

On the Linear Decoration of Intuitionistic Derivations

by

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Abstract

We define an optimal proof-by-proof embedding of intuitionistic sequent calculus into linear logic and analyse the (purely logical) linearity information thus obtained.

1 Introduction

Uniform translations of intuitionistic into linear logic, with their plethoric use of exponentials, are bound to give only ‘universal linearity information’ about proofs. This paper aims at displaying the structure of ‘specific linearity information’ hidden in a given derivation. How can we apply this to intuitionistic proofs? We have to build a translation into linear logic such that reductions of the intuitionistic proof can be simulated by reductions of its linear image. A necessary condition for this to hold, is that the ‘skeleton’ of the original proof is preserved by the translation. We call translations with this property ‘*decorations*’. Specifically, we construct a proof-by-proof embedding of **IL** into **LL** (formulated as sequent calculi) such that: 1/ the skeleton of the original proof is preserved, 2/ each exponential in the image is necessary, as it is required by some instance of a contraction or weakening rule somewhere in the proof.

We define a notion of ‘lower decoration strategy’, which in the case of the neutral fragment of **ILU** (Intuitionistic fragment of Unified Logic (Girard(1993))) is ‘sub-girardian’ in the sense that the result is a subdecoration of the uniform decoration obtained using Girard’s well-known translation of intuitionistic into linear logic. In fact, the linear proof we find is an *optimal* linear version of the original **ILU**-derivation, with essentially the same set of reductions.

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Because in this proof each exponential is imperative, one might conjecture that for each subproof determined by a ‘right’-! (a “box” in terms of proofnets) at least one normalisation strategy exists, in the course of which this specific subproof will be duplicated or erased. But this is *not* the case: even when ‘logically necessary’, exponentials can be ‘computationally superfluous’.

The paper is largely self-contained (though some acquaintance with Danos et al.(1993b) will be useful). In fact, this is the first in a series of papers, all of which make use of linear logic as a proof theoretical tool.

Though for purposes of exposition in this paper we limit ourselves to the implicational fragment of intuitionistic sequent calculus, the notions and techniques introduced can be applied to the *full* intuitionistic calculus. And, indeed, even to sequent calculi for *classical* logic: in Danos et al.(1994) we e.g. define a noetherian normalization procedure for second order **LK**, based upon the linear decoration of classical sequent derivations.

2 Girard’s translation and decorations

Linear logic, introduced in Girard(1987), sprouts from the (semantical) decomposition of intuitionistic implication $A \supset B$ as $(!A) \multimap B$, where \multimap denotes *linear* implication, and “!” is a unary operator (also called *exponential* or *modality*) which in the literature appears in turn under the names ‘of course’, ‘bang’, ‘shriek’, as well as (the less imaginative) ‘exclamation mark’ (we will most of the time use ‘shriek’). This decomposition then forms the core of a sound and faithful embedding of intuitionistic into linear logic, which we will refer to as *Girard’s translation*. Restricting ourselves to the implicational fragment, it is inductively defined by

$$\begin{aligned} A^* &:= A, \quad \text{for atomic } A; \\ (A \supset B)^* &:= !A^* \multimap B^*. \end{aligned}$$

Let us recall the standard sequent calculus formulation of intuitionistic implicational logic (**IL**):

Axioms: $A \Rightarrow A$, for atomic A

Logical rules:

$$\supset R \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \qquad \supset L \quad \frac{\Gamma_1 \Rightarrow A \quad B, \Gamma_2 \Rightarrow C}{\Gamma_1, A \supset B, \Gamma_2 \Rightarrow C}$$

Structural rules:

$$wL \quad \frac{\Gamma \Rightarrow A}{\Gamma, B \Rightarrow A} \qquad cL \quad \frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad exch \quad \frac{\Gamma \Rightarrow A}{\sigma(\Gamma) \Rightarrow A}$$

Cut rule:

$$\frac{\Gamma_1 \Rightarrow A \quad A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}$$

The sound- and faithfulness of the above translation means that a sequent $\Gamma \Rightarrow A$ is derivable in this calculus if and only if the sequent $!\Gamma^* \Rightarrow A^*$ is derivable in linear sequent calculus. In fact, by eliminability of the cut rule in linear logic, we know that if a sequent of the form $!\Gamma^* \Rightarrow A^*$ is derivable in linear logic, then it is cut-free derivable in the following fragment:

Axioms: $A \Rightarrow A$, for atomic A

Logical rules:

$$\multimap R \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \quad \multimap L \quad \frac{\Gamma_1 \Rightarrow A \quad B, \Gamma_2 \Rightarrow C}{\Gamma_1, A \multimap B, \Gamma_2 \Rightarrow C}$$

Structural rules:¹

$$exch \quad \frac{\Gamma \Rightarrow A}{\sigma(\Gamma) \Rightarrow A} \quad !w \quad \frac{\Gamma \Rightarrow B}{\Gamma, !A \Rightarrow B} \quad !c \quad \frac{\Gamma, !A, !A \Rightarrow B}{\Gamma, !A \Rightarrow B}$$

Exponential rules:

$$!R \quad \frac{!\Gamma \Rightarrow C}{!\Gamma \Rightarrow !C} \text{ (promotion)} \quad !L \quad \frac{\Gamma, A \Rightarrow B}{\Gamma, !A \Rightarrow B} \text{ (dereliction)}$$

Faithfulness then is obvious²: if we replace in a derivation of $!\Gamma^* \Rightarrow A^*$ in this fragment all symbols \multimap by \supset , delete all exponentials $!$, then what we obtain is a derivation of $\Gamma \Rightarrow A$ in the intuitionistic sequent calculus (possibly with some meaningless repetitions of sequents due to applications of the rules $!L$ and $!R$ in the linear derivation, which may be removed or ignored).

One shows *soundness* by induction on the length of derivations in the intuitionistic calculus. So e.g. if the last rule applied in the derivation of the sequent $\Gamma, A \supset B \Rightarrow C$ has been $\supset L$, then by inductive hypothesis we have derivations of $!\Gamma_1^* \Rightarrow A^*$ and $!\Gamma_2^*, !B^* \Rightarrow C^*$. Applying $!R$ to the first of these, and using $\multimap L$, we obtain $!\Gamma^*, !A^* \multimap !B^* \Rightarrow C^*$. A cut with the derivable sequent $!(!A^* \multimap B^*) \Rightarrow !A^* \multimap !B^*$ then yields the result.

¹In examples we will sometimes explicitly indicate the use of exchange because this makes it easier to follow a specific occurrence of a formula in a derivation (which, in a way, is what this paper is all about) without having to add indices. It is used however merely as a *notational device*.

²Observe that this is far less obvious in case we consider Girard's translation, defined on the *full* language, as an embedding of intuitionistic logic into *classical* linear logic, though also in that case it holds. See Schellinx(1991).

Note that this implies that Girard’s translation does *not* give us a proper translation of sequent-calculus derivations: the inductive procedure makes it necessary to introduce cuts in the linear derivation. If we define the *skeleton* $sk(\pi)$ of a linear derivation π to be the (classical or intuitionistic) derivation obtained by deleting all exponentials, replacing the connectives by their obvious non-linear counterpart and ignoring repetitions due to the exponential rules !R and !L, then the above observation can be expressed as follows: “*The skeleton of the linear derivation obtained through application of Girard’s translation in general will not be the intuitionistic derivation we started from*”. In fact this is true in a *strong* sense: also after elimination of the cuts introduced (the ‘*correction cuts*’) what we obtain is likely to be a derivation having a skeleton that is different from the original one.

2.1. Definition. A *decoration* of a (classical, intuitionistic) derivation π is a linear derivation π' such that $sk(\pi') = \pi$; by a decoration-strategy for a given (sequent)calculus we mean a uniform procedure (algorithm) that outputs a decoration for any given derivation in the calculus. \square

Though the $(\cdot)^*$ -translation in general does *not* give us a decoration-strategy for **IL**-derivations, elimination of the correction cuts results in a linear derivation having as skeleton an **IL**-derivation of the same sequent for which Girard’s translation *is* a decoration:

2.2. Theorem. *Let $\Gamma \Rightarrow A$ be any sequent derivable in intuitionistic (implicational) sequent calculus. Then there exists a derivation ν such that Girard’s translation provides a decoration-strategy for ν . Moreover, given a derivation π of $\Gamma \Rightarrow A$, we can in fact find such a ν by applying the translation to π and eliminating the ‘correction cuts’.*

PROOF: See Schellinx(1994). \square

The $(\cdot)^*$ -translation fails to be a decoration-strategy for the class of *all* intuitionistic derivations because of its economy in the use of exponentials. When one is willing to ‘shriek’ less economically, it is mostly not too difficult, given any of the standard sequent calculi for classical or intuitionistic logic, to define a decoration in the form of a sound and faithful translation into linear logic. Given e.g. a two-sided sequent calculus for classical predicate logic with all rules in multiplicative form, one may proceed as follows:

for atoms take $A^\diamond := A$; then put

$$\begin{aligned} (\neg A)^\diamond &:= (!?A^\diamond)^\perp \\ (A \wedge B)^\diamond &:= !?A^\diamond \otimes !?B^\diamond \end{aligned}$$

$$\begin{aligned}
(A \vee B)^\diamond &:= ?!A^\diamond \wp ?!B^\diamond \\
(A \rightarrow B)^\diamond &:= ?!A^\diamond \multimap ?!B^\diamond \\
(\forall x A)^\diamond &:= \bigwedge x ?!A^\diamond \\
(\exists x A)^\diamond &:= \bigvee x ?!A^\diamond.
\end{aligned}$$

It is easy to check that a sequent $\Gamma \Rightarrow \Delta$ is derivable in classical logic if and only if the sequent $! \Gamma^\diamond \Rightarrow ?! \Delta^\diamond$ is derivable in linear logic and that in fact this translation gives a decoration.³

Returning to the implicative fragment of intuitionistic logic, here we might take

$$\begin{aligned}
A^\circledast &:= A, \quad \text{for atomic } A; \\
(A \supset B)^\circledast &:= !A^\circledast \multimap !B^\circledast,
\end{aligned}$$

then check that $\Gamma \Rightarrow A$ is derivable in intuitionistic logic if and only if $! \Gamma^\circledast \Rightarrow A^\circledast$ is linearly derivable, and that moreover this time we *do* get decorations.

The corresponding strategy consists in replacing all sequents $\Gamma \Rightarrow A$ in a derivation π by $! \Gamma^\circledast \Rightarrow A^\circledast$, and applying !R just before each rule in which the succedent formula is active, i.e. before implication-right and before applications of implication-left and cut (in the left premiss). We will call the resulting linear derivation the *f(ull)-decoration* of π , written as π^\circledast .

By construction π^\circledast has the *down-property*, i.e. each main formula in an application of !R is active in the rule below.

Observe that in general the number of shrieks thus introduced is somewhat overabundant, to put it mildly: the inductive procedure can not ‘know’ whether a given formula will be subject to non-linear treatment somewhere in the derivation under construction, and therefore has to shriek them all.

However, it is very easy to define a *minimal* decoration-strategy applicable to any *cut-free* derivation: enumerate the applications of structural rules in the given derivation, then, for each instance, ‘shriek’ the relevant formulas, except when they were already ‘marked’ at an earlier stage. A glance at the conclusion of the decorated proof will tell which (sub)formulas have been subjected to structural manipulations in the original derivation.

³Linear decorations of this type are studied in detail in Joinet(1993), Schellinx(1994) and Danos et al.(1993a).

3 Decorated formulas

A *formula* is decorated by prefixing its subformulas with strings of linear modalities. Clearly the number of distinct decorations is infinite. However, if we call any (possibly empty) sequence $M := m_1 m_2 \dots m_n$ with $m_i \in \{!, ?\}$ a “modality”, then it is not very difficult to show that, modulo provable linear equivalence, we have essentially *seven* modalities in linear logic, namely $\cdot, ?, !, !?, ?!, !?! \text{ and } ?!?$, where “ \cdot ” represents the void modality. (See e.g. Joinet(1993).) As for derivations, we will sometimes refer to the obvious classical/intuitionistic formula that underlies a linear formula ϕ as its *skeleton* $sk(\phi)$.

So consequently we have, modulo provable linear equivalence, but a *finite* number of decorations of a given skeleton.

In *intuitionistic* linear logic we can restrict ourselves to “!” (and the void modality). If a linear formula ϕ^+ has been obtained from a linear formula ϕ by prefixing “!” to some number of *positive* subformulas of ϕ , we call ϕ^+ a *positive !-decoration* of ϕ (note that ϕ itself is not necessarily without modalities). One easily shows that $\phi^+ \Rightarrow \phi$ always is derivable in linear logic⁴, and has a canonical derivation which is a decoration of the canonical derivation of the axiom $\phi \Rightarrow \phi$.

We can form the set $\mathcal{D}^+(\phi)$ of all *non-equivalent* positive !-decorations of ϕ , which is naturally ordered by $B \leq A$ if and only if the decoration of A is equal to or extends that of B ⁵. Thus $\mathcal{D}^+(\phi)$ becomes a lattice. The join $A \cup B$ of two elements A, B is obtained by taking the superposition of their decorations, the meet $A \cap B$ by taking the “intersection”; top is the full positive !-decoration of ϕ , and bottom is ϕ itself.

Quite similarly we can consider *negative !-decorations* ϕ^- of some formula ϕ and the lattice of all its non-equivalent negative !-decorations, $\mathcal{D}^-(\phi)$.

Let us restrict ourselves once more to the implicational fragment, and consider a linear derivation⁶ π of a sequent $\Gamma, \phi \Rightarrow B$, or of a sequent $\Gamma \Rightarrow \phi$. Let ϕ^+ be any positive !-decoration of ϕ , ϕ^- any negative !-decoration. By τ we denote the canonical derivation of $\phi^+ \Rightarrow \phi$ or $\phi \Rightarrow \phi^-$. We can use π and τ to construct a linear derivation θ as follows:

⁴One might also look this up in the chapter on elementary syntactical results in Troelstra(1992).

⁵ $B \leq A$ implies that $A \Rightarrow B$ is derivable; the converse however does not hold necessarily.

⁶In diagrams we will use

$$\begin{array}{c} \pi \\ \vdots \\ \Gamma, \phi \Rightarrow B \end{array}$$

as abbreviation for a derivation π with $\Gamma, \phi \Rightarrow B$ as conclusion.

$$\begin{array}{c} \tau \\ \vdots \\ \phi^+ \Rightarrow \phi \quad \Gamma, \phi \Rightarrow B \\ \hline \Gamma, \phi^+ \Rightarrow B \end{array} \quad \text{OR} \quad \begin{array}{c} \pi \\ \vdots \\ \Gamma \Rightarrow \phi \quad \phi \Rightarrow \phi^- \\ \hline \Gamma \Rightarrow \phi^- \end{array}$$

We then have the following property:

3.1. Proposition. *We can eliminate the cut from θ in such a way that for the resulting reduct θ' it holds that $sk(\theta') = sk(\pi)$.*

PROOF: By induction on the complexity of ϕ : if $\phi \equiv p$ for some atom p , then $\phi^- = \phi$, so $\pi = \theta'$; ϕ^+ either is p or $!p$, so $\pi = \theta'$ or we obtain θ' from π by a linear dereliction ($!L$), which does not change the skeleton.

Let $\phi \equiv !\psi$. Then $\phi^+ = !\psi^+$, and $\phi^- = !\psi^-$. For the first case, we will prove the slightly stronger claim that in a derivation θ of the form

$$\begin{array}{c} \tau \\ \vdots \\ \psi^+ \Rightarrow \psi \\ \hline !\psi^+ \Rightarrow \psi \\ \hline !\psi^+ \Rightarrow !\psi \quad \Gamma, (!\psi)^n \Rightarrow B \\ \hline \Gamma, (!\psi^+)^n \Rightarrow B \end{array} \quad \begin{array}{c} \pi \\ \vdots \\ \Gamma, (!\psi)^n \Rightarrow B \end{array}$$

(where $(\phi)^n$ stands for $n \geq 1$ occurrences of the formula ϕ) we can eliminate the (derivable) rule of “semi-multi-cut” in such a way that for the reduct θ' we have that $sk(\theta') = sk(\pi)$. For this we proceed by induction on the length of π . We need the stronger hypothesis for the case where the last rule applied is a contraction on $!\psi$, which in the present form is trivially handled by inductive hypothesis, as are most other cases. The crucial one is the one where the last rule of π is a dereliction on ψ . Then θ has the form

$$\begin{array}{c} \tau \\ \vdots \\ \psi^+ \Rightarrow \psi \\ \hline !\psi^+ \Rightarrow \psi \\ \hline !\psi^+ \Rightarrow !\psi \quad \Gamma, (!\psi)^{n-1}, \psi \Rightarrow B \\ \hline \Gamma, (!\psi)^n \Rightarrow B \end{array} \quad \begin{array}{c} \pi \\ \vdots \\ \Gamma, (!\psi)^{n-1}, \psi \Rightarrow B \\ \hline \Gamma, (!\psi)^n \Rightarrow B \end{array}$$

If $n > 1$ we transform this into

$$\begin{array}{c} \tau \\ \vdots \\ \psi^+ \Rightarrow \psi \\ \hline !\psi^+ \Rightarrow !\psi \quad \Gamma, (!\psi)^{n-1}, \psi \Rightarrow B \\ \hline \Gamma, (!\psi^+)^{n-1}, \psi^+ \Rightarrow B \\ \hline \Gamma, (!\psi^+)^n \Rightarrow B \end{array} \quad \begin{array}{c} \pi \\ \vdots \\ \Gamma, (!\psi)^{n-1}, \psi \Rightarrow B \\ \hline \Gamma, (!\psi)^n \Rightarrow B \end{array}$$

and we get the result by our inductive hypotheses.

For $\phi^- = !\psi^-$ we have θ of the form

$$\frac{\frac{\frac{\pi}{\vdots} \quad \frac{\psi \Rightarrow \psi^-}{! \psi \Rightarrow \psi^-}}{\Gamma \Rightarrow ! \psi} \quad ! \psi \Rightarrow ! \psi^-}{\Gamma \Rightarrow ! \psi^-}$$

and we apply induction on the length of π .

Finally we consider the case that $\phi \equiv \psi \multimap \chi$. Then $\phi^+ = !(\psi^- \multimap \chi^+)$ or $(\psi^- \multimap \chi^+)$ and $\phi^- = (\psi^+ \multimap \chi^-)$. Without loss of generality we may assume that $\phi^+ = \psi^- \multimap \chi^+$, and θ has the form

$$\frac{\frac{\frac{\tau_1}{\vdots} \quad \frac{\tau_2}{\vdots} \quad \frac{\psi \Rightarrow \psi^- \quad \chi^+ \Rightarrow \chi}{\psi^- \multimap \chi^+, \psi \Rightarrow \chi}}{\psi^- \multimap \chi^+ \Rightarrow \psi \multimap \chi} \quad \frac{\Gamma, \psi \multimap \chi \Rightarrow B}{\Gamma, \psi^- \multimap \chi^+ \Rightarrow B}}{\Gamma, \psi^- \multimap \chi^+ \Rightarrow B} \quad \pi$$

We proceed rather straightforwardly by induction on the length of π . Similarly for ϕ^- . \square

By the above proposition we can replace any formula ϕ in the antecedents of sequents in a derivation in intuitionistic linear implicational sequent calculus by any of its positive $!$ -decorations ϕ^+ :

$$\frac{\Gamma, \phi \Rightarrow C}{\Gamma, \phi^+ \Rightarrow C}$$

is a derivable rule in a *strong* sense, i.e. when we justify it by means of a cut with $\phi^+ \Rightarrow \phi$ then we can eliminate that cut and obtain a derivation π^+ of $\Gamma, \phi^+ \Rightarrow C$ such that $sk(\pi^+) = sk(\pi)$. So elimination merely ‘injects’ a number of applications of exponential rules to ‘adjust’ the derivation.

(Note that this is in sharp contrast with our observations in the previous section: also

$$\frac{\Gamma, !\phi \multimap !\psi \Rightarrow C}{\Gamma, !(!\phi \multimap \psi) \Rightarrow C}$$

is a derivable rule, but in this case elimination of the justifying cut may very well change the skeleton.)

4 The ‘lower’ decoration strategy

Given a **IL**-derivation that is not cut-free, we would like to apply the same idea of decoration as the one sketched at the end of section 2: start from a structural *source* (i.e. a weakened or contracted formula), trace it through the proof, and prefix its successive occurrences with a shriek. It will in general not be true that this ‘shrieking’ automatically stops because we reach the conclusion of the derivation. We may end in a cut-formula, which forces us to travel up again in the subderivation determined by the other premiss of the cut. In that case, the formula we are tracing will emerge in this subderivation as succedent of some sequent. There ‘!’ has to be introduced by means of an instance of the *promotion*-rule, or of an *axiom* in which the succedent formula is shrieked. In both cases the linear derivation we are constructing can become *correct* only if all the formulas in the antecedent are shrieked as well. These formulas then in turn will act as sources, and we have to trace all of them, and shriek them throughout the proof. So each structural source will cause a ‘cascade of shrieks’ to cover the original derivation.

It is clear how to proceed, though the formal description is somewhat tiresome. In Joinet(1993) a notion of *path* is defined, comparable to the *identity-classes* of Danos et al.(1993b). Here we will profit from our assumption that all axioms are atomic, and define for formulas F occurring in a derivation π in the calculus the *track* $T_\pi(F)$ of F in π , being (almost) a subtree of π with nodes labeled by a symbol $\sigma(F)$ denoting (among other things) the sign of F (i.e. $\sigma(F) = “+”$, “ \star ” or “ ∇ ” if F occurs *positively*, $\sigma(F) = “-”$ if F occurs *negatively* in the sequent $\Gamma \Rightarrow \Delta$ (i.e. in the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$)), or either \circ or \bullet (in case a (sub)formula “disappears” in a cut). Using the terminology of Regnier(1992), we will call a formula in π *terminal* if it either is a cutformula in π or a formula in the conclusion. Then, to be precise, we define $T_\pi(F)$ for (sub)formulas F of terminal formulas in π ; in case F is a cutformula we put $T_\pi(F) := T_{\pi'}(F)$, where π' denotes the subderivation of π having the cut on F as a last rule. The occurrence of F in the terminal formula is called the *root* of $T_\pi(F)$.

Let us then state a formal inductive definition of $T_\pi(F)$ for the sequent calculus formulation of intuitionistic implicative logic given in section 2.

1. if π is an axiom $F \Rightarrow F$, then, for the *left* occurrence of F , $T_\pi(F)$ is the one-node tree labeled by “ $-$ ”; for the *right* occurrence labeled by “ ∇ ”;
2. if π has been obtained from π' by means of rule $\supset R$, then for all subformulas F of $\Gamma \cup \{A\} \cup \{B\}$, *except* B , we obtain $T_\pi(F)$ by adding a new node labeled by the appropriate sign of occurrence; we obtain $T_\pi(B)$ by adding a new node \star ; $T_\pi(A \supset B)$ is the one-node tree, labeled “ $+$ ”;

3. if π has been obtained from π_1 and π_2 by means of rule $\supset L$, then for all subformulas F of $\Gamma_1 \cup \{A\}$, *except* A , we obtain $T_\pi(F)$ from $T_{\pi_1}(F)$ by adding a new node labeled by the appropriate sign of occurrence (i.e. “+” or “-”); we obtain $T_\pi(A)$ by adding a new node labeled “ \star ”; for all subformulas F of $\Gamma_2 \cup \{B\} \cup \{C\}$ we obtain $T_\pi(F)$ from $T_{\pi_2}(F)$ by adding a new node labeled by the appropriate sign of occurrence, and $T_\pi(A \supset B)$ is the one-node tree labeled by “-” (so a node labelled “ \star ” in $T_\pi(F)$ indicates that it is the immediate successor of a node corresponding to the *lowest* occurrence in π of F as a *succedent* (i.e. in a sequent $\Gamma \Rightarrow F$));
4. if π has been obtained from π' by means of a weakening with formula A , then for all subformulas F of A , $T_\pi(F)$ is the one-node tree labeled by the appropriate sign of occurrence; for all other F we get $T_\pi(F)$ from $T_{\pi'}(F)$ by adding a node labeled by the appropriate sign of occurrence;
5. if π has been obtained from π' by means of exchange, then we get $T_\pi(F)$ from $T_{\pi'}(F)$ by adding a node labeled by the appropriate sign of occurrence;
6. if π has been obtained from π' by means of a contraction on a formulas A , then for all subformulas A of F we obtain $T_\pi(F)$ by joining the trees $T_{\pi'}(F^{(1)})$ and $T_{\pi'}(F^{(2)})$ (where $F^{(1)}$ and $F^{(2)}$ denote the two distinct occurrences of F) to a node labeled by the appropriate sign of occurrence; for all other F we get $T_\pi(F)$ from $T_{\pi'}(F)$ by adding a new node, again labeled by the appropriate sign of occurrence;
7. if π has been obtained from π_1 and π_2 by means of application of cut on a formula A , then for all subformulas F of Γ_1 we obtain $T_\pi(F)$ from $T_{\pi_1}(F)$ by adding a new node labeled by the appropriate sign of occurrence, for all subformulas F of $\Gamma_2 \cup \{B\}$ we obtain $T_\pi(F)$ from $T_{\pi_2}(F)$ by adding a new node, again labeled by the appropriate sign of occurrence; for proper subformulas F of A , $T_\pi(F)$ is the tree obtained by joining $T_{\pi_1}(F)$ and $T_{\pi_2}(F)$ through a bottom-node labeled \circ . $T_\pi(A)$ is obtained by joining $T_{\pi_1}(A)$ and $T_{\pi_2}(A)$ through a bottom-node labeled “ \bullet ” (so a node labelled “ \bullet ” in $T_\pi(F)$ indicates (that F is a cutformula and) that it is the immediate successor of a node corresponding to the *lowest* occurrence in π of F as a succedent).

(A similar definition of course can be given for *any* sequent calculus, containing rules for whatever set of connectives.) The idea is easily grasped by considering some examples (see below).

4.1. Proposition. *For each terminal formula F , $T_\pi(F)$ is either a finite tree with all nodes labeled $\neq \circ$ (“ \bullet ”), or a finite tree with bottom-node \circ (“ \bullet ”). Moreover, if all nodes are labeled $\neq \circ$ (“ \bullet ”), then either all nodes are labeled “+” (“ \star ”, “ ∇ ”), or all nodes are labeled “-”. If the bottom-node is labeled \circ (“ \bullet ”), then the labels in the subtrees defined by the two predecessors of \circ (“ \bullet ”) are all equal to “-” in one of the two, all unequal to “-” in the other. \square*

Clearly *each occurrence* of a (sub)formula F in a given derivation π corresponds to a unique node i in $\mathcal{T}_\pi(F)$.

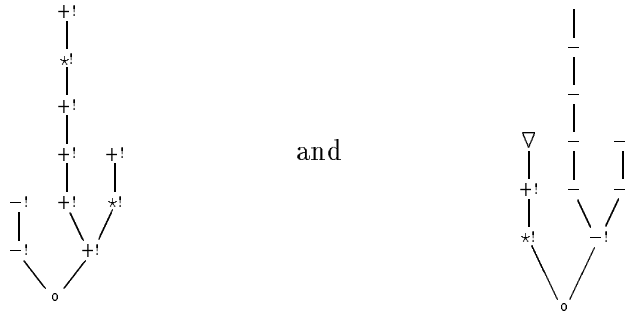
Define for a *negative* occurrence of a formula $N(i)$ the $N(i)$ -*decoration* of $\mathcal{T}_\pi(N)$ as the labeled tree obtained from $\mathcal{T}_\pi(N)$ by the following instructions:

1. replace the label at node i , as well as the labels “ $-$ ” at all successors of i , by “ $-!$ ”;
2. if the bottom-node is *not* labeled “ \circ ” or (“ \bullet ”), then we are done;
3. if it is “ \bullet ” then change the label of its positive predecessor by “ $+!$ ”, and we are done;
4. if it is “ \circ ”, then for all branches of the ‘*positive*’ subtree:
 - (a) if the branch contains a starred node, take the starred node’s predecessor and add “ $!$ ” to its label, as well as to the ‘non-zero’-labels of all its successive successor-nodes;
 - (b) if the branch does *not* contain a starred node, then add “ $!$ ” to the ‘non-zero’-labels of all its nodes.

E.g. the decoration induced by the occurrence of AA introduced by weakening, respectively the decoration induced by the negative occurrence of A in the right-most axiom in the derivation

$$\begin{array}{c}
 \frac{A \Rightarrow A}{A, A \Rightarrow A} \\
 \frac{A \Rightarrow AA \quad A \Rightarrow A}{A, (AA)A \Rightarrow A} \\
 \frac{(AA)A, A \Rightarrow A}{(AA)A \Rightarrow AA} \quad \frac{A \Rightarrow A}{(AA)A, (AA)A \Rightarrow A} \\
 \frac{A \Rightarrow A}{A, AA \Rightarrow A} \quad \frac{(AA)A \Rightarrow AA \quad A \Rightarrow A}{(AA)A \Rightarrow A} \\
 \frac{A \Rightarrow (AA)A}{A \Rightarrow (AA)A} \quad \frac{(AA)A \Rightarrow A}{A \Rightarrow A}
 \end{array}$$

are



An $N(i)$ -decoration induces in the obvious way a distribution of shrieks in the original derivation π : we ‘shriek’ the occurrences of N that correspond to nodes of the $N(i)$ -decoration whose label contain a “ $!$ ”. (We will refer to the result as a ‘pre-decoration’ of π .)

So, for the AA - and A -decoration given above we get the pre-decorations

$$\begin{array}{c}
\frac{A \Rightarrow A}{A, A \Rightarrow A} \\
\frac{A \Rightarrow A}{A \Rightarrow !(AA)} \quad A \Rightarrow A \\
\frac{A, !(AA)A \Rightarrow A}{!(AA)A, A \Rightarrow A} \\
\frac{!(AA)A \Rightarrow !(AA)}{!(AA)A \Rightarrow !(AA)} \quad A \Rightarrow A \\
\frac{A \Rightarrow A}{A, !(AA) \Rightarrow A} \quad \frac{!(AA)A, !(AA)A \Rightarrow A}{!(AA)A \Rightarrow A} \\
\frac{A \Rightarrow !(AA)A}{A \Rightarrow !(AA)A} \quad \frac{!(AA)A \Rightarrow A}{!(AA)A \Rightarrow A} \\
\hline
A \Rightarrow A
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\frac{A \Rightarrow A}{A, A \Rightarrow A} \\
\frac{A \Rightarrow A}{A \Rightarrow AA} \quad A \Rightarrow A \\
\frac{A, (AA)A \Rightarrow A}{(AA)A, A \Rightarrow A} \\
\frac{(AA)A \Rightarrow AA}{(AA)A \Rightarrow AA} \quad !A \Rightarrow A \\
\frac{A \Rightarrow A}{A, AA \Rightarrow !A} \quad \frac{(AA)A, (AA)!A \Rightarrow A}{(AA)A, (AA)!A \Rightarrow A} \\
\frac{A \Rightarrow (AA)!A}{A \Rightarrow (AA)!A} \quad \frac{(AA)!A \Rightarrow A}{(AA)!A \Rightarrow A} \\
\hline
A \Rightarrow A
\end{array}$$

The idea for a decoration-strategy now should be clear: given a derivation we look at the collection of instances of weakening and contraction that have been used; we then start ‘shrieking’ the main formula, say $N(i)$, in the conclusion of such a rule ($N(i)$ is a ‘*primary shriek-source*’), and trace the formula through the derivation. This is done by means of the $N(i)$ -decoration defined above. (Note that we made *one* choice: we stop putting shrieks as soon as we reach a *lowest* occurrence of the formula as *succedent* in a sequent. We will therefore speak of the ‘*lower decoration-strategy*’; clearly there are other possibilities: one might stop only at the *highest* occurrences, or anywhere in between.)

The last shriek we put might be the conclusion of a sequent having G_1, \dots, G_n as premisses. In order to obtain a derivation that is correct in linear logic we have no choice but to ‘shriek’ these; consequently G_1, \dots, G_n will be ‘*secondary shriek-sources*’, and the process continues until there is nothing left to be done.

To put this formally, starting with $N(i)$ we define a finite tree of decorations as follows:

1. top node is the $N(i)$ -decoration;
2. let a node α be given, i.e. some $M(j)$ -decoration;
 - (a) if α has a bottom-node labeled $-!$, then α has no successors;
 - (b) if α has bottom-node labeled \circ or \bullet , then look at the highest nodes labeled $!$ in the positive subtree. If these are $\nabla!$ or predecessors of $\star!$ -nodes, then they correspond to sequents $\Gamma_1 \Rightarrow M(j), \dots, \Gamma_n \Rightarrow M(j)$ in π . The successors of α then are the $G_m(i)$ -decorations, for all $G_m(i) \in \Gamma_1 \cup \dots \cup \Gamma_n$ that so far have not yet appeared in the tree.

(Observe that finiteness is clear, as there are but finitely many formulas in a given derivation.)

We thus obtain a finite tree of N -decorations for each primary shriek-source in the derivation. The *linearly decorated derivation* then is the original shriek derivation with shrieks added in accordance with the superposition of all the corresponding pre-decorations.

It will suffice to look carefully at an example to convince one-self that this is a completely self-evident process⁷, though admittedly somewhat cumbersome to describe formally.

The reader might wish to verify that applying the above to the derivation

$$\begin{array}{c}
 \frac{A \Rightarrow A}{A, A \Rightarrow A} \\
 \frac{A \Rightarrow AA \quad A \Rightarrow A}{A, (AA)A \Rightarrow A} \\
 \frac{A \Rightarrow A \quad A \Rightarrow A}{A, AA \Rightarrow A} \quad \frac{A \Rightarrow A}{(AA)A, A \Rightarrow A} \\
 \frac{A, AA, AA \Rightarrow A}{A, AA, AA \Rightarrow A} \quad \frac{(AA)A \Rightarrow AA \quad A \Rightarrow A}{(AA)A, (AA)A \Rightarrow A} \\
 \frac{A, AA \Rightarrow A}{A \Rightarrow (AA)A} \quad \frac{(AA)A \Rightarrow A}{(AA)A \Rightarrow A} \\
 \hline
 A \Rightarrow A
 \end{array}$$

results in

$$\begin{array}{c}
 \frac{A \Rightarrow A}{A, !A \Rightarrow A} \\
 \frac{!A \Rightarrow !A \quad A \Rightarrow A}{!A, !(AA) \Rightarrow !A} \quad \frac{A \Rightarrow A}{!A \Rightarrow !(AA)A} \\
 \frac{!A, !(AA), !AA \Rightarrow A}{!A, !(AA), !AA \Rightarrow A} \quad \frac{!A, !(AA)A \Rightarrow A}{!(AA)A, !A \Rightarrow A} \\
 \frac{!A, !(AA) \Rightarrow A}{!A \Rightarrow !(AA)A} \quad \frac{!(AA)A \Rightarrow !(AA)A \quad A \Rightarrow A}{!(AA)A, !(AA)A \Rightarrow A} \\
 \frac{!A \Rightarrow !(AA)A}{!A \Rightarrow !(AA)A} \quad \frac{!(AA)A \Rightarrow A}{!(AA)A \Rightarrow A} \\
 \hline
 !A \Rightarrow A
 \end{array}$$

But then, did we, by applying the decoration-strategy, in the end obtain a *correct* derivation in linear logic?

Admittedly, there are some deviations. E.g. we encounter instances of

$$\frac{\Gamma, C \Rightarrow B}{! \Gamma \Rightarrow !(CB)}$$

These, however, can be interpreted as being abbreviations for

$$\frac{\Gamma, C \Rightarrow B}{\Gamma \Rightarrow CB} \\
 \vdots \\
 \frac{! \Gamma \Rightarrow CB}{! \Gamma \Rightarrow !(CB)}$$

More serious seems that in the decorated derivation we get contractions on formulas that, though having the same skeleton, are not identical as linear formulas. This is illustrated by the example.

In general, in a decorated derivation we will encounter instances of contraction of the form

⁷As to the complexity of the procedure, this is easily seen to be linear in the *size* of (i.e. the number of subformulas appearing in) the original derivation.

$$(\dagger) \quad \frac{\Gamma, \delta_1(A), \delta_2(A) \Rightarrow B}{\Gamma, \delta_3(A) \Rightarrow B},$$

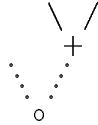
where $\delta_i(A)$ denotes a decoration of a formula A and all $\delta_i(A)$ are not necessarily identical.

We can however observe the following:

4.2. Lemma. *If (\dagger) is an instance of a contraction-rule in a decorated derivation π , then δ_1 and δ_2 differ at most in the decoration of positive subformulas. Moreover $\delta_3(A) = (\delta_1 \cup \delta_2)(A)$ or $\delta_3 = !(\delta_1 \cup \delta_2)(A)$ (where \cup stands for superposition of decorations).*

PROOF: It is sufficient to show that the decoration induced by each node of each finite tree belonging to the primary shriek-sources in π has the property.

Now for the decoration induced by A we have $\delta_1(A) = \delta_2(A) = A$ and $\delta_3(A) = !A$. Otherwise we have a decoration induced by a subformula of A occurring as a primary or secondary source somewhere else in π . If it is a *negative* subformula N , the source is in a subtree of π ending in a premiss of an instance of cut on a formula having A as a subformula, the other premiss being the conclusion of a subderivation of π containing (\dagger) . The instance of contraction corresponds to a splitting in the *positive* subtree of the track of N , and in fact we can take $\mathcal{T}_\pi(N)$ to be of the form



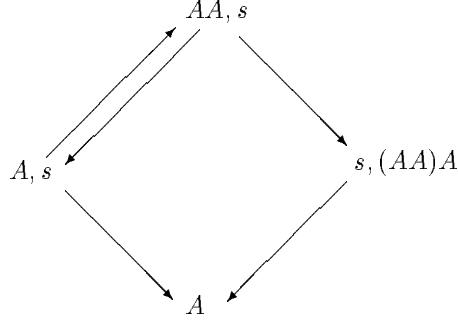
as neither the bottom-node of the splitting, nor any of its successors can be labelled “ \star ”. But then, by definition of N -decoration *all* three nodes of the splitting will be shrieked. So the induced decoration satisfies $\delta_1(A) = \delta_2(A) = \delta_3(A)$.

If the subformula is positive, the source is in the subderivation of π ending with (\dagger) . Then only one of both occurrences of A in the premiss of (\dagger) is decorated, but obviously it is a positive subformula that is shrieked. \square

So the instances (\dagger) of contraction in our decorated derivation have the property that $\delta_3(A)$ is a positive $!$ -decoration of both $\delta_1(A)$ and $\delta_2(A)$. Therefore the observations of section 2 apply: (\dagger) is not only derivable in linear logic, it is completely ‘harmless’, as, if we would wish so, we could apply cuts with $\delta_3(A) \Rightarrow \delta_i(A)$, eliminate these, and obtain a linear derivation in which all contractions are literally correct, with the same skeleton as our decoration.

The vertex set of $\mathcal{G}(\pi)$ is denoted by \mathcal{V}_π . \square

As an example the reader might verify that the following represents the exponential graph of $\partial(\pi)$ above:



In Danos et al.(1993b) we showed how, given a linear derivation π , one can remove exponentials from π in such a way that the result remains correct. This rewriting determines a lattice of linear derivations (with top π and as bottom a unique normal form π^\flat), having the property that all its elements have the same behaviour under reduction (the same ‘dynamics’). Moreover, in the case of ‘mono’-derivations (which include the images of the uniform decorations of intuitionistic proofs), any subdecoration of π will have an associated “exponential graph” containing that of π^\flat , i.e. no significant further improvement will be possible.

4.6. Lemma. *Let $!A$ be a subformula of a terminal formula in $\partial(\pi)$. Then there is a directed path γ in $\mathcal{G}(\partial(\pi))$ from a vertex labeled “ s ” to the vertex labeled $T_\pi(!A)$.*

PROOF: This is immediate from the construction of $\partial(\pi)$. Formally we proceed by induction on the length of a branch in a finite tree of decorations starting from a primary shriek-source: if $T_\pi(!A)$ contains that source, our claim evidently holds; otherwise we have a sequent $!\Gamma, !A \Rightarrow !B$ being the conclusion of an instance of $!R$ in $\partial(\pi)$. By inductive hypothesis there is a directed path in $\mathcal{G}(\partial(\pi))$ from a vertex labeled “ s ” to $T_\pi(!B)$. But as we have an arrow from $T_\pi(!B)$ to $T_\pi(!A)$ in $\mathcal{G}(\partial(\pi))$, there is a directed path from the same source to $T_\pi(!A)$. \square

As $\partial(\pi)$ is a subdecoration of π^\circledast , we can, in the obvious way consider $\mathcal{G}(\partial(\pi))$ as a subgraph of $\mathcal{G}(\pi^\circledast)$. More so:

4.7. Lemma. $\mathcal{G}(\partial(\pi))$ is a full subgraph of $\mathcal{G}(\pi^\otimes)$, i.e. if $T_\pi(!A), T_\pi(!B) \in \mathcal{V}_{\partial(\pi)}$, and there is an arrow between the corresponding vertices in $\mathcal{G}(\pi^\otimes)$, then that arrow exists also in $\mathcal{G}(\partial(\pi))$.

PROOF: The arrow is there because of the conclusion of an instance of !R in π^\otimes . As both π^\otimes and $\partial(\pi)$ have the down-property, the corresponding sequent in $\partial(\pi)$ is also the conclusion of !R, q.e.d. \square

Let us, for a vertex labeled “ s ”, write $c\vec{x}(s)$ to denote the union of all directed paths in $\mathcal{G}(\pi)$ starting from the given vertex. We then obtain:

4.8. Theorem. $\mathcal{G}(\partial(\pi)) = \bigcup_{s \in \mathcal{V}_{\pi^\otimes}} c\vec{x}(s)$.

PROOF: By lemma’s 4.6 and 4.7. \square

Now $\bigcup_{s \in \mathcal{V}_{\pi^\otimes}} c\vec{x}(s)$ is precisely $\mathcal{G}((\pi^\otimes)^\flat)$, so, by the results of Danos et al.(1993b), $\partial(\pi)$ is an optimal linearization of π , which is essentially equal to $(\pi^\otimes)^\flat$ (though in general not identical to it).

5 ILU and (sub-)Girardian decorations

If we call a decoration of a derivation in intuitionistic implicational logic *sub-Girardian* in case it does not put shrieks in front of subformulas that are not shrieked in the $(\cdot)^*$ -translation, then, as we saw in section 1, a (minimal) decoration strategy for the usual sequent formulation can never lead to decorations that are *always* sub-Girardian.

This is related to the fact that Girard’s translation is not a decoration-strategy. The ‘root of all evil’ is that intuitionistic sequent calculus allows application of the rule $\supset L$ where the main formula in the right premiss has been active in a structural rule. We can do without that property: the collection of derivations that do *not* use it is complete for intuitionistic logic, which in fact is part of the content of theorem 2.2. This suggests a formulation of intuitionistic implicational sequent calculus in which the use of $\supset L$ on such formulas is forbidden, and for which as a consequence Girard’s translation should be a decoration-strategy.

Such a formulation appears as a straightforward abstraction of the structure of linear derivations of sequents of the form $!\Gamma^* \Rightarrow A^*$.⁸

⁸Observe that the instances of implication-left that we will get rid of have no equivalent in the natural deduction formulation of **IL**. Therefore this modified sequent calculus will be closer to natural deduction and the simply typed λ -calculus than the standard formulation.

Axioms: $A; \Rightarrow A$ (for A atomic);

Logical rules:

$$\supset R \frac{\Pi; \Gamma, A \Rightarrow B}{\Pi; \Gamma \Rightarrow A \supset B} \quad \supset L \frac{; \Gamma_1 \Rightarrow A \quad B; \Gamma_2 \Rightarrow C}{A \supset B; \Gamma_1, \Gamma_2 \Rightarrow C}$$

Structural rules:

$$wL \frac{\Pi; \Gamma \Rightarrow A}{\Pi; \Gamma, B \Rightarrow A} \quad cL \frac{\Pi; \Gamma, A, A \Rightarrow B}{\Pi; \Gamma, A \Rightarrow B} \quad dL \frac{B; \Gamma \Rightarrow A}{; B, \Gamma \Rightarrow A}$$

Π here is a multi-set containing *at most* one (the *head-* or *stoup-*)formula. In its linear interpretation it represents a formula that is not (yet) shrieked. The structural rule dL is the equivalent of the linear dereliction rule $!L$.

What we obtained is the (cut-free) *neutral* fragment of intuitionistic implicational logic as it appears in Girard's system of Unified Logic (**LU**, Girard(1993)). We will refer to this fragment here as **ILU**. By construction its connection with the $(\cdot)^*$ -translation is flawless: $\Pi; \Gamma \Rightarrow A$ is derivable if and only if $\Pi^*, !\Gamma^* \Rightarrow A^*$ is derivable in linear logic. Moreover for a **ILU**-derivation π Girard's translation determines a decoration-strategy, producing a linear derivation π^* (the *g(irard)-decoration* of π) with the down-property.

But we have more:

5.1. Proposition. *The collection of g-decorated ILU-derivations is closed under linear cut.*

PROOF: One shows by induction on the complexity of the formula A that

1/ if π^* is a g-decorated **ILU**-derivation of a sequent $\Pi^*, !\Gamma^* \Rightarrow A^*$ then, for any g-decorated **ILU**-derivation τ^* , elimination of a linear cut

$$\frac{\begin{array}{c} \pi^* \\ \vdots \\ \Pi^*, !\Gamma^* \Rightarrow A^* \end{array} \quad \begin{array}{c} \tau^* \\ \vdots \\ A^*, !\Delta^* \Rightarrow B^* \end{array}}{\Pi^*, !\Gamma^*, !\Delta^* \Rightarrow B^*}$$

does not take us out of the collection of g-decorated **ILU**-derivations;

2/ if π^* is a g-decorated **ILU**-derivation of a sequent $!\Gamma^* \Rightarrow A^*$ then, for any g-decorated **ILU**-derivation τ^* , we can eliminate the linear multi-cut

$$\frac{\begin{array}{c} \pi^* \\ \vdots \\ !\Gamma^* \Rightarrow A^* \\ !\Gamma^* \Rightarrow !A^* \end{array} \quad \begin{array}{c} \tau^* \\ \vdots \\ \Pi^*, !\Delta^*, (!A^*)^n \Rightarrow B^* \end{array}}{\Pi^*, !\Gamma^*, !\Delta^* \Rightarrow B^*}$$

in such a way that we stay within the collection of g-decorated **ILU**-derivations. \square

As a corollary we get that **ILU** is closed under the
Cut rules:

$$\text{head - cut} \quad \frac{\Pi; \Gamma_1 \Rightarrow A \quad A; \Gamma_2 \Rightarrow B}{\Pi; \Gamma_1, \Gamma_2 \Rightarrow B}$$

$$\text{mid - cut} \quad \frac{; \Gamma_1 \Rightarrow A \quad \Pi; A, \Gamma_2 \Rightarrow C}{\Pi; \Gamma_1, \Gamma_2 \Rightarrow C}$$

5.2. Remark. The cut elimination procedure for an **ILU**-derivation π is the **ILU**-equivalent of the cut elimination procedure for the linear derivation π^* . Observe however, that the correspondence between the two procedures is in fact *not* step-by-step. Due to the fact that in **ILU** the exponential rule !R is ‘invisible’, in certain cases an elementary reduction step in **ILU** will correspond to *two* consecutive steps in the linear equivalent.

It is not difficult to show sound- and completeness of **ILU** with respect to intuitionistic implicational logic directly for the cut-free fragment. However, we will give an instructive argument using closure of **ILU** under cut:

5.3. Proposition. *If $\mathbf{ILU} \vdash \Pi; \Gamma \Rightarrow A$, then $\mathbf{IL} \vdash \Pi, \Gamma \Rightarrow A$. Conversely, if $\mathbf{IL} \vdash \Gamma \Rightarrow A$, then $\mathbf{ILU} \vdash ; \Gamma \Rightarrow A$.*

PROOF: Note that only the second claim is worth of our attention. Here we proceed by induction on the length of **IL**-derivations. We encounter a problem in case the last rule applied has been $\supset L$. By inductive hypothesis we have **ILU**-derivations of $; \Gamma \Rightarrow A$ and $; \Delta, B \Rightarrow C$, and we would like to get a derivation of $; \Gamma, \Delta, A \supset B \Rightarrow C$. As B is not a head-formula, we cannot use the **ILU**-rule directly. However, it is easy to derive $; A, A \supset B \Rightarrow B$ in **ILU**. So using mid-cuts we construct

$$\frac{\frac{\frac{; \Gamma \Rightarrow A \quad ; A, A \supset B \Rightarrow B}{; \Gamma, A \supset B \Rightarrow B} \quad ; \Delta, B \Rightarrow C}{; \Gamma, \Delta, A \supset B \Rightarrow C}}$$

and we are done by closure of **ILU** under cut.

(In fact, the cuts introduced correspond “on the nose” with the ‘correction cut’ introduced in the application of Girard’s translation to **IL**-derivations; their

5.5. Corollary. *Let π be an **ILU**-derivation. Then $\partial(\pi)$ is a subdecoration of the g -decoration π^* of π (which in turn is a subdecoration of the f -decoration π^\circledast of π , considered as an **IL**-proof). \square*

Therefore $\mathcal{G}(\partial(\pi))$ can be considered, in the obvious way, as a subgraph of $\mathcal{G}(\pi^*)$, as well as of $\mathcal{G}(\pi^\circledast)$. The arguments at the end of section 4 clearly go through in this case, hence

5.6. Theorem. *Let π be in **ILU**. Then $\bigcup_{s \in \mathcal{V}_{\pi^*}} c\vec{x}(s) = \mathcal{G}(\partial(\pi)) = \bigcup_{s \in \mathcal{V}_{\pi^\circledast}} c\vec{x}(s)$. \square*

Again we have that $\bigcup_{s \in \mathcal{V}_{\pi^*}} c\vec{x}(s)$ is precisely $\mathcal{G}((\pi^*)^\circ)$. As, moreover, the $*$ -decoration is (using the terminology of Danos et al.(1993b)) a ‘strong mono’-decoration for **ILU**-derivations, we may conclude that $\delta(\pi)$ is an optimal linearization of π , with essentially the same set of reductions.

5.7. Remarks. (1) One can alternatively introduce “**ILU**” as a restriction on the form of **IL**-derivations that is conserved under cut-elimination:

*An **IL**-derivation is in “**ILU**” iff each negative formula occurrence active in a \supset -L-rule is newly born (i.e. has just been introduced, in a logical rule, or is the negative formula occurrence in an identity axiom).*

To be precise, this alternative definition corresponds to a version of **ILU** in which one demands that stoup-formulas are *always* active (or in the conclusion of the derivation). Cf. Danos et al.(1994).

(2) The proximity of the sequent calculus **ILU** to natural deduction for intuitionistic logic is put to use in Herbelin(1994), where a λ -calculus with explicit substitution is introduced, that is isomorphic to (a small variation on) **ILU**.

6 Decorated derivations and normalization

When normalizing derivations in sequent calculus (i.e. eliminating the cuts) we come across counterparts of the structural rules of weakening and contraction: because in the derivation by means of an application of weakening a formula may suddenly appear as if out of thin air, there are elementary reduction steps in which complete subderivations are *erased*; and similarly, because in the derivation distinct occurrences of a formula can be contracted into a single one, we get elementary reduction steps in which subderivations are *duplicated*.

For any derivation in intuitionistic implicative logic we showed how to trace the linear consequences of occurrences of weakening and contraction in order to obtain a minimally decorated linear equivalent. Each ‘shriek’ occurring in such a decorated derivation has *at least one* structural cause. In this section we will look into the converse, namely whether a minimal decoration $\partial(\pi)$ of a derivation π can provide us with information as to the behaviour of π under reduction.⁹

Whereas weakening and contraction are about *formulas*, the notions of erasure and duplication are about (sub)*derivations*. However, there are obvious candidates for the title of ‘*marked subderivation*’ in a decorated derivation, namely those subderivations that end with an application of !R. Using the analogy with the similar notion in proofnets, we will speak of *boxes*.

6.1. Definition. A subderivation π_1 of a derivation π (written as $\pi_1 \prec \pi$) is said to be *boxed* (or is said to be a *box*) in π iff it is a subderivation of a sequent $\Gamma \Rightarrow A$ that is *externally* decorated in $\partial(\pi)$ (i.e. all formulas in $\Gamma \cup \{A\}$ start with a shriek). A subderivation $\pi_s \prec \pi$ is said to be a *pseudo-box* (or *source*) in π if its last rule is an instance of weakening or contraction. (Note that a pseudo-box might be a box.) \square

With each box π_i we can uniquely associate an instance c_i of the cut rule in π . We will say that c_i is the *reflecting cut* for π_i . As an example, let us decorate (an **ILU**-version of) the derivation of page 11:

$$\begin{array}{c}
\frac{A; \Rightarrow A}{A; !A \xrightarrow{a} A} \\
\frac{A; !A \xrightarrow{a} A}{A; \Rightarrow !AA} \\
\frac{A; \Rightarrow A}{; !A \xrightarrow{1} !(AA)} \quad A; \Rightarrow A \\
\frac{; !A \xrightarrow{1} !(AA) \quad A; \Rightarrow A}{!(AA)A; !A \Rightarrow A} \\
\frac{!(AA)A; !A \Rightarrow A}{!(AA)A; \Rightarrow !AA} \\
\frac{A; \Rightarrow A}{A; !(AA) \xrightarrow{b} A} \quad ; !(AA)A \xrightarrow{2} !(AA) \quad A; \Rightarrow A \\
\frac{A; !(AA) \xrightarrow{b} A}{; A, !(AA) \Rightarrow A} \quad ; !(AA)A; !(AA)A \Rightarrow A \\
\frac{; A, !(AA) \Rightarrow A}{; !A \xrightarrow{3} !(AA)A} \quad ; !(AA)A, !(AA)A \Rightarrow A \\
\frac{; !A \xrightarrow{3} !(AA)A}{; !A \Rightarrow A} \quad ; !(AA)A \xrightarrow{c} A
\end{array}$$

We see that we have three boxes π_1, π_2 and π_3 (indicated above by 1, 2 and 3) and three sources π_a, π_b and π_c (indicated by a, b and c) corresponding to the three primary shriek-sources in π : weakening of A , weakening of AA and contraction on $(AA)A$. (None of the pseudo-boxes is in fact a box.)

If we take the tree of decorations associated with a primary shriek-source N in π , then each (initial segment of a) branch of this tree determines a *chain* Σ

⁹Throughout this section we will assume our derivations to be in **ILU**.

consisting in a source followed by zero (a *trivial* chain) or more boxes in π . In the example we find the trivial chain π_a (the weakening on A has no consequences), the chain $\pi_b \rightarrow \pi_1$, the chain $\pi_b \rightarrow \pi_2 \rightarrow \pi_3$ and the chain $\pi_c \rightarrow \pi_3$.

6.2. Definition. A box π_i in π is said to be *contained* in a box π_j (and π_j is said to *contain* π_i) if $\pi_i \prec \pi_j$. A chain $\Sigma \equiv \pi_s \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_n$ is called *linear* if $\pi_i \prec \pi_j$ for no $i \neq j$. We say that $\pi_i \rightarrow \pi_{i+1}$ is a *linear link* in Σ if π_i, π_{i+1} neither are contained in, nor contain, other boxes from Σ . \square

In our example the two-element chains are linear (as are *all* two-element chains), but the three-element chain is not (as $\pi_b \prec \pi_3$).

Observe that a chain defines, in the obvious way, a *path* through the derivation π ¹⁰.

6.3. Definition. A non-trivial chain $\Sigma \equiv \pi_s \rightarrow \pi_1 \rightarrow \dots \rightarrow \pi_n$ is called *strong* if all its reflecting cuts are distinct. It is called *adequate* if its induced path passes an instance of contraction always either by a side formula or by the *same* active formula. \square

Clearly, all two-element chains are strong. They are the *only* strong chains in our example. It is not true that all linear chains are strong.

Also it is quite obvious that strong chains always are adequate, while the converse is false. We will see in what follows that adequate chains (and therefore also strong chains) always are linear.

If we apply an elementary reduction step (of the procedure of cut elimination) to a derivation π and it turns out that in performing this step $\pi_1 \prec \pi$ is erased or duplicated, then π_1 is boxed in π . This is clear, as then π either has the form

$$\frac{\frac{\frac{\pi_1}{\vdots} \quad \frac{\Pi; \Delta \Rightarrow B}{\vdots}}{\vdots} \quad \frac{\Pi; \Delta, A \Rightarrow B}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots} \quad \text{or} \quad \frac{\frac{\frac{\pi_1}{\vdots} \quad \frac{\Pi; \Delta, A, A \Rightarrow B}{\vdots}}{\vdots} \quad \frac{\Pi; \Delta, A \Rightarrow B}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}$$

and reduces to respectively

$$\frac{\frac{\frac{\vdots}{\vdots}}{\vdots} \quad \frac{\Pi; \Delta \Rightarrow B}{\vdots}}{\vdots} \quad \text{and} \quad \frac{\frac{\frac{\frac{\pi_1}{\vdots} \quad \frac{\vdots}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}}{\vdots} \quad \frac{\frac{\frac{\vdots}{\vdots} \quad \frac{\Pi; \Delta, A, A \Rightarrow B}{\vdots}}{\vdots} \quad \frac{\Pi; \Delta, A \Rightarrow B}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}}{\vdots} \quad \frac{\vdots}{\vdots}$$

¹⁰The concepts of ‘chain’ and ‘path’ are similar to that of ‘trace’, which plays an important role in the study of proofnets, cf. Regnier(1992).

If $\pi_i \prec \pi$ is boxed and moreover is element of an *adequate chain* Σ in π , then we can show the converse, i.e. there exists a series of reductions (a *reduction strategy*) starting from π , that eventually either will erase or duplicate π_i . An important step towards a proof of this is the following

6.4. Proposition. *Let Σ be an adequate chain in π starting from a source \mathbf{s} (so $\Sigma \equiv \pi_s \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_n$ for some $n \geq 1$). Suppose $\pi_i \rightarrow \pi_{i+1}$ is a linear link in Σ . Then we can eliminate the reflecting cut c_i and decorate the reduct π' in such a way that we obtain an adequate chain $\Sigma' \equiv \pi_s \rightarrow \dots \rightarrow \pi_{i-1} \rightarrow \pi^* \rightarrow \pi_{i+2} \rightarrow \dots \rightarrow \pi_n$. If $\pi_i \equiv \pi_s$, then π_1 will be erased or duplicated, depending on whether the source is a weakening or a contraction.*

PROOF: To establish these claims one performs a long induction, considering all the possible configurations in which c_i can appear in π as in a proof of cut elimination. Here we will skip most of the details, and merely consider some important cases, e.g. indicating where the assumptions of linearity and adequacy are used.

- If $\pi_i \equiv \pi_s$, as basic cases we encounter precisely the two instances of erasure/duplication given above: the adequate chain $\pi_s \rightarrow \pi_1 \rightarrow \dots \rightarrow \pi_n$ becomes the adequate chain $\pi_{s'} \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_n$, where s' is a weakening or contraction (of/on a formula in Γ), depending on whether \mathbf{s} is a weakening or a contraction. (By linearity no boxes in Σ are contained in π_1 .)
- 1. Consider the following situation:

$$\begin{array}{c}
 \boxed{\begin{array}{c} \pi_{i+1} \\ ; \Gamma \Rightarrow A \end{array}} \quad \boxed{\begin{array}{c} \pi_i \\ ; \Delta_1, A \Rightarrow C \end{array}} \quad \begin{array}{c} D; \Delta_2 \Rightarrow B \\ \hline C \supset D; \Delta, A \Rightarrow B \end{array} \\
 \hline
 C \supset D; \Gamma, \Delta \Rightarrow B
 \end{array}$$

We reduce the reflecting cut as follows:

$$\frac{\boxed{\begin{array}{c} \pi^* \\ \hline ; \Gamma \Rightarrow A \quad ; \Delta_1, A \Rightarrow C \\ \hline ; \Gamma, \Delta_1 \Rightarrow C \end{array}} \quad D; \Delta_2 \Rightarrow B}{C \supset D; \Gamma, \Delta \Rightarrow B}$$

so π_i and π_{i+1} are ‘merged’, resulting in *one* box π^* .

2. Similarly, performing the reduction, we get a ‘merge’ in the following situation:

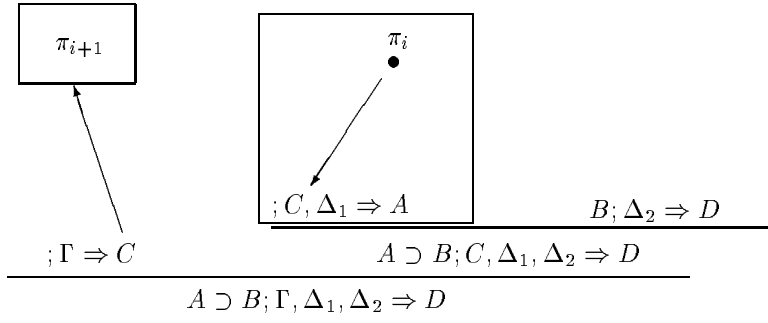
$$\frac{\boxed{\begin{array}{c} \pi_{i+1} \\ ; \Gamma \Rightarrow A \end{array}} \quad \boxed{\begin{array}{c} \pi_i \\ ; \Delta_1, A \Rightarrow C \end{array}} \quad \Pi; \Delta_2, C \Rightarrow B}{\Pi; \Delta, A \Rightarrow B}$$

as one may readily check.

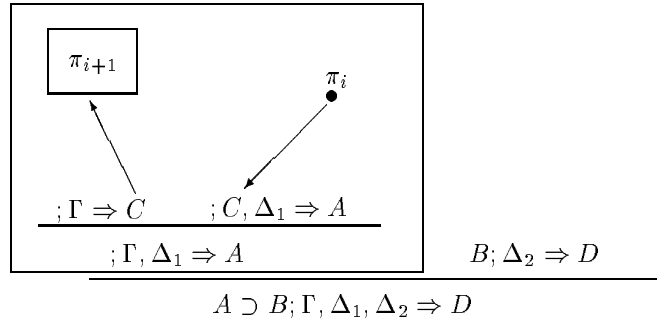
- In all other cases one can perform the reduction without effecting the adequate chain. However, one has to be careful, and both linearity of the link and adequacy of the chain are used.

In case of a duplication of a box in the chain due to a reduction of a cut on a contracted formula, we find, due to adequacy, a copy of the box via which the chain can pass unaffected, and still adequate. But there are other ‘traps’: let us consider two important cases in detail.

1. Consider the situation:

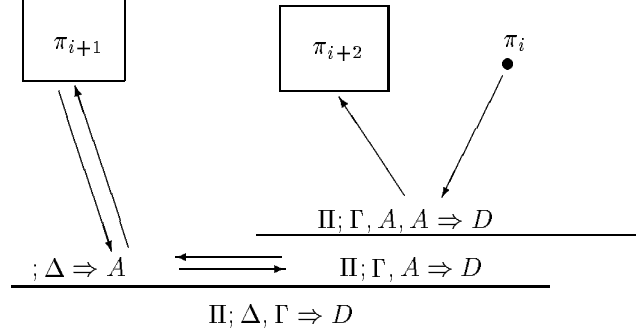


If the link is not linear, it might be the case that the subderivation of π with conclusion $;C, \Delta_1 \Rightarrow A$ is a box π_j in Σ , while at the same time we have no choice but to reduce the reflecting cut of π_i as follows:

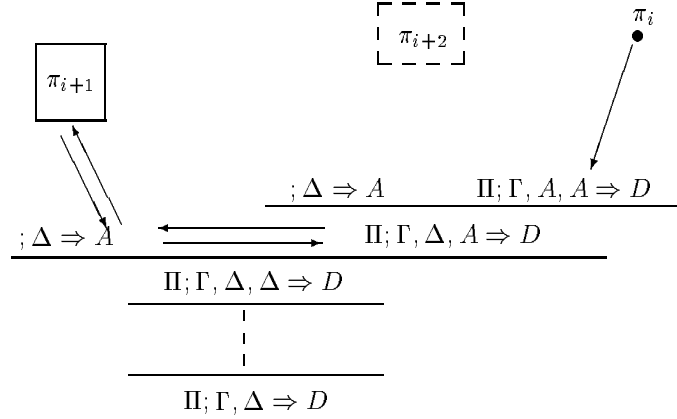


Decoration now gives us a box $\pi_{j'}$, with conclusion $; \Gamma, \Delta_1 \Rightarrow A$. Clearly $\pi_j \neq \pi_{j'}$. Our assumption avoids this difficulty.

2. Another possible configuration is:



If Σ is not adequate, then we might have $c_i \equiv c_{i+1}$ and an induced path γ that passes from the source via the *right* occurrence of A , but then continues from π_{i+1} to π_{i+2} via the *left* occurrence of A . After reduction, however, we get a derivation π' in which we can only pass via the right occurrence of A , and π_2 simply ceases to be a box:



□

6.5. Lemma. *Let $\dots \rightarrow \pi_i \rightarrow \pi_{i+1} \rightarrow \pi_{i+2} \rightarrow \dots$ be a subchain of three (consecutive) boxes in an adequate chain. Then neither of the three contains nor is contained in any of the others.*

PROOF: This is evident if $c_{i+1} \not\equiv c_i$. So let $c_{i+1} \equiv c_i$. Suppose π_i has conclusion $;\Gamma, B \Rightarrow A$ (with B as secondary shriek-source), while π_{i+1} has conclusion $;\Delta, C \Rightarrow B$ (with C as secondary shriek-source). Let $\phi(B)$ denote a formula ϕ having B as a subformula. Then by adequacy, for some ϕ , $C \multimap \phi(B)$ has to be a subformula of the cutformula, and C emerges in the derivation of the other

premiss of the reflecting cut as the succedent of the conclusion of a subderivation which is the left premiss of an application of implication-left with C and $\phi(B)$ as main formulas; π_i then has to be contained in the subderivation of the right premiss. \square

6.6. Proposition. *Adequate chains are linear.*

PROOF: Suppose $\Sigma \equiv \pi_s \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_n$ is an adequate *non-linear* chain. Then there is a smallest k such that π_k is contained in or contains a box π_i for some $i < k$. Consider $\Sigma^- \equiv \pi_s \rightarrow \dots \rightarrow \pi_i \rightarrow \dots \rightarrow \pi_k$. By lemma 6.5 we have that $k - i > 2$. Therefore there is a linear link between π_i and π_k . By proposition 6.4 we can eliminate its reflecting cut, and obtain an adequate chain with one box less between π_i and π_k . Iterating this eventually will contradict lemma 6.5. \square

We now have found sufficiently many properties of decorated **ILU**-derivations to prove the following

6.7. Theorem. *Let Σ be an adequate chain in π starting from a source \mathbf{s} , so $\Sigma \equiv \pi_s \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_n$ for some $n \geq 1$. If the source is a weakening, then there is a reduction strategy σ that will erase each of $\pi_1, \pi_2, \dots, \pi_n$ (precisely in that order); if the source is a contraction then there is a σ that will duplicate each of π_1, \dots, π_n (precisely in that order).*

PROOF: Let c_i be the reflecting cut for π_i . The strategy will consist in eliminating c_1, \dots, c_n precisely in that order. All the work in fact has been done in the proofs of propositions 6.6 and 6.4. \square

In the example on page 22, if we apply an elementary reduction step to the **ILU**-skeleton and then decorate the reduct π' , the result is:

$$\begin{array}{c}
 \frac{A; \Rightarrow A}{A; !A \xrightarrow{a} A} \\
 \frac{A; !A \xrightarrow{a} A}{A; \Rightarrow !AA} \\
 \frac{!A \xrightarrow{1} !(!AA) \quad A \Rightarrow A}{!(!AA)A; !A \Rightarrow A} \\
 \frac{!(!AA)A; !A \Rightarrow A}{!(!AA)A; \Rightarrow !AA} \\
 \frac{A; \Rightarrow A}{A; !(!AA) \xrightarrow{b'} A} \quad \frac{A; \Rightarrow A}{A; !(!AA) \xrightarrow{b} A} \quad \frac{!(!AA)A; !A \Rightarrow A}{!(!AA)A; \Rightarrow !AA} \\
 \frac{A; !(!AA) \xrightarrow{b'} A}{; A, !(!AA) \Rightarrow A} \quad \frac{A; !(!AA) \xrightarrow{b} A}{; A, !(!AA) \Rightarrow A} \quad \frac{!(!AA)A; !A \Rightarrow A}{; !(!AA)A; !(!AA)A \Rightarrow A} \\
 \frac{; A, !(!AA) \xrightarrow{3} !(!AA)A}{; !A \xrightarrow{3} !(!AA)A} \quad \frac{; A, !(!AA)A \Rightarrow A}{; !A, A \Rightarrow A} \quad \frac{; !A, A \Rightarrow A}{; !A \Rightarrow A}
 \end{array}$$

The duplication of π_3 in fact duplicates the pseudo-box π_b , and we see that our original chain $\pi_b \rightarrow \pi_1$ can be found in the reduct as chain $\pi_{b'} \rightarrow \pi_1$. The

remaining non-trivial chain, $\pi_b \rightarrow \pi_2 \rightarrow \pi_3$, is still there. Note that all non-trivial chains in π' are *strong*, so by 6.7 we have, for each of the boxes in π' (and a fortiori for each of the boxes in π) a reduction strategy that eventually will erase or duplicate it. However, this is not a general property: *the problematic situations indicated in our sketch of the proof of proposition 6.4 can be built into concrete examples of derivations with boxed subderivations, that, whatever strategy of reduction applied, neither are erased nor duplicated, but e.g. simply are ‘un-boxed’.*

This is always caused by the identification of formulas in *contractions*: the problematic situations do not occur in contraction-free derivations, i.e. in *affine* implicational logic. As trivially in contraction-free derivations all chains are adequate we find as a corollary to 6.7:

6.8. Proposition. *Let π be a contraction-free derivation, and Σ a chain $\pi_s \rightarrow \pi_1 \rightarrow \dots \rightarrow \pi_n$, with $n \geq 1$. Then there is a reduction strategy that will erase each of π_1, \dots, π_n (precisely in that order). \square*

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