theoretic means.

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## Appendix: LK, classical logic

*Identity axiom and cut rule:*

$$(Ax) \quad A \vdash A \qquad (cut) \quad \frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

*Axioms for the constants:*

$$\begin{array}{ll} (\top_m) & \vdash \top_m \qquad (\top_a) \quad \Gamma \vdash \top_a, \Delta \\ (\perp_m) & \perp_m \vdash \qquad (\perp_a) \quad \Gamma, \perp_a \vdash \Delta \end{array}$$

*Negation rules:*

$$(L\neg) \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad (R\neg) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

*Multiplicative logical rules:*

$$\begin{array}{ll} (L\multimap) \quad \frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta_1, \Delta_2} & (R\multimap) \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \\ (L\vee_m) \quad \frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee_m B \vdash \Delta_1, \Delta_2} & (R\vee_m) \quad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee_m B, \Delta} \\ (R\wedge_m) \quad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge_m B, \Delta_1, \Delta_2} & (L\wedge_m) \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge_m B \vdash \Delta} \end{array}$$

*Additive logical rules:*

$$\begin{array}{lll} (R\overset{a}{\rightarrow}) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A \overset{a}{\rightarrow} B, \Delta} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \overset{a}{\rightarrow} B, \Delta} & (L\overset{a}{\rightarrow}) \quad \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \overset{a}{\rightarrow} B \vdash \Delta} \\ (R\vee_a) \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee_a B, \Delta} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee_a B, \Delta} & (L\vee_a) \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee_a B \vdash \Delta} \\ (L\wedge_a) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge_a B \vdash \Delta} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge_a B \vdash \Delta} & (R\wedge_a) \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge_a B, \Delta} \end{array}$$

*Rules for quantifiers ( $y$  not free in  $\Gamma, \Delta$ ):*

$$\begin{array}{ll} (L\forall) \quad \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} & (R\forall) \quad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \\ (L\exists) \quad \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} & (R\exists) \quad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \end{array}$$

*Structural rules:*

$$\begin{array}{ll} (LW) \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} & (RW) \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \\ (LC) \quad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} & (RC) \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \end{array}$$