Probabilistic coherence spaces as a model of higher-order probabilistic computation

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Abstract

We study a probabilistic version of coherence spaces and show that these objects provide a model of Linear Logic. We build a model of the pure lambda-calculus in this setting and show how to interpret a probabilistic version of the functional language PCF. We give a probabilistic interpretation of the semantics of probabilistic PCF closed terms of ground type. Last we suggest a generalization of this approach, using Banach spaces.

Introduction

There are various motivations for introducing probabilistic features in the denotational semantics of programming languages, such as giving a systematic account of program execution in an environment subject to random evolution or providing a denotational understanding of randomized algorithms.

For designing such models, two main directions have been explored so far.

- In the "standard" domain-theoretic approach, the idea has been to define a probabilistic analogue of the power domain constructions previously introduced in [Plo76] for interpreting non-deterministic languages. Such a *probabilistic power domain* construction has been first considered by Saheb-Djahromi [SD80] and further studied by Jones and Plotkin in [JP89] where it is used as a computational monad in the sense of Moggi [Mog89]. In this setting, one associates a domain with each type, and a program from type A to type B is interpreted as a continuous function f from the domain X associated with A to the probabilistic power domain of the domain Y associated with B. The intuition is clear: f maps any value of X to a (sub-)probability distribution (or, more generally, a (sub-)probability measure) describing the probability of obtaining a given result in Y. Composing such maps and interpreting programming constructs is possible, thanks to the additional structure of the power domain functor (as already mentioned, it is a computational monad).
- In the game-theoretic framework, a probabilistic version of Hyland-Ong [HO00] and Nickau [Nic94] game semantics has been introduced by the first author and Harmer in [DH00]. The low-level description[1] of interactions provided by games allows indeed to view *probabilistic strategies* interpreting probabilistic programs of a given type A as stochastic processes on the plays of the game associated with A. This probabilistic intuition is perfectly compatible with the standard game interpretation and its non-deterministic version developed in [HM99], and the factorization and full abstraction properties of deterministic and non-deterministic game models have been successfully extended to this probabilistic setting.

There is however another tradition in the denotational semantics of functional programming languages and logical systems, dating back to the *coherence space* model introduced by Girard in [Gir86, Gir87],

[1] The very idea of game semantics is to give an account of execution at all types in terms of ground type elementary interactions, just as compilation consists in transforming an abstract program into a sequence of basic operations acting on elementary tokens. This operational viewpoint on games is illustrated in [DHR96].

and similar models such as *hypercoherences*, developed by the second author [Ehr93], or Loader's *totality spaces* [Loa94]. The object interpreting a type A in these models can often be seen as a domain whose elements are certain subsets (the *cliques*) of a given set (the *web*) associated with the type, these cliques being ordered under inclusion. This web is usually endowed with an additional structure (a binary graph structure for coherence spaces, a hypergraph structure for hypercoherences...) which is used for defining which subsets of the web are cliques.

These *web-based* models provide interpretations of functional languages and intuitionistic logic proofs, of course, but also of Linear Logic [Gir87]. Though much less successful than game models in terms of full completeness, they have been powerful tools for discovering new syntaxes: coherence spaces played an essential role in the discovery of Linear Logic.

Adding numerical coefficients to such webbed objects by replacing subsets (cliques) by scalar valued functions defined on the web is a natural step to take... and it has been taken by Girard even before he discovered qualitative and coherence spaces. In [Gir88], he interpreted each type A as a set (a web), and each closed program of type A as a map from that web to sets, to be understood as possibly infinite numerical coefficients. Endowing these webs with an additional structure, it is possible to keep these coefficients finite, as shown in [Ehr02, Ehr05]. The principle of these latter constructions is pervasive in linear logic: everything is defined in terms of a fundamental *linear duality*. For instance, for defining real Köthe spaces, given a set I (a web), one says that $x \in \mathbb{R}^I$ and $x' \in \mathbb{R}^I$ are in duality if $\sum_{i\in I} |x_i x'_i| < \infty$. A Köthe space of web I is a set of elements of $\mathbb{R}^I$ which is equal to its bidual.

As briefly explained in [Gir04], it is quite natural to give a probabilistic flavour to the definition above by slightly modifying the duality. Since probabilities are non-negative numbers it is reasonable to restrict to x's belonging to $(\mathbb{R}^+)^I$, and to say that $x \in (\mathbb{R}^+)^I$ and $x' \in (\mathbb{R}^+)^I$ are in duality if $\sum_{i\in I} x_i x'_i \leq 1$. This appears as a natural "fuzzy" generalization of coherence spaces, if one keeps in mind that a coherence space of web I can equivalently be defined as a set of subsets of I which is equal to its bidual for the following notion of duality: $u \subseteq I$ and $u' \subseteq I$ are in duality if $u \cap u'$ has at most one element. Therefore, these new objects are called *probabilistic coherence spaces* (PCSs for short). The multiplicative ($\otimes$ and $\parr$) and additive ($\oplus$ and $\&$) constructions on PCSs are presented in [Gir04]. We show that PCSs, with suitably defined morphisms, provide a model of full classical linear logic, and hence a cartesian closed category (the Kleisli category of the exponential comonad).

Although the definitions of PCSs and of Köthe spaces are formally similar, we shall see that the two notions have quite different properties, in particular:

- Just as in the power domain and game approaches, each PCS can be seen as a continuous domain, and morphisms in the cartesian closed category of PCSs are Scott-continuous and admit therefore fixpoints. So, general recursion can be interpreted in PCSs, whereas this is impossible in the CCC of Köthe spaces.

- On the other hand, the *cocontraction* rule of differential linear logic [Ehr02, ER06] can be interpreted in Köthe spaces whereas it cannot in PCSs.

It is therefore possible to interpret a probabilistic version PPCF of PCF [Plo77], where the language is extended with a programming primitive, which randomly yields a non-negative integer with a prescribed probability distribution. For this purpose, the ground type of integers is interpreted as the set of natural numbers, together with all the families $x \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} x_n \leq 1$, that is, all sub-probability distributions on $\mathbb{N}$. This is, *mutatis mutandis*, the same interpretation of natural numbers as in the probabilistic game model of [DH00]. But, in sharp contrast with that probabilistic game interpretation, in PCSs, the simple probabilistic intuition is lost at higher types[2]: the families of non-negative real numbers interpreting terms at higher types are no more sub-probability distributions in general.

We choose the leftmost-outermost reduction strategy as operational semantics for PPCF. We show first that, in a precise sense, the semantics of terms is invariant under reduction. This result could easily be generalized to arbitrary reductions (not only those of our strategy), with the proviso however that the probabilistic reduction rule should be applied only when the probabilistic redex stands in linear position

[2]This phenomenon is not new. For instance, in the hyper coherence model of [Ehr93], strongly stable functions between products of ground types are sequential, whereas at higher types, there is no simple interpretation of strong stability in terms of sequentiality. On the other hand, in game-theoretic models, such as the sequential algorithms model of Berry and Curien [BC82], the sequentiality intuitions are preserved at all types, but these models are "less abstract" in the sense that they keep more "intentional" information about interpreted programs.

(typically in head position). This is quite different from the situation in other probabilistic lambda-calculi, such as the one considered in [DPHW05], where the probabilistic reduction can be performed at any place in a term.

Next, we show that PCSs have nevertheless a clear probabilistic meaning. We prove that the interpretation of a closed PPCF term M of type integer is the sub-probability distribution on the integers mapping n to the probability that M reduces to $\underline{n}$ (the integer n of the language), in our leftmost outermost strategy, presented in a small step way, as a stochastic matrix on terms, that is, as a Markov process[3]. The proof is an adaptation of Plotkin's logical relation proof of adequacy for the Scott's semantics of PCF in [Plo77]. It is certainly an exciting challenge to try to understand the probabilistic meaning of PCSs at higher types. One could probably address this issue by defining a logical relation between the probabilistic game model and the present PCS model, but this is postponed to future work.

Introducing a notion of *substructure* for PCSs, a very restrictive notion of morphisms for which PCSs are closed under directed colimits, and showing that the logical constructions are continuous wrt. these colimits, we show that all types admit least fixpoints. In particular, we exhibit a PCS structure on a relational model of the pure lambda-calculus that the second author recently introduced with Bucciarelli and Manzonetto [BEM07, BEM09].

Last, we suggest an *intrinsic*[4] version of this semantics, associating a Banach space with any PCS, and showing that PCS morphisms give rise to linear and continuous maps between the associated Banach spaces. This defines a functor from the category of PCSs to the category of *coherent Banach spaces* of [Gir99][5]. We show however that this functor is not full, and propose to consider *ordered Banach spaces* (an ordered Banach space is a Banach space together with a positive cone thereof) as a possible intrinsic version of PCSs. These objects indeed combine the algebraic and topological features of Banach spaces with the order-theoretic features of cpos, but the corresponding theoretical investigations are postponed to further work.

Notations

We use $\mathbb{N}$ for the set of non-negative integers, $\mathbb{N}^+$ for the set of positive integers ($\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$). If A is a set, $\mathcal{M}_{\mathrm{fin}}(A)$ denotes the set of finite multisets on A. We use $[a_1, \dots, a_n]$ for the multiset whose elements are $a_1, \dots, a_n$, taking multiplicities into account and we use $m + m'$ for the disjoint union of multisets m and m'. We denote by $\delta_{a,b}$ the Kronecker symbol, whose value is 1 if $a = b$, and 0 otherwise.

We extend the ordinary operations and notations on real numbers to families of real numbers, pointwise. For instance, if $x \in \mathbb{R}^A$, we use $|x|$ for the family $(|x_a|)_{a\in A}$ of absolute values.

We denote by $\mathbb{R}^+$ the set of non-negative real numbers, and by $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$ the completed non-negative real half-line, which is a rig (a rig, or semi-ring, is an algebraic structure defined like a ring apart that it is not required that each element x has an additive inverse $-x$). Remember that, in that rig, $0 \times \infty = 0$.

Contents

[3]Here again, our choice of strategy is certainly not essential, and any left reduction would probably lead to the same result, but this would require introducing a non-deterministic stochastic reduction matrix: at each state (term) we would have as many probability distributions for the next state as available left redexes, these choices being essentially irrelevant by the Church-Rosser property, which certainly holds for a version of the calculus extended with convex linear combinations, as in [Vau07]. In spite of the irrelevance of this non-determinism, this would certainly make the whole story much more involved.

[4]That is, where objects are not defined in terms of a web, which, from an algebraic viewpoint, can be seen as an arbitrary choice of basis in a vector space.

[5]These objects, which are triples (V, V', v) where V and V' are Banach spaces and v is a continuous bilinear form on $V \times V'$, can be seen as particular instances of the triples introduced in the *Chu construction* [Bar79].

1 Probabilistic coherence spaces

1.1 General definitions

1.1.1 Basic duality. Let A be a set. We denote by e_a the element of $(\mathbb{R}^+)^A$ defined by $(e_a)_{a'} = \delta_{a,a'}$. Given $a \in A$ and $x \in (\mathbb{R}^+)^A$, we use x_a to denote $x(a) \in \mathbb{R}^+$. We define $\pi^a : (\mathbb{R}^+)^A \to \mathbb{R}^+$ by $\pi^a(x) = x(a)$.

If $x, x' \subseteq (\mathbb{R}^+)^A$, we set

$$\langle x, x' \rangle = \sum_{a \in A} x_a x'_a \in \overline{\mathbb{R}^+}\,.$$

If $P \subseteq (\mathbb{R}^+)^A$, let

$$P^\perp = \{x' \in (\mathbb{R}^+)^A \mid \forall x \in P\ \langle x, x'\rangle \leq 1\}\,.$$

This set could be called the *polar* of P.

One checks easily that $P \subseteq Q \Rightarrow Q^\perp \subseteq P^\perp$ and that $P \subseteq P^{\perp\perp}$. Therefore $P^{\perp\perp\perp} = P^\perp$.

Let $Q \subseteq (\mathbb{R}^+)^A$ and let $P = Q^\perp$. Observe first that

$$\forall x \in P\, \forall y \in (\mathbb{R}^+)^A \quad y \leq x \Rightarrow y \in P\,.$$

Given $a \in A$, the set $\pi^a(P) \subseteq \mathbb{R}^+$ is therefore an initial segment of the non-negative real half-line, and we define $\mathsf{c}_P(a) \in \overline{\mathbb{R}^+}$ as the lub of $\pi^a(P)$. For any $\lambda \in \mathbb{R}^+$ such that $\lambda < \mathsf{c}_P(a)$, one can find $x \in P$ such that $x_a = \lambda$, and hence $\lambda e_a \in P$ (since $\lambda e_a \leq x$). It follows that $\mathsf{c}_P(a)e_a \in P$. Indeed, for any $x' \in Q$, and any $\lambda < \mathsf{c}_P(a)$ we have $1 \geq \langle \lambda e_a, x'\rangle = \lambda x'_a$ and therefore $\mathsf{c}_P(a)x'_a \leq 1$. Moreover, by definition of $\mathsf{c}_P(a)$, we must have $\lambda \leq \mathsf{c}_P(a)$ for any λ such that $\lambda e_a \in P$. We have shown that

$$\mathsf{c}_P(a) = \sup\{\lambda > 0 \mid \lambda e_a \in P\} \quad \text{and} \quad \mathsf{c}_P(a)e_a \in P\,.$$

1.1.2 Probabilistic coherence spaces.

A *probabilistic coherence space* (PCS) is a pair $X = (|X|, \mathsf{P}X)$ where $|X|$ is a countable set and $\mathsf{P}X \subseteq (\mathbb{R}^+)^{|X|}$ is such that

1. $\mathsf{P}X^{\perp\perp} \subseteq \mathsf{P}X$ (that is, $\mathsf{P}X^{\perp\perp} = \mathsf{P}X$)
2. $\forall a \in |X| \exists \lambda > 0\ \lambda e_a \in \mathsf{P}X$, that is $\mathsf{c}_X(a) > 0$ (where we set $\mathsf{c}_X(a) = \mathsf{c}_{\mathsf{P}X}(a)$)
3. and $\forall a \in |X|$, the set $\pi^a(\mathsf{P}X) \subseteq \mathbb{R}^+$ is bounded, that is $\mathsf{c}_X(a) < \infty$.

Remark: We do not require $\mathsf{P}X \subseteq [0,1]^{|X|}$, which might seem a desirable (or at least intuitively appealing) condition. We shall understand why when the exponentials will come in. There is *a priori* no direct probabilistic interpretation of an element of $\mathsf{P}X$ (as a discrete sub-probability measure). It is only the evaluation of $\langle x, x'\rangle$ which yields a probability, to be understood as the probability of success of the interaction between x and x'. In that sense, the model is really probabilistic, though a direct probabilistic interpretation of types is available only at ground types (booleans, natural numbers), see 3.1.

Conditions (2) and (3) are there for keeping finite all the real numbers involved; they are not explicitly stated in the definition of PCSs in [Gir04].

Lemma 1 *If X is a PCS, then $X^\perp = (|X|, \mathsf{P}X^\perp)$ is also a PCS and $\mathsf{c}_{X^\perp}(a) = \mathsf{c}_X(a)^{-1}$, for any $a \in |X|$.*

Proof. We only have to prove conditions (2) and (3) for $X^\perp$, which will follow if we show that $\mathsf{c}_{X^\perp}(a) = \mathsf{c}_X(a)^{-1}$. We have $\mathsf{c}_X(a) \in \mathsf{P}X$ and $\mathsf{c}_{X^\perp}(a) \in \mathsf{P}X^\perp$, hence $\mathsf{c}_X(a)\mathsf{c}_{X^\perp}(a) \leq 1$, that is $\mathsf{c}_{X^\perp}(a) \leq \mathsf{c}_X(a)^{-1}$. Moreover, for any $x' \in \mathsf{P}X^\perp$, we have $x'_a \leq \mathsf{c}_{X^\perp}(a)$, that is $x'_a\mathsf{c}_{X^\perp}(a)^{-1} \leq 1$, hence $\mathsf{c}_{X^\perp}(a)^{-1}e_a \in \mathsf{P}X$. Therefore $\mathsf{c}_{X^\perp}(a)^{-1} \leq \mathsf{c}_X(a)$. □

We define the *norm* of $x \in \mathsf{P}X$ as

$$\|x\|_X = \sup_{x' \in \mathsf{P}X^\perp} \langle x, x'\rangle$$

so that $\|x\|_X \in [0,1]$, see Section 4 for more details.

1.2 Morphisms of PCSs

1.2.1 Tensor product.

Let X and Y be PCSs. If $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$, we define $x \otimes y \in (\mathbb{R}^+)^{|X| \times |Y|}$ by $(x \otimes y)_{a,b} = x_a y_b$. Let

- $|X \otimes Y| = |X| \times |Y|$
- and $\mathsf{P}(X \otimes Y) = \{x \otimes y \mid x \in \mathsf{P}X \text{ and } y \in \mathsf{P}Y\}^{\perp\perp}$.

Then $X \otimes Y = (|X \otimes Y|, \mathsf{P}(X \otimes Y))$ is a PCS. Condition (1) is obvious, because $\mathsf{P}(X \otimes Y)$ is of the shape $P^\perp$. Conditions (2) and (3) will follow from the fact that

$$\mathsf{c}_{X \otimes Y}(a, b) = \mathsf{c}_X(a)\mathsf{c}_Y(b)\,.$$

Since $\mathsf{c}_X(a)e_a \in \mathsf{P}X$ and $\mathsf{c}_Y(b)e_b \in \mathsf{P}Y$, we have $(\mathsf{c}_X(a)e_a) \otimes (\mathsf{c}_Y(b)e_b) = \mathsf{c}_X(a)\mathsf{c}_Y(b)e_{a,b} \in \mathsf{P}(X \otimes Y)$, therefore $\mathsf{c}_X(a)\mathsf{c}_Y(b) \leq \mathsf{c}_{X \otimes Y} a, b$.

On the other hand, given $x' \in \mathsf{P}X^\perp$ and $y' \in \mathsf{P}Y^\perp$, and $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$, one checks easily that

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle \leq 1$$

and hence $x' \otimes y' \in \mathsf{P}(X \otimes Y)^\perp$ (this means that the MIX rule holds in our model). In particular, we have $\mathsf{c}_X(a)^{-1}\mathsf{c}_Y(b)^{-1}e_{a,b} \in \mathsf{P}(X \otimes Y)^\perp$ and hence $\forall z \in \mathsf{P}(X \otimes Y)\ z_{a,b} \leq \mathsf{c}_X(a)\mathsf{c}_Y(b)$, that is $\mathsf{c}_{X \otimes Y}(a, b) \leq \mathsf{c}_X(a)\mathsf{c}_Y(b)$.

1.2.2 The PCS of morphisms.

Let $X \multimap Y = (X \otimes Y^\perp)^\perp$. A morphism u from X to Y is, by definition, an element of $\mathsf{P}(X \multimap Y)$. So u can be seen as a matrix with $|Y|$ lines and $|X|$ columns (since $u \in (\mathbb{R}^+)^{|X| \times |Y|}$).

Given any matrix $u \in (\mathbb{R}^+)^{|X| \times |Y|}$, we can define the map $(\mathbb{R}^+)^{|X|} \to \overline{\mathbb{R}^+}^{|Y|}$

$$\begin{aligned} \mathsf{fun}(u) : (\mathbb{R}^+)^{|X|} &\to \overline{\mathbb{R}^+}^{|Y|} \\ x &\mapsto u \cdot x = \left(\sum_{a \in |X|} u_{a,b} x_a \right)_{b \in |Y|} \end{aligned}$$

Also, we define the transpose ${}^{\mathsf{t}}u \in (\mathbb{R}^+)^{|Y| \times |X|}$ of such a matrix u in the usual way: $({}^{\mathsf{t}}u)_{b,a} = u_{a,b}$.

Lemma 2 *Let $u \in (\mathbb{R}^+)^{|X| \times |Y|}$, $x \in \mathsf{P}X$ and $y' \in \mathsf{P}Y^\perp$. Then*

$$\langle u, x \otimes y' \rangle = \langle u \cdot x, y' \rangle = \langle {}^{\mathsf{t}}u \cdot y', x \rangle = \langle {}^{\mathsf{t}}u, y' \otimes x \rangle\,.$$

Straightforward computation in the rig $\overline{\mathbb{R}^+}$.

Lemma 3 *Let $u \in \overline{\mathbb{R}^+}^{|X| \times |Y|}$. The following conditions are equivalent.*

1. $u \in \mathsf{P}(X \multimap Y)$,
2. ${}^{\mathsf{t}}u \in \mathsf{P}(Y^\perp \multimap X^\perp)$,
3. $\forall x \in \mathsf{P}X\ u \cdot x \in \mathsf{P}Y$
4. *and* $\forall y' \in \mathsf{P}X^\perp\ {}^{\mathsf{t}}u \cdot y' \in \mathsf{P}X^\perp$

Proof. The proof is essentially a direct application of Lemma 2. For instance, let us prove that (1)⇒(3). With the notations of the lemma, we must show that $u \cdot x \in \mathsf{P}Y = \mathsf{P}Y^{\perp\perp}$. So let $y' \in \mathsf{P}Y^\perp$. We have $\langle u \cdot x, y' \rangle = \langle u, x \otimes y' \rangle \in [0, 1]$ by assumption.

Let us check also that (3)⇒(1). One must show first that $u \in (\mathbb{R}^+)^{|X| \times |Y|}$, that is, $u_{a,b} < \infty$ for each a, b. So let $a \in |X|$ and $b \in |Y|$ and let $\lambda > 0$ be such that $\lambda e_a \in \mathsf{P}X$. One has $u \cdot \lambda e_a \in \mathsf{P}Y$. Let $\mu > 0$ be such that $\mu e_b \in \mathsf{P}Y^\perp$, we have $\langle u \cdot \lambda e_a, \mu e_b \rangle \in [0, 1]$, that is $\lambda \mu u_{a,b} \in [0, 1]$, so $u_{a,b} \in \mathbb{R}^+$. Next, let $x \in \mathsf{P}X$ and $y' \in \mathsf{P}Y^\perp$. We have $\langle u, x \otimes y' \rangle = \langle u \cdot x, y' \rangle$ by Lemma 2 again, and so $\langle u, x \otimes y' \rangle \in [0, 1]$ since $u \cdot x \in \mathsf{P}Y$. □

1.2.3 Identity, composition and isomorphisms.

The identity matrix $\mathrm{Id} \in (\mathbb{R}^+)^{|X|\times|X|}$ defined by $\mathrm{Id}_{a,a'} = \delta_{a,a'}$ belongs to $\mathsf{P}(X \multimap X)$. If $u \in \mathsf{P}(X \multimap Y)$ and $v \in \mathsf{P}(Y \multimap Z)$, we define $vu \in \overline{\mathbb{R}^+}^{|X|\times|Z|}$ as usual by

$$(vu)_{a,c} = \sum_{b\in|Y|} v_{b,c} u_{a,b}$$

Given $x \in \mathsf{P}X$, we have $vu \cdot x = v \cdot (u \cdot x)$. But $u \cdot x \in \mathsf{P}Y$ since $u \in \mathsf{P}(X \multimap Y)$ and so $v \cdot (u \cdot x) \in \mathsf{P}Z$ since $v \in \mathsf{P}(Y \multimap Z)$. This shows that $vu \in \mathsf{P}(X \multimap Z)$ by Lemma 3.

Lemma 4 *Let X and Y be PCSs. We have $\mathsf{P}(X \otimes Y)^\perp = \mathsf{P}(X \multimap Y^\perp)$.*

Immediate consequence of Lemma 3.

Let **Pcoh** be the category whose objects are the PCSs and where $\mathbf{Pcoh}(X,Y) = \mathsf{P}(X \multimap Y)$, identities and morphism composition being defined in the above matricial way.

As in any category, we have a canonical notion of isomorphism. Among these isomorphisms, some of them will be quite important, we call them *web-isomorphisms*. A web-isomorphism from X to Y is an isomorphism $f \in \mathbf{Pcoh}(X,Y)$ such that there is a (obviously unique) bijection $\varphi : |X| \to |Y|$ such that $f_{a,b} = \delta_{\varphi(a),b}$. In other words, the underlying bijection φ has the following property: for any $y \in (\mathbb{R}^+)^{|Y|}$, one has $y \in \mathsf{P}Y$ iff $(y_{\varphi(a)})_{a\in|X|} \in \mathsf{P}X$.

Of course, if $f \in \mathbf{Pcoh}(X,Y)$ is a web-isomorphism, then ${}^{\mathrm{t}}f \in \mathbf{Pcoh}(Y^\perp, X^\perp)$ is a web-isomorphism.

Remark: We conjecture that all isomorphisms in **Pcoh** are web-isomorphisms, but this property does not play any role in the present article, so this question is postponed to further studies.

1.3 Order-theoretic considerations.

Let X be a PCS. It will be useful to consider $\overline{\mathbb{R}^+}^{|X|}$ as a partially ordered set, with the usual pointwise order: $x \le y$ if $x_a \le y_a$ for all $a \in |X|$. The main property of $\mathsf{P}X$ from this viewpoint is the following.

Proposition 5 *$\mathsf{P}X$ is a bounded-complete and ω-continuous cpo and, for any $x' \in \mathsf{P}X^\perp$, the map $x \mapsto \langle x, x'\rangle$ is Scott-continuous from $\mathsf{P}X$ to $[0,1]$.*

Proof. We prove first that $\mathsf{P}X$ is a cpo. Let $D \subseteq \mathsf{P}X$ be directed. The pointwise lub $y = \sup D$ belongs to $(\mathbb{R}^+)^{|X|}$ since all sets $\pi^a(\mathsf{P}X)$ are bounded. We show that $y \in \mathsf{P}X$, so let $x' \in \mathsf{P}X^\perp$. It is clear that

$$\sup_{x\in D} \langle x, x'\rangle \le \langle y, x'\rangle$$

so let us prove the converse inequation. Assume, towards a contradiction, that $\sup_{x\in D}\langle x, x'\rangle < \langle y, x'\rangle$. Let $\lambda \in \mathbb{R}^+$ be such that $\sup_{x\in D}\langle x, x'\rangle < \lambda < \langle y, x'\rangle$. We can find a finite subset I of $|X|$ such that $\sum_{a\in I} y_a x'_a > \lambda$. But since I is finite, we have $\sum_{a\in I} y_a x'_a = \sup_{x\in D} \sum_{a\in I} x_a x'_a \le \sup_{x\in D}\langle x, x'\rangle < \lambda$ (by continuity of addition and multiplication on the real numbers); contradiction. This shows also that the map $x \mapsto \langle x, x'\rangle$ is Scott-continuous.

Let $x, y \in \mathsf{P}X$. Remember that y is way below x (written $y \ll x$) if, for any directed subset D of $\mathsf{P}X$ such that $x \le \sup D$, there exist $y' \in D$ such that $y \le y'$.

Given $x, y, z \in \mathsf{P}X$ such that $x, y \le z$, defining $\max(x,y) \in (\mathbb{R}^+)^{|X|}$ by $\max(x,y)_a = \max(x_a, y_a)$, we have $\max(x,y) \le z$ and hence $\max(x,y) \in \mathsf{P}X$, so $\mathsf{P}X$ is bounded-complete. It is easy to prove that $x, y \ll z \Rightarrow \max(x,y) \ll z$.

Last we observe that there is a countable set $B \subseteq \mathsf{P}X$ such that, for any $x \in \mathsf{P}X$, the set $\{y \in B \mid y \ll x\}$ is directed and has x as lub. It suffices to take for B the elements of $\mathsf{P}X$ which have a finite support and take rational values. Indeed, for any $r \in \mathbb{Q}$ such that $0 \le r < x_a$, one has $re_a \ll x$ and $x = \sup\{re_a \mid a \in |X| \text{ and } r \in \mathbb{Q} \cap [0, x_a)\}$. □

Proposition 6 *Let $u \in \mathsf{P}(X \multimap Y)$. Then the function $\mathsf{fun}(u)$ is a Scott-continuous function from $\mathsf{P}X$ to $\mathsf{P}Y$.*

Proof. Given a directed set $D \subseteq \mathsf{P}X$, we must show that $u \cdot \sup D = \sup_{x\in D}(u \cdot x)$. So let $b \in |Y|$, we have to show that $(u \cdot \sup D)_b = \sup_{x\in D}(u \cdot x)_b$. Let $\mu > 0$ be such that $\mu e_b \in \mathsf{P}Y^\perp$. We have

$$\begin{aligned}
\mu(u \cdot \sup D)_b &= \langle u \cdot \sup D, \mu e_b\rangle \\
&= \langle \sup D, {}^{\mathrm{t}}u \cdot (\mu e_b)\rangle \\
&= \sup_{x\in D}\langle x, {}^{\mathrm{t}}u \cdot (\mu e_b)\rangle \quad \text{by Proposition 5} \\
&= \mu \sup_{x\in D}\langle u \cdot x, e_b\rangle
\end{aligned}$$

and this concludes the proof. □

1.4 Tensor product

1.4.1 Preliminary properties. We have already defined the PCS $X \otimes Y$ in 1.2.1. The next preliminary lemmas will be quite useful. They are of an algebraic nature and will be used for exhibiting the monoidal structure and the properties of the category **Pcoh**. These computations have to be done before proving these categorical properties, and it is not clear to us how they could be expressed in a more abstract, categorical way.

Lemma 7 *Let X, Y and Z be PCSs. The matrix $\alpha \in (\mathbb{R}^+)^{|((X\otimes Y)\multimap Z)\multimap(X\multimap(Y\multimap Z))|}$ defined by $\alpha_{((a,b),c),(a',(b',c'))} = \delta_{a,a'}\delta_{b,b'}\delta_{c,c'}$ is a web-isomorphism from $(X \otimes Y) \multimap Z$ to $X \multimap (Y \multimap Z)$.*

Proof. Let $w \in \mathsf{P}((X \otimes Y) \multimap Z)$. We prove that $\alpha \cdot w \in \mathsf{P}(X \multimap (Y \multimap Z))$. Let $x \in \mathsf{P}X$, we have to show that $(\alpha \cdot w) \cdot x \in \mathsf{P}(Y \multimap Z)$. But this is clear since, for any $y \in \mathsf{P}Y$, one has $((\alpha \cdot w) \cdot x) \cdot y = w \cdot (x \otimes y)$.

Conversely, let $w \in \mathsf{P}(X \multimap (Y \multimap Z))$ and let β be the transpose of the matrix α. We must show that $\beta \cdot w \in \mathsf{P}((X \otimes Y) \multimap Z)$, that is ${}^{\mathrm{t}}(\beta \cdot w) \in \mathsf{P}(Z^\perp \multimap (X \multimap Y^\perp))$. So let $z' \in \mathsf{P}Z^\perp$, $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$. We have

$$\langle({}^{\mathrm{t}}(\beta \cdot w) \cdot z') \cdot x, y\rangle = \langle(w \cdot x) \cdot y, z'\rangle$$

as shown by an easy computation, and we conclude since, by assumption, $\langle(w \cdot x) \cdot y, z'\rangle \in [0, 1]$. □

Lemma 8 *Let $w \in (\mathbb{R}^+)^{|(X\otimes Y)\multimap Z|}$. Then $w \in \mathsf{P}((X \otimes Y) \multimap Z)$ iff*

$$\forall x \in \mathsf{P}X \forall y \in \mathsf{P}Y \quad w \cdot (x \otimes y) \in \mathsf{P}Z\,.$$

Proof. Assume that $w \cdot (x \otimes y)$ for each $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$. By Lemma 7, for proving that $w \in \mathsf{P}((X \otimes Y) \multimap Z)$, it suffices to show that $\alpha \cdot w \in \mathsf{P}(X \multimap (Y \multimap Z))$. But this is clear since

$$\forall x \in \mathsf{P}X \forall y \in \mathsf{P}Y \quad ((\alpha \cdot w) \cdot x) \cdot y = w \cdot (x \otimes y) \in \mathsf{P}Z\,.$$

□

1.4.2 The tensor product as a functor. Let $u \in \mathbf{Pcoh}(X_1, X_2)$ and $v \in \mathbf{Pcoh}(Y_1, Y_2)$, we define $u \otimes v \in (\mathbb{R}^+)^{|(X_1\otimes Y_1)\multimap(X_2\otimes Y_2)|}$ by $(u \otimes v)_{(a_1,b_1),(a_2,b_2)} = u_{a_1,a_2} v_{b_1,b_2} \in \mathbb{R}^+$.

Let $x_1 \in \mathsf{P}X_1$ and $y_1 \in \mathsf{P}Y_1$, we have $u \cdot x_1 \in \mathsf{P}X_2$ and $v \cdot y_1 \in \mathsf{P}Y_2$, hence $(u \cdot x_1) \otimes (v \cdot y_1) \in \mathsf{P}(X_2 \otimes Y_2)$. But $(u \otimes v) \cdot (x_1 \otimes y_1) = (u \cdot x_1) \otimes (v \cdot y_1)$, so $(u \otimes v) \cdot (x_1 \otimes y_1) \in \mathsf{P}(X_2 \otimes Y_2)$. Therefore, by Lemma 8, one has $u \otimes v \in \mathsf{P}((X_1 \otimes Y_1) \multimap (X_2 \otimes Y_2))$.

A standard computation shows that, if $u_1 \in \mathsf{P}(X_1 \multimap X_2)$, $u_2 \in \mathsf{P}(X_2 \multimap X_3)$, $v_1 \in \mathsf{P}(Y_1 \multimap Y_2)$, and $v_2 \in \mathsf{P}(Y_2 \multimap Y_3)$, then

$$(u_2 u_1) \otimes (v_2 v_1) = (u_2 \otimes v_2)(u_1 \otimes v_1)\,.$$

One has also $\mathrm{Id}_X \otimes \mathrm{Id}_Y = \mathrm{Id}_{X\otimes Y}$ and so $\otimes$ is a functor. Moreover, if $f \in \mathbf{Pcoh}(X_1, X_2)$ and $g \in \mathbf{Pcoh}(Y_1, Y_2)$ are web-isomorphisms, then $f \otimes g : \mathbf{Pcoh}(X_1 \otimes Y_1, X_2 \otimes Y_2)$ is a web-isomorphism.

1.4.3 Pcoh as a monoidal category.

This endows the category **Pcoh** with a monoidal structure. The neutral object is $1 = (\{*\}, [0,1])$ (identifying $(\mathbb{R}^+)^{\{*\}}$ with $\mathbb{R}^+$).

Let $\alpha \in (\mathbb{R}^+)^{|(X\otimes(Y\otimes Z))\multimap((X\otimes Y)\otimes Z)|}$ be defined by $\alpha_{(a,(b,c)),((a',b'),c')} = \delta_{a,a'}\delta_{b,b'}\delta_{c,c'}$. By Lemmas 7 and 4, ${}^{\mathrm{t}}\alpha$ is a web-isomorphism from $((X \otimes Y) \otimes Z)^\perp$ to $(X \otimes (Y \otimes Z))^\perp$ and so α is a web-isomorphism from $X \otimes (Y \otimes Z)$ to $(X \otimes Y) \otimes Z$.

One shows immediately that $\sigma \in (\mathbb{R}^+)^{|(X\otimes Y)\multimap(Y\otimes X)|}$ defined by $\sigma_{(a,b),(b',a')} = \delta_{a,a'}\delta_{b,b'}$ is a web-isomorphism from $X \otimes Y$ to $Y \otimes X$. One exhibits similarly isomorphisms expressing that 1 is neutral for $\otimes$. It is routine to check that all these data endow **Pcoh** with the structure of a symmetric monoidal category.

Monoidal closeness results immediately from Lemma 7.

Last, $\star$-autonomy, with respect to the dualizing object $\perp = 1^\perp = 1$ is obvious when one observes that the PCSs $X^\perp$ and $X \multimap \perp$ are isomorphic.

The De Morgan dual of $\otimes$ is the cotensor, also called *par*; it is defined by $X \parr Y = (X^\perp \otimes Y^\perp)^\perp = X^\perp \multimap Y$. If $a \in |X|$ and $b \in |Y|$, we have $\mathsf{c}_{X\parr Y}(a,b) = \mathsf{c}_X(a)\mathsf{c}_Y(b)$. The identity matrix defines a morphism in $\mathbf{Pcoh}(X \otimes Y, X \parr Y)$. Indeed, given $x \in \mathsf{P}X$, $y \in \mathsf{P}Y$, $x' \in \mathsf{P}X^\perp$ and $y' \in \mathsf{P}Y^\perp$, we have $\langle (x \otimes y) \cdot x', y' \rangle = \langle x, x' \rangle \langle y, y' \rangle \leq 1$. This means that the MIX rule of linear logic (see e.g. [Gir87]) holds (in the strongest sense actually, because $1 = \perp$).

1.5 Additive structure

It will play a crucial role in the construction of our model of the pure lambda-calculus. Let $(X_i)_{i\in I}$ be a countable family of PCSs. We define a PCS $X = \&_{i\in I} X_i$ by taking $|\&_{i\in I} X_i| = \cup_{i\in I}(\{i\} \times |X_i|)$. Given $x \in (\mathbb{R}^+)^{|X|}$, we define $\pi^i(x) \in (\mathbb{R}^+)^{|X_i|}$ by $\pi^i(x)_a = x_{i,a}$. We set $\mathsf{P}X = \{x \in (\mathbb{R}^+)^{|X|} \mid \forall i \in I,\ \pi^i(x) \in \mathsf{P}X_i\}$.

The fact that $\mathsf{P}X$ so defined satisfies $\mathsf{P}X^{\perp\perp} \subseteq \mathsf{P}X$ results from the following:

$$\mathsf{P}X^\perp = \{x' \in (\mathbb{R}^+)^{|X|} \mid \sum_{i\in I} \|\pi^i(x')\|_{X_i^\perp} \leq 1\}\,.$$

Also, it is clear that condition (2) and (3) hold, since $\mathsf{c}_X(i,a) = \mathsf{c}_{X_i}(a)$ for each $i \in I$ and $a \in |X_i|$.

For each $i \in I$, we define $\mathsf{pr}^i \in (\mathbb{R}^+)^{|X|\times|X_i|}$ by

$$\mathsf{pr}^i_{(j,a),b} = \begin{cases} 1 & \text{if } i = j \text{ and } a = b \\ 0 & \text{otherwise} \end{cases}$$

Proposition 9 *For each $i \in I$, one has $\mathsf{pr}^i \in \mathbf{Pcoh}(\&_{i\in I} X_i, X_i)$ and $\mathsf{pr}^i \cdot x = \pi^i(x)$ for each $x \in \mathsf{P}(\&_{i\in I} X_i)$. The PCS $\&_{i\in I} X_i$, equipped with the projections pr^i is the cartesian product of the family $(X_i)_{i\in I}$ in the category* **Pcoh**.

Proof. The first part, which expresses the properties of the pr^i's is clear from the definition of $\&_{i\in I} X_i$.

So let Y be a PCS and let $t^i \in \mathbf{Pcoh}(Y, X_i)$, for each $i \in I$. Let $t \in (\mathbb{R}^+)^{|Y|\times|\&_{i\in I} X_i|}$ be defined by $t_{b,(i,a)} = (t_i)_{b,a}$ for $b \in |Y|$ and $i \in I$ and $a \in |X_i|$. Given $y \in \mathsf{P}Y$, on has $\pi^i(t \cdot y) = t^i \cdot y \in \mathsf{P}X_i$ for each $i \in I$. Therefore $t \cdot y \in \mathsf{P}(\&_{i\in I} X_i)$. Hence $t \in \mathsf{P}(Y \multimap \&_{i\in I} X_i)$, that is, $t \in \mathbf{Pcoh}(Y, \&_{i\in I} X_i)$. It is obvious that $\mathsf{pr}^i \cdot t = t^i$ for each i and that t is the unique morphism from Y to $\&_{i\in I} X_i$ with this property. □

Therefore, the operation $(X_i)_{i\in I} \mapsto \&_{i\in I} X_i$ is a functor. Explicitly, given a collection of morphisms $u_i \in \mathbf{Pcoh}(X_i, Y_i)$, there is a uniquely determined morphism $\&_i u_i : \mathbf{Pcoh}(\&_{i\in I} X_i, \&_{i\in I} Y_i)$ which satisfies $\mathsf{pr}^i(\&_{i\in I} u_i) = u_i \mathsf{pr}^i$. Given $(i,a) \in |\&_{i\in I} X_i|$ and $(j,b) \in |\&_{j\in I} Y_j|$, one has

$$(\mathop{\&}_i u_i)_{(i,a),(j,b)} = \begin{cases} (u_i)_{a,b} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, if the u_i's are web isomorphisms, then $\&_{i\in I} u_i$ is a web-isomorphism.

One sets of course $\oplus_{i\in I} X_i = (\&_{i\in I} X_i^\perp)^\perp$, this is the sum of the family $(X_i)_{i\in I}$ with injections in^i obtained by transposing the pr^i's.

1.5.1 Special cases. Let X be a PCS and let I be a countable set. We denote by X^I the PCS $\&_{i\in I} X_i$ and by $X^{(I)}$ the PCS $\oplus_{i\in I} X_i$, where $X_i = X$ for each i. Hence $(X^I)^\perp = (X^\perp)^{(I)}$.

In particular, $1^{(\mathbb{N})} = \{x \in (\mathbb{R}^+)^{\mathbb{N}} \mid \sum_{i=1}^{\infty} x_i \leq 1\}$ will be used for interpreting the type of integers; it is an analogue of the flat domain of integers.

1.6 Exponentials

1.6.1 Multinomial coefficients. Let I be a set and $m \in \mathcal{M}_{\mathrm{fin}}(I)$ (the set of all finite multisets of elements of I; if m is such a multiset, $m(i)$ is the number of occurrences of i in m). Let $\#m = \sum_{i\in I} m(i) \in \mathbb{N}$. We set $m! = \prod_{i\in I} m(i)! \in \mathbb{N}$, which is well defined since the multiset m is finite.

Let $m \in \mathcal{M}_{\mathrm{fin}}(I)$. We define the multinomial coefficient $[m] \in \mathbb{N}$ as

$$[m] = \frac{\#m!}{\prod_{i\in I} m(i)!}\,.$$

Writing $m = [i_1, \dots, i_n]$ with $i_1, \dots, i_n \in I$, this coefficient $[m]$ is the number of functions $f : \{1, \dots, n\} \to \{i_1, \dots, i_n\}$ which enumerate m in the sense that $[f(1), \dots, f(n)] = m$.

Let $(x_i)_{i\in I} \in (\mathbb{R}^+)^I$ be such that $\sum_{i\in I} x_i \in \mathbb{R}^+$ and let $n \in \mathbb{N}$. The *multinomial identity* expresses that

$$\left(\sum_{i\in I} x_i\right)^n = \sum_{\substack{m\in[I]\\ \#m=n}} [m] \prod_{i\in I} x_i^{m(i)}\,. \tag{1}$$

If $M \in \mathcal{M}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{fin}}(I))$, we define $\Sigma M \in \mathcal{M}_{\mathrm{fin}}(I)$ by

$$\Sigma M = \sum_{m\in\mathcal{M}_{\mathrm{fin}}(I)} M(m)m\,.$$

Since M is a finite multiset, this sum is finite.

Let J be another set and let $m \in \mathcal{M}_{\mathrm{fin}}(I)$ and $p \in \mathcal{M}_{\mathrm{fin}}(J)$. We define $L(m, p)$ as the set of all $r \in \mathcal{M}_{\mathrm{fin}}(I \times J)$ such that

$$\begin{aligned} \forall i \in I \ \sum_{j\in J} r(i,j) &= m(i) \\ \text{and } \forall j \in J \ \sum_{i\in I} r(i,j) &= p(j)\,. \end{aligned}$$

Observe that $\forall r \in L(m, p)\ \#r = \#m = \#p$, and therefore, $L(m, p) \neq \emptyset \Rightarrow \#m = \#p$. Observe also that the set $L(m, p)$ is always finite, since it is a subset of $\{r \in \mathcal{M}_{\mathrm{fin}}(\mathsf{supp}(m) \times \mathsf{supp}(p)) \mid \#r = \#m = \#p\}$ which is a finite set.

Given $r \in L(m, p)$, we set

$$\begin{bmatrix} p \\ r \end{bmatrix} = \prod_{j\in J} \frac{p(j)!}{\prod_{i\in I} r(i,j)!}$$

which is an integer ≥ 1 since, for each j, one has $p(j) = \sum_{i\in I} r(i, j)$.

We give a combinatorial interpretation of this coefficient. Let $r \in L(m, p)$ and let n be the common cardinality of the multisets m, p and r. We can write $m = [i_1, \dots, i_n]$, $p = [j_1, \dots, j_n]$ and $r = [(i_1, j_1), \dots, (i_n, j_n)]$ with $i_1, \dots, i_n \in I$ and $j_1, \dots, j_n \in J$ (of course the i_l's are not pairwise distinct in general and neither are the j_l's). Then $\left[\begin{smallmatrix} p \\ r \end{smallmatrix}\right]$ is the number of maps $f : \{1, \dots, n\} \to \{i_1, \dots, i_n\}$ which enumerate m in the sense that $[f(1), \dots, f(n)] = m$ and satisfy $[(f(1), j_1), \dots, (f(n), j_n)] = r$.

1.6.2 The exponential. We define now a PCS $!X$. First, $|!X|$ is $\mathcal{M}_{\mathrm{fin}}(|X|)$, the set of all finite multisets of elements of $|X|$.

Given $x \in \mathbb{R}^{|X|}$ and $m \in |!X|$, we set $x^m = \prod_{a\in|X|} x_a^{m(a)}$ (this a finite product since m is a finite multiset). Next, one sets $x^! = (x^m)_{m\in|!X|} \in \mathbb{R}^{|!X|}$. Then, the PCS $!X$ is defined by setting

$$\mathsf{P}(!X) = \{x^! \mid x \in \mathsf{P}X\}^{\perp\perp}\,.$$

Let $m \in |!X|$, $k = \#m$ and $\{a_1, \ldots, a_n\} = \mathsf{supp}(m)$.

Let $\lambda > 0$ be such that $\lambda e_{a_i} \in \mathsf{P}X$ for each $i = 1, \ldots, n$. If $n > 0$ then $x = \frac{\lambda}{n} \sum_{i=1}^{n} e_{a_i} \in \mathsf{P}X$ and hence $x^! \in \mathsf{P}(!X)$. But $x^m = (\frac{\lambda}{n})^k$ and hence $(\frac{\lambda}{n})^k e_m \in \mathsf{P}(!X)$. Since $0 \in \mathsf{P}X$, we have $0^! \in \mathsf{P}(!X)$. But $0^!_{[]} = 1$ and hence $e_{[]} \in \mathsf{P}(!X)$. This shows that condition (2) holds for $!X$.

For each $x \in \mathsf{P}X$, we have $x_{a_i} \leq \mathsf{c}_X(a_i)$ for $i = 1, \ldots, n$. We have $x^m \leq \prod_{i=1}^{n} \mathsf{c}_X(a_i)^{m(a_i)}$, so condition (3) holds for $!X$.

Remark: We have given a rough lower bound for $\mathsf{c}_{!X}(m)$. But there is an easy better one, based on the following simple fact.

Lemma 10 *Let $p_1, \ldots, p_n$ be positive integers. The maximal value of the function*

$$\begin{array}{rcl} f : [0,1]^n & \to & [0,1] \\ (z_1, \ldots, z_n) & \mapsto & z_1^{p_1} \cdots z_n^{p_n} \end{array}$$

on the set $\{(z_1, \ldots, z_n) \in [0,1]^n \mid z_1 + \cdots + z_n = 1\}$ is

$$\frac{p_1^{p_1} \cdots p_n^{p_n}}{(p_1 + \cdots + p_n)^{p_1 + \cdots + p_n}}$$

and is reached at point $(p_1 + \cdots + p_n)^{-1}(p_1, \ldots, p_n)$.

From this, we derive that

$$\frac{m(a_1)^{m(a_1)} \cdots m(a_n)^{m(a_n)}}{\#m^{\#m}} \prod_{i=1}^{n} \mathsf{c}_X(a_i)^{m(a_i)} \leq \mathsf{c}_{!X}(m) \leq \prod_{i=1}^{n} \mathsf{c}_X(a_i)^{m(a_i)} .$$

Let I be a countable set. For $X = 1^{(I)}$, the lower bound is reached and for $X = 1^I$, the upper bound is reached. Consider for instance the case where $I = \{\mathbf{t}, \mathbf{f}\}$, then X is the PCS of booleans. The corresponding coherence space **Bool** has I as web, with $\mathbf{t}$ and $\mathbf{f}$ incoherent. Let $m = [\mathbf{t}, \mathbf{t}, \mathbf{f}]$, then $\mathsf{c}_{!X}(m) = 2^2/3^3 = 4/27$. The fact that this number is < 1 corresponds to the fact that m does not belong to the web of the coherence space !**Bool** in Girard's model of coherence space (because the support of m is not a clique).

Let $t \in \mathbf{Pcoh}(X, Y)$, we define $!t \in (\mathbb{R}^+)^{|!X| \times |!Y|}$ by setting

$$(!t)_{m,p} = \sum_{r \in L(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} t^r .$$

This sum is finite since $L(m, p)$ is finite.

Lemma 11 *For any $x \in \mathsf{P}X$, one has $!t \cdot x^! = (t \cdot x)^!$.*

Proof. Let $x \in \mathsf{P}X$ and let $p \in |!Y|$. Let $L(p)$ be the set of all $l \in \mathcal{M}_{\mathrm{fin}}(|X|)^{|Y|}$ such that $\forall b \in |Y|\ \#l(b) = p(b)$. Given such an l, we set $l_1 = \sum_{b \in |Y|} l(b) \in \mathcal{M}_{\mathrm{fin}}(|X|)$. Then, we identify l with the element l' of $L(l_1, p)$ defined by $l'(a, b) = l(b)(a)$. Observe that, with these notations, $\left[\begin{smallmatrix} p \\ l' \end{smallmatrix}\right] = \prod_{b \in |Y|} [l(b)]$.

One has, computing in $\mathbb{R}^+$,

$$
\begin{aligned}
(t\cdot x)^!_p &= \prod_{b\in|Y|}\left(\sum_{a\in|X|} t_{a,b}x_a\right)^{p(b)} \\
&\qquad \text{(remember that this product is finite)} \\
&= \prod_{b\in|Y|}\left(\sum_{\substack{l(b)\in\mathcal{M}_{\mathrm{fin}}(|X|)\\ \#l(b)=p(b)}} [l(b)]\left(\prod_{a\in|X|} t_{a,b}^{l(b)(a)}\right) x^{l(b)}\right) \\
&\qquad \text{by the multinomial identity (1)} \\
&= \sum_{l\in L(p)}\left(\prod_{b\in|Y|}[l(b)]\right) t^{l'}x^{l_1} \\
&\qquad \text{by definition of } L(p) \text{ as a set of functions} \\
&= \sum_{l\in L(p)} \begin{bmatrix} p \\ r \end{bmatrix} t^{l'}x^{l_1} \\
&= \sum_{m\in\mathcal{M}_{\mathrm{fin}}(|X|)}\left(\sum_{\substack{l\in L(p)\\ l_1=m}} \begin{bmatrix} p \\ r \end{bmatrix} t^{l'}\right) x^m \\
&= \sum_{m\in\mathcal{M}_{\mathrm{fin}}(|X|)}\left(\sum_{r\in L(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} t^{r}\right) x^m \\
&= (!t\cdot x^!)_p
\end{aligned}
$$

□

Lemma 12 *Let $u\in(\mathbb{R}^+)^{|!X\multimap Y|}$. Then one has $u\in\mathsf{P}(!X\multimap Y)$ as soon as $\forall x\in\mathsf{P}X\ u\cdot x^!\in\mathsf{P}Y$.*

Proof. It suffices to show that ${}^{\mathrm{t}}u\in\mathsf{P}(Y^\perp\multimap(!X)^\perp)$, that is $\forall y'\in\mathsf{P}Y^\perp\ {}^{\mathrm{t}}u\cdot y'\in\mathsf{P}(!X)^\perp$. But this is clear, since $\forall x\in\mathsf{P}X\ \langle {}^{\mathrm{t}}u\cdot y',x^!\rangle=\langle y',u\cdot x^!\rangle\in[0,1]$ by assumption. □

Lemma 13 *For any $t\in\mathbf{Pcoh}(X,Y)$, one has $!t\in\mathbf{Pcoh}(!X,!Y)$.*

Direct consequence of Lemma 12 and of the fact that $!t\cdot x^!=(t\cdot x)^!$.

1.6.3 Entire functions.

Lemma 14 *Let $S,T\in\mathbf{Pcoh}(!X,Y)$. If, for all $x\in\mathsf{P}X$, one has $S\cdot x^!=T\cdot x^!$, then $S=T$.*

Proof. Let $b\in|Y|$. Let $\mu>0$ be such that $\mu e_b\in\mathsf{P}Y^\perp$.

Let $m\in|!X|$. Let $a_1,\dots,a_n$ be an enumeration of the set $\mathsf{supp}(m)$. Let $\lambda>0$ be such that $\lambda e_{a_i}\in\mathsf{P}X$ for $i=1,\dots,n$. Let $\theta:[0,\frac{\lambda}{n}]^n\to(\mathbb{R}^+)^{|X|}$ be defined by $\theta(z)=\sum_{i=1}^n z_i e_{a_i}$, then $\forall z\,\theta(z)\in\mathsf{P}X$. We consider the map

$$
\begin{aligned}
\varphi:\left[0,\frac{\lambda}{n}\right]^n &\to [0,1] \\
z &\mapsto \langle S\cdot\theta(z)^!,\mu e_b\rangle=\langle T\cdot\theta(z)^!,\mu e_b\rangle
\end{aligned}
$$

Let $\mathcal{M}=\mathcal{M}_{\mathrm{fin}}(\{a_1,\dots,a_n\})$. We have, for all $z\in\left[0,\frac{\lambda}{n}\right]^n$,

$$\varphi(z)=\mu\sum_{m'\in\mathcal{M}} S_{m',b}z^{m'}=\mu\sum_{m'\in\mathcal{M}} T_{m',b}z^{m'}$$

Since $\left(0, \frac{\lambda}{n}\right)^n$ is open in $\mathbb{R}^n$, we have $S_{m',b} = T_{m',b}$ for each $m' \in \mathcal{M}$ and in particular for $m' = m$. So $S = T$. □

Given $S \in \mathsf{P}((!X) \multimap Y)$, let $\mathsf{Fun}(S) : \mathsf{P}X \to \mathsf{P}Y$ be defined by $\mathsf{Fun}(S)(x) = S \cdot x^! = \mathsf{fun}(S)(x^!)$. We have seen that if $\mathsf{Fun}(S) = \mathsf{Fun}(T)$ then $S = T$.

Let us say that a function $f : \mathsf{P}X \to \mathsf{P}Y$ is *entire* if there exists $S \in \mathsf{P}((!X) \multimap Y)$ such that $f(x) = S \cdot x^!$ for all $x \in \mathsf{P}X$. As we have seen, there is only one such S (this S is analogue to the trace of a stable function in [Gir87]).

1.6.4 Functoriality of the exponential

Proposition 15 *The operation $X \mapsto !X$ and $t \mapsto !t$ is a functor from* **Pcoh** *to* **Pcoh**.

Proof. We use Lemma 14. We have $!\mathrm{Id}_X \cdot x^! = x^!$ and hence $!\mathrm{Id}_X = \mathrm{Id}_{!X}$. Given $s \in \mathbf{Pcoh}(X,Y)$ and $t \in \mathbf{Pcoh}(Y,Z)$, for any $x \in \mathsf{P}X$, one has

$$
\begin{aligned}
!(ts) \cdot x^! &= (ts \cdot x)^! \\
&= (t \cdot (s \cdot x))^! \\
&= !t \cdot (s \cdot x)^! \\
&= !t \cdot !s \cdot x^! \\
&= !t!s \cdot x^!
\end{aligned}
$$

and hence $!(ts) = !t!s$ by Lemma 14. □

Observe that, if $t \in \mathbf{Pcoh}(X,Y)$ is a web-isomorphism, then $!t$ is also a web-isomorphism.

1.6.5 Comonad structure of the exponential. The counit (also called *dereliction*) is $\mathrm{d}^X \in (\mathbb{R}^+)^{|!X| \times |X|}$ given by $\mathrm{d}^X_{m,a} = \delta_{m,[a]}$; we prove that it belongs to $\mathbf{Pcoh}(!X, X)$. For this, it suffices to check that ${}^{\mathrm{t}}\mathrm{d}^X \in \mathbf{Pcoh}(X^\perp, (!X)^\perp)$. So let $x' \in \mathsf{P}X^\perp$. We have to show that ${}^{\mathrm{t}}\mathrm{d}^X \cdot x' \in \mathsf{P}(!X)^\perp$. So let $x \in \mathsf{P}X$. We have $\langle {}^{\mathrm{t}}\mathrm{d}^X \cdot x', x^! \rangle = \langle x', \mathrm{d}^X \cdot !x \rangle = \langle x', x \rangle \in [0,1]$. Observe that the reasoning is simply based on the fact that $\forall x \in \mathsf{P}X,\ \mathrm{d}^X \cdot x^! = x$.

The comultiplication (also called *digging*) is $\mathrm{p}^X \in (\mathbb{R}^+)^{|!X| \times |!!X|}$ given by $\mathrm{p}^X_{m,M} = \delta_{m,\Sigma M}$. We check that, $\mathrm{p}^X \in \mathbf{Pcoh}(!X, !!X)$. As above, it suffices to check that, if $x \in \mathsf{P}X$, then $\mathrm{p}^X \cdot x^! \in \mathsf{P}(!!X)$. Given $M \in |!!X|$, we have $(\mathrm{p}^X \cdot x^!)_M = x^{\Sigma M} = ((x^!)^!)_M$ since indeed $((x^!)^!)_M = (x^!)^M = \prod_{m \in |!X|} (x^!_m)^{M(m)} = \prod_{m \in |!X|} (x^m)^{M(m)} = x^{\Sigma M}$. We have seen that $\mathrm{p}^X \cdot x^! = (x^!)^! \in \mathsf{P}(!!X)$, as required.

Checking that the three comonad equations are satisfied, namely

- $\mathrm{d}^{!X}\,\mathrm{p}^X = \mathrm{Id}_{!X}$
- $!(\mathrm{d}^X)\,\mathrm{p}^X = \mathrm{Id}_{!X}$
- and $\mathrm{p}^{!X}\,\mathrm{p}^X = !(\mathrm{p}^X)\,\mathrm{p}^X$

can be done using again Lemma 14. For instance, for the last equation, we have $\mathrm{p}^{!X}\,\mathrm{p}^X \cdot x^! = \mathrm{p}^{!X} \cdot (x^!)^! = ((x^!)^!)^!$ and $!(\mathrm{p}^X)\,\mathrm{p}^X \cdot x^! = !(\mathrm{p}^X) x^! = (\mathrm{p}^X \cdot x^!)^! = ((x^!)^!)^!$.

The naturality of d^X and p^X is proved in the same way.

1.6.6 Cartesian closeness of the Kleisli category. Remember that this Kleisli category $\mathbf{Pcoh}_!$ is defined as follows:

- its objects are the PCSs,
- $\mathbf{Pcoh}_! = \mathbf{Pcoh}(!X, Y)$,
- the identity map is $\mathrm{d}^X \in \mathbf{Pcoh}_!(X,X)$
- and last, given $S \in \mathbf{Pcoh}_!(X,Y)$ and $T \in \mathbf{Pcoh}_!(Y,Z)$, composition is given by $T \circ S = T\,!S\,\mathrm{p}^X$.

One has $\mathsf{Fun}(\mathrm{d}^X)(x) = \mathrm{d}^X \cdot x^! = x$ and $\mathsf{Fun}(T \circ S)(x) = (T\,!S\ \mathrm{p}^X) \cdot x^! = T \cdot (!S \cdot (x^!)^!) = T \cdot (S \cdot x^!)^! = (\mathsf{Fun}(T) \circ \mathsf{Fun}(S))(x)$. So any morphism $S \in \mathbf{Pcoh}_!(X, Y)$ can be identified with the associated entire map $\mathsf{Fun}(S)$, and this identification is compatible with composition. We identify therefore $\mathbf{Pcoh}_!$ with the category whose objects are the PCSs and where a morphism from X to Y is an entire function from $\mathsf{P}X$ to $\mathsf{P}Y$.

This Kleisli category $\mathbf{Pcoh}_!$ is cartesian closed, because the PCSs $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$ are naturally isomorphic. This isomorphism is the web-isomorphism induced by the usual bijection between the webs

$$\begin{aligned} f : |!X \otimes !Y| &\to |!(X \mathbin{\&} Y)| \\ ([a_1, \dots, a_p], [b_1, \dots, b_q]) &\mapsto [(1, a_1), \dots, (1, a_p), (2, b_1), \dots, (2, b_q)] \end{aligned}$$

Let us denote by φ the corresponding matrix, $\varphi \in (\mathbb{R}^+)^{|(!X \otimes !Y) \multimap !(X \& Y)|}$, given by $\varphi_{(m,p),q} = \delta_{f(m,p),q}$. We check that φ is indeed an isomorphism. Let ψ be the inverse (or transpose, in this case the notions coincide) of φ.

Given $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$, we have $\psi \cdot (x \oplus y)^! = x^! \otimes y^!$ and since all the elements of $\mathsf{P}(X \mathbin{\&} Y)$ are of the shape $x \oplus y$ with $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$, this shows that $\psi \in \mathbf{Pcoh}(!(X \mathbin{\&} Y), !X \otimes !Y)$. We want now to show that $\varphi \in \mathbf{Pcoh}(!X \otimes !Y, !(X \mathbin{\&} Y))$.

By Lemma 4, and using the notations of that lemma, it suffices to show that

$$\alpha \cdot \varphi \in \mathsf{P}(!X \multimap (!Y \multimap !(X \mathbin{\&} Y)))\,.$$

This is easy to prove, using twice Lemma 12, and the fact that

$$((\alpha \cdot \varphi) \cdot x^!) \cdot y^! = \varphi \cdot (x^! \otimes y^!) = (x \oplus y)^!\,.$$

The object of morphisms from X to Y is then $X \Rightarrow Y = (!X \otimes Y^\perp)^\perp = !X \multimap Y$. By the above isomorphism, we have as usual $\mathbf{Pcoh}_!(Z \mathbin{\&} X, Y) = \mathbf{Pcoh}(!(Z \mathbin{\&} X), Y) \simeq \mathbf{Pcoh}(!Z \otimes !X, Y) \simeq \mathbf{Pcoh}(!Z, !X \multimap Y) = \mathbf{Pcoh}_!(Z, X \Rightarrow Y)$, showing that $\mathbf{Pcoh}_!$ is cartesian closed.

We can identify $\mathsf{P}(X \mathbin{\&} Y)$ with $\mathsf{P}X \times \mathsf{P}Y$ and $\mathsf{P}(X \Rightarrow Y)$ with the set of entire functions from $\mathsf{P}X$ to $\mathsf{P}Y$. Under these identifications, the cartesian closed structure is standard in the sense that the evaluation map $\mathrm{ev} : \mathsf{P}((X \Rightarrow Y) \mathbin{\&} X) \to \mathsf{P}Y$ is given by $\mathrm{ev}(f, x) = f(x)$, and, if $f : \mathsf{P}(Z \mathbin{\&} X) \to \mathsf{P}Y$ is entire, the curryfication $\mathsf{Cur}(f) : \mathsf{P}Z \to \mathsf{P}(X \Rightarrow Y)$, which is an entire map, is given by $\mathsf{Cur}(f)(z)(x) = f(z, x)$.

1.6.7 Scott-continuity of morphisms.

Lemma 16 *The function from $\mathsf{P}X$ to $\mathsf{P}(!X)$ which maps x to $x^!$ is Scott-continuous.*

Proof. Let $m \in |!X|$. If $x \le y$ are elements of $\mathsf{P}X$, then $x^m \le y^m$, so the map $x \mapsto x^!$ is monotone. Let $D \subseteq \mathsf{P}X$ be directed. Then $\sup_{x \in D} x^m = (\sup D)^m$, by continuity of the map

$$\begin{aligned} \mathbb{R}^{\mathsf{supp}(m)} &\to \mathbb{R} \\ z &\mapsto z^m \end{aligned}$$

□

Proposition 17 *Any entire map is Scott-continuous.*

Proof. Use Lemmas 6 and 16. □

In particular, any entire $f : \mathsf{P}X \to \mathsf{P}X$ admits a least fixpoint which is $\sup_{n \in \mathbb{N}} f^n(0) \in \mathsf{P}X$.

We apply this observation to a particular morphism. Let X be a PCS. Let

$$\begin{aligned} \mathcal{Y} : \mathsf{P}((X \Rightarrow X) \Rightarrow X) &\to (\mathsf{P}X)^{\mathsf{P}(X \Rightarrow X)} \\ F &\mapsto \lambda f^{X \Rightarrow X}\, f(F(f)) \end{aligned}$$

By cartesian closeness of $\mathbf{Pcoh}_!$, this function is an entire endomap on $\mathsf{P}((X \Rightarrow X) \Rightarrow X)$. Let $\mathsf{Fix}_X \in \mathsf{P}((X \Rightarrow X) \Rightarrow X)$ be the least fixpoint of $\mathcal{Y}$. Observe that $\mathcal{Y}^n(0)(f) = f^n(0)$. Therefore we have the following result.

Proposition 18 *For any entire map* $f : \mathsf{P}X \to \mathsf{P}X$*, the value* $\mathsf{Fix}_X(f)$ *is the least fixpoint of* f*.*

So the operation which sends an entire endomap to its least fixpoint is itself entire. It will be used for interpreting the fixpoint construction of our probabilistic version of PCF.

2 Fixpoints of types

Our main goal here is to show that the category **Pcoh** contains a reflexive object, that is, a model of the pure lambda-calculus. We shall define this object as the least fixpoint of the operation $X \mapsto (!(X^{\mathbb{N}^+}))^\perp$. This operation however is not a covariant functor with respect to entire maps or even to linear maps, so we shall restrict our attention to embedding-projection pairs (just as in the construction of the model D_∞ by Scott, see [Bar84]; see also [Gir86] for the use of the same notion in coherence spaces), and more precisely, to "inclusions" of PCSs. This is clearly quite a restrictive notion of morphism between PCSs.

Given two sets S, T with $S \subseteq T$, we define $\zeta_{S,T} \in (\mathbb{R}^+)^{S\times T}$ and $\rho_{S,T} \in (\mathbb{R}^+)^{T\times S}$ by $(\zeta_{S,T})_{a,b} = (\rho_{S,T})_{b,a} = \delta_{a,b}$ for $a \in S$ and $b \in T$.

2.1 Substructures and limits of directed systems of PCSs

Let X and Y be PCSs. We say that X is a *sub-PCS* of Y or that X is *included in* Y, and write $X \subseteq Y$, if $|X| \subseteq |Y|$ and

$$\begin{aligned} &\forall x \in \mathsf{P}X \qquad \zeta_{|X|,|Y|} \cdot x \in \mathsf{P}Y \\ &\forall y \in \mathsf{P}Y \qquad \rho_{|X|,|Y|} \cdot y \in \mathsf{P}X\,. \end{aligned}$$

So

$$X \subseteq Y \quad \Leftrightarrow \quad |X| \subseteq |Y|,\ \zeta_{|X|,|Y|} \in \mathbf{Pcoh}(X, Y) \text{ and } \rho_{|X|,|Y|} \in \mathbf{Pcoh}(Y, X) \tag{2}$$

If $X_1 \subseteq X_2 \subseteq X_3$, then $X_1 \subseteq X_3$ with

$$\zeta_{|X_2|,|X_3|}\zeta_{|X_1|,|X_2|} = \zeta_{|X_1|,|X_3|} \quad \text{and} \quad \rho_{|X_1|,|X_2|}\rho_{|X_2|,|X_3|} = \rho_{|X_1|,|X_3|}\,. \tag{3}$$

The crucial property is that linear negation is *covariant* with respect to this notion of inclusion.

Lemma 19 *If* $X \subseteq Y$*, then* $X^\perp \subseteq Y^\perp$*.*

This is due to the following obvious facts:

$$^{\mathrm{t}}\zeta_{|X|,|Y|} = \rho_{|X^\perp|,|Y^\perp|} \quad \text{and} \quad ^{\mathrm{t}}\rho_{|X|,|Y|} = \zeta_{|X^\perp|,|Y^\perp|}\,.$$

Lemma 20 *If* $X \subseteq Y$ *and* $a \in |X|$*, then* $\mathsf{c}_X(a) = \mathsf{c}_Y(a)$*.*

Proof. Since $\mathsf{c}_X(a)e_a \in \mathsf{P}X \subseteq \mathsf{P}Y$, we have $\mathsf{c}_X(a) \le \mathsf{c}_Y(a)$. For the same reason, since $X^\perp \subseteq Y^\perp$, we have $\mathsf{c}_{X^\perp}(a) \le \mathsf{c}_{Y^\perp}(a)$. But remember that $\mathsf{c}_{X^\perp}(a) = \mathsf{c}_X(a)^{-1}$ and $\mathsf{c}_{Y^\perp}(a) = \mathsf{c}_Y(a)^{-1}$. □

We denote by $\mathbf{Pcoh}_\subseteq$ the category whose objects are the PCSs and whose morphisms are the inclusions of PCSs, so that $\mathbf{Pcoh}_\subseteq$ is actually a partially ordered class, whose least element is 0, the empty-web PCS. Of course, inclusions of PCSs are a very restrictive notion of morphism, sufficient however for our purpose. An immediate generalization would be to consider maps which are composites of inclusions and web-isomorphisms, corresponding to more general embedding-retraction pairs. This is not necessary here and the benefit of this simplification is that we can consider the class of PCSs as a "cpo".

2.1.1 Inductive limits of directed families in $\mathbf{Pcoh}_\subseteq$. A *directed* family of PCSs is a collection of PCSs $(X_\gamma)_{\gamma\in\Gamma}$ indexed by a directed partially ordered set Γ, and such that $\forall\gamma,\delta \in \Gamma\ \gamma \le \delta \Rightarrow X_\gamma \subseteq X_\delta$.

Let $S = \cup_{\gamma\in\Gamma}|X_\gamma|$. Let $\zeta_\gamma = \zeta_{|X_\gamma|,S} \in (\mathbb{R}^+)^{|X_\gamma|\times S}$ and $\rho_\gamma = \rho_{|X_\gamma|,S} \in (\mathbb{R}^+)^{S\times|X_\gamma|}$. If $\gamma \le \delta$, we set $\zeta_{\gamma,\delta} = \zeta_{|X_\gamma|,|X_\delta|}$ and $\rho_{\gamma,\delta} = \rho_{|X_\gamma|,|X_\delta|}$.

Then, we define a PCS $\cup_{\gamma\in\Gamma} X_\gamma$ by setting

- $|\cup_{\gamma\in\Gamma} X_\gamma| = S = \cup_{\gamma\in\Gamma}|X_\gamma|$

- and $\mathsf{P}(\cup_{\gamma\in\Gamma} X_\gamma) = \{\zeta_\gamma \cdot x \mid \gamma \in \Gamma \text{ and } x \in \mathsf{P}X_\gamma\}^{\perp\perp}$.

We check that $\cup_{\gamma\in\Gamma} X_\gamma$ so defined is a PCS. The inclusion $\mathsf{P}(\cup_{\gamma\in\Gamma} X_\gamma)^{\perp\perp} \subseteq \mathsf{P}(\cup_{\gamma\in\Gamma} X_\gamma)$ results from the definition of $\mathsf{P}(\cup_{\gamma\in\Gamma} X_\gamma)$ as a dual. So we are left with checking conditions (2) and (3) of the definition of PCSs. Let $a \in S$ and let $\gamma \in \Gamma$ be such that $a \in |X_\gamma|$. Observe first that, for any $\delta \in \Gamma$ such that $a \in |X_\delta|$, one can find a $\eta \in \Gamma$ such that $\gamma, \delta \leq \eta$, and therefore we have $\mathsf{c}_{X_\gamma}(a) = \mathsf{c}_{X_\eta}(a) = \mathsf{c}_{X_\delta}(a)$ by Lemma 20.

Since $\mathsf{c}_{X_\gamma}(a)e_a \in \mathsf{P}X_\gamma$, we have $\zeta_\gamma \cdot \mathsf{c}_{X_\gamma}(a)e_a \in \mathsf{P}(\cup_{\delta\in\Gamma} X_\delta)$ and therefore $\mathsf{c}_{\cup_{\delta\in\Gamma} X_\delta}(a) \geq \mathsf{c}_{X_\gamma}(a) > 0$. Conversely, we have $\mathsf{c}_X(a)^{-1}e_a \in (\mathsf{P}X_\gamma)^\perp$. We show that $\mathsf{c}_X(a)^{-1}e_a \in (\cup_{\delta\in\Gamma} \mathsf{P}X_\delta)^\perp$. Let $\delta \in \Gamma$ and let $y \in \mathsf{P}X_\delta$. We have $(\zeta_\delta \cdot y)_a = y_a \leq \mathsf{c}_{X_\delta}(a) = \mathsf{c}_{X_\gamma}(a)$. Therefore $\langle \mathsf{c}_X(a)^{-1}e_a, \zeta_\delta \cdot y\rangle \leq 1$ as required.

This shows that $\cup_{\gamma\in\Gamma} X_\gamma$ is a PCS, which has a countable web as soon as Γ and the $|X_\gamma|$'s are countable.

Let $\gamma \in \Gamma$. We check that $X_\gamma \subseteq \cup_{\delta\in\Gamma} X_\delta$. Obviously, for any $x \in \mathsf{P}X_\gamma$, we have $\zeta_\gamma \cdot x \in \mathsf{P}(\cup_{\delta\in\Gamma X} X_\gamma)$. Let $x' \in \mathsf{P}X_\gamma^\perp$ and let $y \in \mathsf{P}X_\delta$ for some $\delta \in \Gamma$. Let $\eta \in \Gamma$ be such that $\gamma, \delta \leq \eta$. We have

$$\begin{aligned} \langle {}^{\mathrm{t}}\rho_\gamma \cdot x', \zeta_\delta \cdot y\rangle &= \langle {}^{\mathrm{t}}\rho_\eta {}^{\mathrm{t}}\rho_{\gamma,\eta} \cdot x', \zeta_\eta \zeta_{\delta,\eta} \cdot y\rangle \\ &= \langle {}^{\mathrm{t}}\rho_{\gamma,\eta} \cdot x', \rho_\eta \zeta_\eta \zeta_{\delta,\eta} \cdot y\rangle \\ &= \langle {}^{\mathrm{t}}\rho_{\gamma,\eta} \cdot x', \zeta_{\delta,\eta} \cdot y\rangle \in [0,1] \end{aligned}$$

since ${}^{\mathrm{t}}\rho_{\gamma,\eta} \cdot x' \in \mathsf{P}X_\eta^\perp$ and $\zeta_{\delta,\eta} \cdot y \in \mathsf{P}X_\eta$.

Proposition 21 $(\cup_{\gamma\in\Gamma} X_\gamma, (\zeta_\gamma)_{\gamma\in\Gamma})$ *is the colimit cocone of the diagram* $((X_\gamma)_{\gamma\in\Gamma}, (\zeta_{\gamma,\delta})_{\gamma\leq\delta})$ *in the category* **Pcoh**.

Proof. Let Y be a PCS and let $(u_\gamma)_{\gamma\in\Gamma}$ be a cocone to Y based on that diagram, that is, a family of matrices with $u_\gamma \in \mathbf{Pcoh}(X_\gamma, Y)$ for each $\gamma \in \Gamma$ and such that

$$\forall \gamma, \delta \in \Gamma \quad \gamma \leq \delta \Rightarrow u_\delta \zeta_{\gamma,\delta} = u_\gamma\,. \tag{4}$$

Given $\gamma, \delta \in \Gamma$ such that $\gamma \leq \delta$ and given $a \in |X_\gamma|$ and $c \in |Y|$, by (4), we have $(u_\delta)_{a,c} = (u_\gamma)_{a,c}$. Therefore, we can define a matrix $u \in (\mathbb{R}^+)^{|\cup_{\gamma\in\Gamma} X_\gamma| \times |Y|}$ by setting $u_{a,c} = (u_\gamma)_{a,c}$ where $\gamma \in \Gamma$ is such that $a \in |X_y|$ (the value of $(u_\gamma)_{a,c}$ does not depend on the choice of γ since Γ is directed). Observe that $u\zeta_\gamma = u_\gamma$ for all $\gamma \in \Gamma$.

Let $y' \in \mathsf{P}Y^\perp$, we prove that ${}^{\mathrm{t}}u \cdot y' \in \mathsf{P}(\cup_{\gamma\in\Gamma} X_\gamma)^\perp$. So let $\gamma \in \Gamma$ and let $x \in \mathsf{P}X_\gamma$. We have $\langle {}^{\mathrm{t}}u \cdot y', \zeta_\gamma \cdot x\rangle = \langle y', u\zeta_\gamma \cdot x\rangle = \langle y', u_\gamma \cdot x\rangle \in [0,1]$. This shows that $u \in \mathbf{Pcoh}(\cup_{\gamma\in\Gamma} X_\gamma, Y)$. Moreover, it is clear that u is the unique element of $u \in \mathbf{Pcoh}(\cup_{\gamma\in\Gamma} X_\gamma, Y)$ such that $u\zeta_\gamma = u_\gamma$ for all $\gamma \in \Gamma$. □

We give now a "projective" account of this colimit. This is based on the order-theoretic considerations of Section 1.3.

Proposition 22 *Let* $y \in (\mathbb{R}^+)^{|\cup_{\gamma\in\Gamma} X_\gamma|}$*. One has*

$$y \in \mathsf{P}(\bigcup_{\gamma\in\Gamma} X_\gamma) \quad \Leftrightarrow \quad \forall \gamma \in \Gamma\ \rho_\gamma \cdot y \in \mathsf{P}X_\gamma$$

Proof. Assume first that $y \in \mathsf{P}(\cup_{\gamma\in\Gamma} X_\gamma)$. Let $x' \in \mathsf{P}X_\gamma^\perp$. We have ${}^{\mathrm{t}}\rho_\gamma \cdot x' \in \mathsf{P}(\cup_{\delta\in\Gamma} X_\delta)^\perp$ because $X_\gamma \subseteq \cup_{\delta\in\Gamma} X_\delta$, and hence $X_\gamma^\perp \subseteq (\cup_{\delta\in\Gamma} X_\delta)^\perp$. Therefore, $\langle y, {}^{\mathrm{t}}\rho_\gamma \cdot x'\rangle \in [0,1]$, that is $\langle \rho_\gamma \cdot y, x'\rangle \in [0,1]$. Since this holds for all $x' \in \mathsf{P}X_\gamma$, we have shown that $\rho_\gamma \cdot y \in \mathsf{P}X_\gamma$.

Conversely, assume that $\rho_\gamma \cdot y \in \mathsf{P}X_\gamma$ for each $\gamma \in \Gamma$. Let $y(\gamma) = \zeta_\gamma \rho_\gamma \cdot y$. We have $y(\gamma) \in \mathsf{P}(\cup_{\delta\in\Gamma} X_\delta)$. Moreover, for $a \in |\cup_{\delta\in\Gamma} X_\delta|$, we have $y(\gamma)_a = y_a$ if $a \in |X_\gamma|$ and $y(\gamma)_a = 0$ otherwise. So the family $(y(\gamma))_{\gamma\in\Gamma}$ is directed in $\mathsf{P}(\cup_{\delta\in\Gamma} X_\delta)$ and its lub is y. By Proposition 5, we conclude that $y \in \mathsf{P}(\cup_{\delta\in\Gamma} X_\delta)$. □

Proposition 23 *If* Y *is a PCS and if we have* $X_\gamma \subseteq Y$ *for all* $\gamma \in \Gamma$*, then* $\cup_{\gamma\in\Gamma} X_\gamma \subseteq Y$*. That is,* $\cup_{\gamma\in\Gamma} X_\gamma \subseteq Y$ *is the colimit of* $(X_\gamma)_{\gamma\in\Gamma}$ *in the category (partially ordered class)* $\mathbf{Pcoh}_\subseteq$.

Proof. Let $X = \cup_{\gamma\in\Gamma} X_\gamma$. By assumption, we have $|X| \subseteq |Y|$. Let $\zeta \in (\mathbb{R}^+)^{|X|\times|Y|}$ be the matrix of this inclusion and $\rho \in (\mathbb{R}^+)^{|Y|\times|X|}$ be its transpose.

By Proposition 21, there is a unique $\theta \in \mathbf{Pcoh}(X,Y)$ such that $\theta\zeta_\gamma = \zeta_{|X_\gamma|,|Y|}$ for each $\gamma \in \Gamma$. By these equations, we have $\theta = \zeta$. Let $y \in \mathsf{P}Y$, for concluding, we must show that $\rho \cdot y \in \mathsf{P}X$. We apply Proposition 22, so let $\gamma \in \Gamma$. We have $\rho_\gamma \cdot (\rho \cdot y) = \rho_{X_\gamma,Y} \cdot y$, and we know that $\rho_{X_\gamma,Y} \cdot y \in \mathsf{P}X_\gamma$ because $X_\gamma \subseteq Y$. Since this holds for all $\gamma \in \Gamma$, we have $\rho \cdot y \in \mathsf{P}X$. □

Proposition 24 *The construction $\cup_{\gamma\in\Gamma}$ is selfdual. More precisely, given a directed system $(X_\gamma)_{\gamma\in\Gamma}$ of PCSs, one has*

$$(\bigcup_{\gamma\in\Gamma} X_\gamma)^\perp = \bigcup_{\gamma\in\Gamma} X_\gamma^\perp\,.$$

Proof. Let $\delta \in \Gamma$. We have $X_\delta \subseteq \cup_{\gamma\in\Gamma} X_\gamma$, hence $X_\delta^\perp \subseteq (\cup_{\gamma\in\Gamma} X_\gamma)^\perp$, and therefore $\cup_{\gamma\in\Gamma} X_\gamma^\perp \subseteq (\cup_{\gamma\in\Gamma} X_\gamma)^\perp$ by Proposition 23. Therefore, since these two PCSs have the same web, they are equal. □

2.1.2 Continuous functors on $\mathbf{Pcoh}_\subseteq$.

Let $k \in \mathbb{N}$. We denote by $\mathbf{Pcoh}_\subseteq^k$ the k-fold product partially ordered class (considered as a category). We use a vector notation $\vec{\_}$ for denoting the objects of this class, and $\subseteq$ for the partial order of this class.

A functor $F : \mathbf{Pcoh}_\subseteq^k \to \mathbf{Pcoh}_\subseteq$ is *continuous* if it commutes with directed colimits of PCSs, that is $F(\cup_{\gamma\in\Gamma}\vec{X}_\gamma) = \cup_{\gamma\in\Gamma} F(\vec{X}_\gamma)$.

Let 0 be the empty-web PCS. Given a continuous functor $G : \mathbf{Pcoh}_\subseteq \to \mathbf{Pcoh}_\subseteq$, the sequence $(G^n(0))_{n\in\mathbb{N}}$ is a directed system of PCSs, whose colimit $\mathsf{FIX}(G) = \cup_{n\in\mathbb{N}} G^n(0)$ satisfies

$$G(\mathsf{FIX}(G)) = \mathsf{FIX}(G)\,, \tag{5}$$

by continuity of G. More generally, given a continuous functor $F : \mathbf{Pcoh}_\subseteq^{k+1} \to \mathbf{Pcoh}_\subseteq$, the operation $\vec{X} \mapsto \mathsf{FIX}(F(\vec{X}, \_))$ is easily shown to be a continuous functor $\mathbf{Pcoh}_\subseteq^k \to \mathbf{Pcoh}_\subseteq$, using the universal property of the colimit in $\mathbf{Pcoh}_\subseteq$.

2.1.3 Continuity of logical functors.

We show that the various operations on objects we have introduced are continuous functors.

Lemma 25 *If $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, then $X_1 \otimes Y_1 \subseteq X_2 \otimes Y_2$, and one has*

$$\zeta_{|X_1\otimes Y_1|,|X_2\otimes Y_2|} = \zeta_{|X_1|,|X_2|} \otimes \zeta_{|Y_1|,|Y_2|} \quad \textit{and} \quad \rho_{|X_1\otimes Y_1|,|X_2\otimes Y_2|} = \rho_{|X_1|,|X_2|} \otimes \rho_{|Y_1|,|Y_2|}\,.$$

Moreover, if $(X_\gamma)_{\gamma\in\Gamma}$ is a directed systems of PCSs and Y is a PCS, then

$$\bigcup_{\gamma\in\Gamma}(Y \otimes X_\gamma) = Y \otimes (\bigcup_{\gamma\in\Gamma} X_\gamma)\,.$$

Proof. By functoriality of $\otimes$, we know that $\zeta_{X_1,X_2} \otimes \zeta_{Y_1,Y_2} \in \mathbf{Pcoh}(X_1 \otimes Y_1, X_2 \otimes Y_2)$ and $\rho_{X_1,X_2} \otimes \rho_{Y_1,Y_2} \in \mathbf{Pcoh}(X_2 \otimes Y_2, X_1 \otimes Y_1)$. So, by (2), it suffices to check that the two announced equations hold, and this is very easy.

As to the second part of the lemma, we could use a simple categorical argument: as a left adjoint, the functor $Y \otimes \_$ commutes with arbitrary colimits in $\mathbf{Pcoh}$. More concretely, we know that

$$\bigcup_{\gamma\in\Gamma}(Y \otimes X_\gamma) \subseteq Y \otimes (\bigcup_{\gamma\in\Gamma} X_\gamma)$$

by Proposition 23. But the web of these PCSs are equal, so the PCSs are equal. □

The next two lemmas are proved in the same way.

Lemma 26 *If $X_1 \subseteq X_2$, then $!X_1 \subseteq !X_2$, with*

$$\zeta_{!X_1,!X_2} = !\zeta_{X_1,X_2} \quad \textit{and} \quad \rho_{!X_1,!X_2} = !\rho_{X_1,X_2}\,.$$

Moreover, if $(X_\gamma)_{\gamma\in\Gamma}$ is a directed system of PCSs, then

$$!(\bigcup_{\gamma\in\Gamma} X_\gamma) = \bigcup_{\gamma\in\Gamma} !X_\gamma\,.$$

Lemma 27 *If $X_1 \subseteq X_2$, then $X_1^I \subseteq X_2^I$, with*

$$\zeta_{X_1^I,X_2^I} = \zeta^I_{X_1,X_2} \quad \text{and} \quad \rho_{X_1^I,X_2^I} = \rho^I_{X_1,X_2}\,.$$

Moreover, if $(X_\gamma)_{\gamma\in\Gamma}$ is a directed system of PCSs, then

$$(\bigcup_{\gamma\in\Gamma} X_\gamma)^I = \bigcup_{\gamma\in\Gamma} X_\gamma^I\,.$$

Last, remember that the operation $X \mapsto X^\perp$ is a continuous functor on $\mathbf{Pcoh}_\subseteq$ by Proposition 24.

2.2 A model of the pure lambda-calculus in Pcoh

Let us write $X \simeq_\mathsf{w} Y$ if the PCSs X and Y are web-isomorphic. Given any PCS X,

$$X \mathbin{\&} X^\mathbb{N} \simeq_\mathsf{w} X^\mathbb{N}\,. \tag{6}$$

This web-isomorphism s is given by

$$\mathsf{s}_{(1,a),(j,b)} = \delta_{j,0}\delta_{a,b} \quad \text{and} \quad \mathsf{s}_{(2,(i,a)),(j,b)} = \delta_{j,i+1}\delta_{a,b}\,.$$

Let $F : \mathbf{Pcoh}_\subseteq \to \mathbf{Pcoh}_\subseteq$ be the continuous functor defined by $F(X) = (!(X^\mathbb{N}))^\perp$. Let $D_\infty = \mathsf{FIX}(F)$.

Proposition 28 *There is a web-isomorphism of PCSs $\varphi : D_\infty \to (D_\infty \Rightarrow D_\infty)$.*

Proof. We have

$$\begin{array}{rcll} D_\infty \Rightarrow D_\infty & = & (!D_\infty \otimes D_\infty^\perp)^\perp & \text{by definition of } \_ \Rightarrow \_ \\ & = & (!D_\infty \otimes !D_\infty^\mathbb{N})^\perp & \text{by definition of } D_\infty \text{ and by (5)} \\ & \simeq_\mathsf{w} & (!(D_\infty \mathbin{\&} D_\infty^\mathbb{N}))^\perp & \text{by the iso of 1.6.6} \\ & \simeq_\mathsf{w} & (!D_\infty^\mathbb{N})^\perp & \text{by (6)}\,. \end{array}$$

□

In [BEM07], we showed that $|D_\infty|$ is an extensional model of the pure lambda-calculus in the cartesian closed category $\mathbf{Rel}_!$ (the Kleisli category of the comonad $S \to !S = \mathcal{M}_{\mathrm{fin}}(S)$ on the category $\mathbf{Rel}$ of sets and relations, which is a well known model of linear logic). We have just extended that result, showing that D_∞, which is just $|D_\infty|$ equipped with a canonical PCS structure, is a model of the pure lambda-calculus in the cartesian closed category $\mathbf{Pcoh}_!$.

Therefore, it is also a model of the pure probabilistic lambda-calculus which is the pure lambda-calculus extended, e.g. with an operation $\mathsf{ran}(\lambda, M, N)$ where $\lambda \in [0,1]$ and M and N are terms. The reduction rule associated with this construction is that $\lambda\vec{\zeta}\,(\mathsf{ran}(\lambda, P, Q))\,\vec{R}$ reduces to $\lambda\vec{\zeta}\,(P)\,\vec{R}$ with probability λ and to $\lambda\vec{\zeta}\,(Q)\,\vec{R}$ with probability $1-\lambda$; this probabilistic reduction can be performed only if the probabilistic redex $\mathsf{ran}(\lambda, M, N)$ is in head position (or, more generally, in linear position). The precise connection between the probabilistic operational semantics of this lambda-calculus and its denotational semantics in D_∞ will be addressed in future work.

For the time being, we consider the same problem, in the setting of PCF, which is simpler thanks to the presence of a ground type for which the probabilistic interpretation of the semantics is clear.

3 Probabilistic PCF

We introduce the language PPCF, a probabilistic extension of the functional language PCF [Plo77].

The language is simply typed:

- ι is a type
- and if σ and τ are types, so is $\sigma \Rightarrow \tau$.

Terms are defined by the following syntax. We are given an infinite countable set of variables.

- Any variable ζ is a term;
- if P is a term, ζ is a variable and σ is a type, then $\lambda\zeta^\sigma\, P$ is a term;
- if P and Q are terms, so is $(P)\, Q$;
- if P is a term then so is $\mathsf{fix}(P)$;
- if $n \in \mathbb{N}$ then $\underline{n}$ is a term;
- if P is a term then $\mathsf{succ}(P)$ and $\mathsf{pred}(P)$ are terms;
- if P, Q and R are terms, so is $\mathsf{if}(P, Q, R)$;
- for any $\vec{\lambda} \in [0,1]^{\mathbb{N}}$ with $\sum_{n=0}^{\infty} \lambda_n = 1$, $\mathsf{ran}(\vec{\lambda})$ is a term.

As we shall see, $\mathsf{ran}(\vec{\lambda})$ is a term of type ι which reduces to $\underline{n}$ with probability λ_n. This construction is of course far too infinitary for a "real" programming language (in our syntax, the set of terms is not countable).

The syntax can be made more realistic by replacing the $\mathsf{ran}(\vec{\lambda})$ construction by a constant coin of type ι which reduces to $\underline{0}$ with probability $1/2$ and to $\underline{1}$ with probability $1/2$. This does not change the results we prove in the sequel.

A *typing context* is a sequence $\Gamma = (\zeta_1 : \sigma_1, \dots, \zeta_k : \sigma_k)$ where the variables ζ_i are distinct. The typing rules are as follows.

$$\frac{}{\Gamma, \zeta : \sigma \vdash \zeta : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \Rightarrow \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash (M)\, N : \tau} \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x^\sigma\, M : \sigma \Rightarrow \tau}$$

$$\frac{\Gamma \vdash M : \sigma \Rightarrow \sigma}{\Gamma \vdash \mathsf{fix}(M) : \sigma} \qquad \frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash P : \iota \qquad \Gamma \vdash Q : \iota}{\Gamma \vdash \mathsf{if}(M, P, Q) : \iota}$$

$$\frac{}{\Gamma \vdash \underline{n} : \iota} \qquad \frac{}{\Gamma \vdash \mathsf{ran}(\vec{\lambda}) : \iota} \qquad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \mathsf{succ}(M) : \iota} \qquad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \mathsf{pred}(M) : \iota}$$

In the conditional construction, the restriction that the two branches should be of type ι is convenient for the forthcoming proofs, and is not restrictive from an expressiveness viewpoint.

3.1 Denotational semantics in Pcoh

The category **Pcoh** is a model of PCF in which the additional probabilistic construction $\mathsf{ran}(\vec{\lambda})$ can also be interpreted. Since the morphisms of this category are functions, this interpretation is quite easy to describe.

With any type σ, we associate a PCS $[\sigma]$, by induction on σ. We set $[\iota] = \mathbf{N} = 1^{(\mathbb{N})}$. Remember that this PCS is given by $|\mathbf{N}| = \mathbb{N}$ and $\mathsf{P}\mathbf{N} = \{x \in [0,1]^{\mathbb{N}} \mid \sum_{i=0}^{\infty} x_i \leq 1\}$. Next, we set of course $[\sigma \Rightarrow \tau] = [\sigma] \Rightarrow [\tau]$. Given a context $\Gamma = (\zeta_1 : \sigma_1, \dots, \zeta_n : \sigma_n)$, we set $[\Gamma] = [\sigma_1] \mathbin{\&} \cdots \mathbin{\&} [\sigma_n]$.

Given a term M, a context Γ and a type σ such that $\Gamma \vdash M : \sigma$, we define $[M]_\Gamma \in \mathbf{Pcoh}_!([\Gamma], [\sigma])$, by induction on M, as an entire function $\mathsf{P}([\Gamma]) \to \mathsf{P}([\sigma])$, that is as a function $\mathsf{P}([\Gamma_1]) \times \cdots \times \mathsf{P}([\Gamma_n]) \to \mathsf{P}([\sigma])$.

- $[\zeta_i]_\Gamma(\vec{x}) = x_i$;
- $[(P)\, Q]_\Gamma(\vec{x}) = [P]_\Gamma(\vec{x})([Q]_\Gamma(\vec{x}))$;
- $[\lambda\zeta^\sigma\, P]_\Gamma(\vec{x}) \in \mathsf{P}([\sigma] \Rightarrow [\tau])$ (where τ is such that $\Gamma, \zeta : \sigma \vdash P : \tau$) is the entire function $\mathsf{P}([\sigma]) \to \mathsf{P}([\tau])$ given by $[\lambda\zeta^\sigma\, P]_\Gamma(\vec{x})(y) = [P]_{\Gamma, \zeta:\sigma}(\vec{x}, y)$;
- if $\Gamma \vdash P : \sigma \Rightarrow \sigma$, then $[\mathsf{fix}(P)]_\Gamma(\vec{x}) = \mathsf{Fix}_{[\sigma]}([P]_\Gamma(\vec{x}))$;
- $y = [\mathsf{succ}(P)]_\Gamma(\vec{x}) \in \mathsf{P}\mathbf{N}$ is given by $y_0 = 0$ and $y_{i+1} = x_i$, where $x = [P]_\Gamma(\vec{x})$;
- $y = [\mathsf{pred}(P)]_\Gamma(\vec{x}) \in \mathsf{P}\mathbf{N}$ is given by $y_i = x_{i+1}$, where $x = [P]_\Gamma(\vec{x})$;

- if $\Gamma \vdash R : \iota$, $\Gamma \vdash P : \iota$ and $\Gamma \vdash Q : \iota$, setting $x = [R]_\Gamma(\vec{x})$, $y = [P]_\Gamma(\vec{x})$ and $z = [Q]_\Gamma(\vec{x})$, then

$$[\mathsf{if}(R, P, Q)]_\Gamma(\vec{x}) = x_0 y + x_{>0} z$$

  where $x_{>0} = \sum_{i=1}^{\infty} x_i$, and we have $[\mathsf{if}(R, P, Q)]_\Gamma(\vec{x}) \in \mathsf{P}([\iota])$ since $\sum_{i=0}^{\infty} x_i \leq 1$ (because $x \in \mathsf{P}([\iota])$) and $y, z \in \mathsf{P}([\iota])$;

- last $[\mathsf{ran}(\vec{\lambda})]_\Gamma(\vec{x}) = \vec{\lambda} \in \mathsf{P}\mathbf{N}$.

So, for instance, $[\mathsf{if}(\mathsf{ran}(\vec{\lambda}), P, Q)]_\Gamma(\vec{x}) = \lambda_0 [P]_\Gamma(\vec{x}) + (1 - \lambda_0)[Q]_\Gamma(\vec{x})$.

3.2 Reduction strategy

We restrict our attention to a particular reduction strategy, which is the leftmost-outermost strategy; we describe it in a small-step way. Given terms M and M' and given $\lambda \in [0, 1]$, we write $M \xrightarrow{\lambda} M'$ (meaning that M reduces to M' in one step, with probability λ) in one of the following situations:

- $M = \mathsf{pred}(\underline{0})$, $M' = \underline{0}$ and $\lambda = 1$,
- $M = \mathsf{pred}(\underline{n+1})$, $M' = \underline{n}$ and $\lambda = 1$,
- $M = \mathsf{pred}(N)$, $M' = \mathsf{pred}(N')$ and $N \xrightarrow{\lambda} N'$,
- $M = \mathsf{succ}(\underline{n})$, $M' = \underline{n+1}$ and $\lambda = 1$,
- $M = \mathsf{succ}(N)$, $M' = \mathsf{succ}(N')$ and $N \xrightarrow{\lambda} N'$,
- $M = \mathsf{ran}(\vec{\lambda})$, $M' = \underline{n}$ and $\lambda = \lambda_n$ (the *probabilistic reduction rule*),
- $M = \mathsf{if}(\underline{0}, L, R)$, $M' = L$ and $\lambda = 1$,
- $M = \mathsf{if}(\underline{n+1}, L, R)$, $M' = R$ and $\lambda = 1$,
- $M = \mathsf{if}(N, L, R)$, $M' = \mathsf{if}(N', L, R)$ and $N \xrightarrow{\lambda} N'$,
- $M = \lambda\zeta\, N$, $M' = \lambda\zeta\, N'$ and $N \xrightarrow{\lambda} N'$,
- $M = \mathsf{fix}(N)$, $M' = (N)\, M$ and $\lambda = 1$,
- $M = (\lambda\zeta\, N)\, L$, $M' = N\,[L/\zeta]$ and $\lambda = 1$,
- $M = (N)\, L$, $M' = (N')\, L$, $N \xrightarrow{\lambda} N'$ and N is not of the shape $N = \lambda\zeta\, P$ (we say that N *is not an abstraction*).

We say that M is in *head normal form* if it is not reducible for this strategy.

We write $M \to_{\mathsf{d}} M'$ if $M \xrightarrow{\lambda} M'$ without using the probabilistic reduction rule (and hence $\lambda = 1$). Observe that, if $M \to_{\mathsf{d}} M'$ and $M \to_{\mathsf{d}} M''$, then $M' = M''$, and so $\to_{\mathsf{d}}$ is a deterministic reduction.

Lemma 29 (subject reduction) *If $\Gamma \vdash M : \sigma$ and $M \xrightarrow{\lambda} M'$ then $\Gamma \vdash M' : \sigma$.*

Lemma 30 (invariance of the interpretation) *If $\Gamma \vdash M : \sigma$, then the following holds in the PCS* $\mathbf{Pcoh}([\Gamma], [\sigma])$

$$[M]_\Gamma = \sum_{M \xrightarrow{\lambda} M'} \lambda [M']_\Gamma \,.$$

Both results are proved by a straightforward induction on M.

The next substitution lemma will be important in the proof of Proposition 39, and crucially uses the definition of the reduction strategy.

Lemma 31 *Assume that $\Gamma, \zeta : \sigma \vdash M : \tau$, that $\Gamma \vdash P : \sigma$ and that $M \to_{\mathsf{d}} M'$. Then $M\,[P/\zeta] \to_{\mathsf{d}} M'\,[P/\zeta]$.*

Proof. By induction on M. The only non straightforward case is when $M = (N)\,L$, the term N does not start with an abstraction and $N \to_{\mathsf{d}} N'$; in that case, we have $M' = (N')\,L$. Then N cannot be a variable (since $N \to_{\mathsf{d}} N'$), and hence $N\,[P/\zeta]$ cannot be an abstraction since N is not an abstraction. By inductive hypothesis, we have $N\,[P/\zeta] \to_{\mathsf{d}} N'\,[P/\zeta]$ and hence $(N\,[P/\zeta])\,L\,[P/\zeta] \to_{\mathsf{d}} (N'\,[P/\zeta])\,L\,[P/\zeta]$ since $N\,[P/\zeta]$ is not an abstraction. □

3.3 Stochastic matrices and transition paths

Stochastic matrices are used for describing discrete time Markov processes. Let S be a set. A stochastic matrix on S is an element P of $[0,1]^{S\times S}$ such that

$$\forall s \in S \quad \sum_{t\in S} P_{s,t} = 1\,.$$

Intuitively, S is a set of states, and $P_{s,t}$ is the probability of evolving from state s to state t in one step.

If $\mu \in [0,1]^S$ is a probability distribution on S (that is $\sum_{s\in S}\mu_s = 1$) considered as a row vector (with possibly infinitely many components), then the row vector $\mu S = (\sum_{s\in S}\mu_s P_{s,t})_{t\in S}$ is a probability distribution on S, which describes the probability of states after one step of evolution starting from the probability of states described by μ. If $s \in S$, let r_s be the probability distribution defined by $(\mathsf{r}_s)_t = \delta_{s,t}$. We use the notation c_s, when the same vector is considered as a column vector, and more generally c_U for the characteristic vector of the set $U \subseteq S$, considered as a column vector.

If P and Q are stochastic matrices on S, then the usual matricial product PQ is well defined and is a stochastic matrix on S. In particular, P^n is a stochastic matrix, and $P^n_{s,t}$ is the probability of evolving from state s to state t in n steps.

3.3.1 Absorbing states. A state $t \in S$ is *absorbing* if $P_{t,t} = 1$ (so $P_{t,u} = 0$ for $u \neq t$), that is $\mathsf{r}_t\,P = \mathsf{r}_t$. Let S_0 is the set of all absorbing states of S.

Lemma 32 *Let $t \in S_0$. Then, for any $s \in S$, the sequence $(P^n_{s,t})_{n\in\mathbb{N}}$ is monotone. Let $P^\infty_{s,t} = \sup_{n=0}^{\infty} P^n_{s,t}$, one has*

$$\sum_{t\in S_0} P^\infty_{s,t} \leq 1\,.$$

The proof is straightforward.

3.3.2 Transition paths. We use the term *transition path* to refer to any sequence $w = (t_1,\dots,t_k)$ of elements of S such that

$$P_{t_i,t_{i+1}} > 0 \quad \text{for all } i = 1,\dots,k-1.$$

This implies that $t_1,\dots,t_{k-1} \notin S_0$. Observe that some states can be repeated in transition paths, but absorbing states cannot be repeated (they can only occur in last position).

Then we write $w : t_1 \rightsquigarrow t_k$, and we define the *probability of* w as

$$\mathsf{p}(w) = \prod_{i=1}^{k-1} P_{t_i,t_{i+1}} \in (0,1]\,.$$

The *length* $\mathsf{lg}(w)$ of w is $k-1$. In particular, for any $s \in S$, the one-element sequence (s) is the only transition path of length 0 from s to s, and it satisfies $\mathsf{p}((s)) = 1$. If $w = (s = s_1,\dots,s_{k+1} = s') : s \rightsquigarrow s'$ and $w' = (s' = s_{k+1},\dots,s_{k+l+1} = s'') : s' \rightsquigarrow s''$, then $ww' : s \rightsquigarrow s''$ is the sequence $(s_1,\dots,s_{k+1},\dots,s_{k+l})$. Observe that $\mathsf{p}(ww') = \mathsf{p}(w)\,\mathsf{p}(w')$ and that $\mathsf{lg}(ww') = \mathsf{lg}(w) + \mathsf{lg}(w') = k + l$.

Lemma 33 *Let $s,u \in S$ with u non absorbing. Then*

$$P^k_{s,u} = \sum_{\substack{w:s\rightsquigarrow u\\ \mathsf{lg}(w)=k}} \mathsf{p}(w)\,.$$

The hypothesis that u is not absorbing is essential since, when u is absorbing, one has $P^k_{u,u} = 1$ for all k, whereas the only transition path from u to u is (u), of length 0.

Proof. By induction on k, the base case being obvious. By inductive hypothesis, we have

$$\begin{aligned} P^{k+1}_{s,t} &= \sum_{\substack{v\in S\\ P_{s,v}>0}} \sum_{\substack{w:v\rightsquigarrow u\\ \mathsf{lg}(w)=k}} P_{s,v}\,\mathsf{p}(w) \\ &= \sum_{\substack{v\in S_0\\ P_{s,v}>0}} \sum_{\substack{w:v\rightsquigarrow u\\ \mathsf{lg}(w)=k}} P_{s,v}\,\mathsf{p}(w) + \sum_{\substack{v\in S\setminus S_0\\ P_{s,v}>0}} \sum_{\substack{w:v\rightsquigarrow u\\ \mathsf{lg}(w)=k}} P_{s,v}\,\mathsf{p}(w)\,. \end{aligned}$$

Since u is not absorbing, $w : v \rightsquigarrow u$ implies that v is not absorbing (even when $\mathsf{lg}(w) = 0$), and so the value of the first of these two sums is 0. We conclude because all transition paths of length $k+1$ from s to u are of the shape $(s,v)w$ with $w : v \rightsquigarrow u$, $\mathsf{lg}(w) = k$ and $P_{s,v} > 0$ and $v \notin S_0$. □

We can now establish the main result of this section.

Lemma 34 *Let $s \in S$ and $t \in S_0$. Then*

$$P^{\infty}_{s,t} = \sum_{w:s\rightsquigarrow t} \mathsf{p}(w)\,.$$

Proof. By Lemma 32, it suffices to show that

$$P^{k}_{s,t} = \sum_{\substack{w:s\rightsquigarrow t\\ \mathsf{lg}(w)\le k}} \mathsf{p}(w)$$

and this is done by induction on k. The base case $k = 0$ is clear because then both sides of the equation are equal to $\delta_{s,t}$. For the inductive step, we have

$$\begin{aligned} P^{k+1}_{s,t} &= (P^kP)_{s,t} \\ &= \sum_{u\in S} P^k_{s,u}P_{u,t} \\ &= \sum_{u\in S_0} P^k_{s,u}P_{u,t} + \sum_{u\in S\setminus S_0} P^k_{s,u}P_{u,t} \\ &= \sum_{\substack{w:s\rightsquigarrow t\\ \mathsf{lg}(w)\le k}} \mathsf{p}(w) + \sum_{\substack{u\in S\setminus S_0\\ w:s\rightsquigarrow u,\, \mathsf{lg}(w)=k}} \mathsf{p}(w)P_{u,t} \end{aligned}$$

by inductive hypothesis, and by Lemma 33 (we also use the fact that if $u \in S_0$ and $P_{u,t} > 0$ then $u = t$). The result follows easily. □

3.4 The stochastic matrix of terms

We organize the set of all PPCF terms as a Markov process: let S be the set of all PPCF terms, we define a matrix $\mathsf{Red} \in [0,1]^{\mathsf{S}\times\mathsf{S}}$ by

$$\mathsf{Red}_{M,M'} = \begin{cases} \lambda & \text{if } M \xrightarrow{\lambda} M' \\ 1 & \text{if } M = M' \text{ is in head normal form} \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this matrix is stochastic. If M is a head normal form, then M is an absorbing state for Red.

Lemma 35 *Assume that $\Gamma \vdash M : \sigma$. Then, for any M' such that $\mathsf{Red}_{M,M'} > 0$, one has $\Gamma \vdash M' : \sigma$. Moreover, one has*

$$[M]_\Gamma = \sum_{M'\in\mathsf{S}} \mathsf{Red}_{M,M'}[M']_\Gamma\,.$$

This is just a restatement of Lemma 30. Iterating this property, we have $[M]_\Gamma = \sum_{M' \in \mathsf{S}} \mathsf{Red}^k_{M,M'} [M']_\Gamma$ for all $k \in \mathbb{N}$. Assume that $\vdash M : \iota$ and let $n \in \mathbb{N}$. We have therefore

$$([M])_n \geq \sup_{k \in \mathbb{N}} \mathsf{Red}^k_{M,\underline{n}} = \mathsf{Red}^\infty_{M,\underline{n}} \ .$$

Remember that, by Lemma 32, $(\mathsf{Red}^k_{M,\underline{n}})_{k \in \mathbb{N}}$ is a monotone sequence in $[0,1]$, since $\underline{n}$ is an absorbing state in S, and that the lub of that sequence is $\mathsf{Red}^\infty_{M,\underline{n}}$. Observe also that, in the present setting, a transition path $w : M \rightsquigarrow M'$ (where M' is in head normal form), is a sequence $w = (M = M_1, \dots, M_k = M')$, with $M_i \xrightarrow{\lambda_i} M_{i+1}$ for $i = 1, \dots, k-1$, and $\mathsf{p}(w) = \lambda_1 \cdots \lambda_{k-1} > 0$.

3.4.1 A logical relation. Our goal is now to prove the converse inequation. For this purpose, we adapt the logical relation technique of [Plo77] (see also [AC98]). By induction on σ, we define a relation $\mathcal{R}^\sigma$ from $[\sigma]$ to the set of all *closed* terms M of type σ.

- $x \,\mathcal{R}^\iota\, M$ if, for all $n \in \mathbb{N}$, one has $x_n \leq \mathsf{Red}^\infty_{M,\underline{n}} = \sup_{k \in \mathbb{N}} \mathsf{Red}^k_{M,\underline{n}} = \sum_{w : M \rightsquigarrow \underline{n}} \mathsf{p}(w)$;
- $f \,\mathcal{R}^{\sigma \Rightarrow \tau}\, M$ if, for all $x \in [\sigma]$ and all P such that $\vdash P : \sigma$, if $x \,\mathcal{R}^\sigma\, P$ then $f(x) \,\mathcal{R}^\tau\, (M)\, P$.

3.4.2 Closure properties of the logical relation. We first need to prove, by induction on types, a few closure properties of this relation.

Lemma 36 *Assume that $\vdash M : \sigma$ and that $M \to_{\mathsf{d}} M'$. Then $x \,\mathcal{R}^\sigma\, M' \Leftrightarrow x \,\mathcal{R}^\sigma\, M$.*

Proof. For $\sigma = \iota$, the result follows from the fact that $\mathsf{Red}^k_{M',\underline{n}} = \mathsf{Red}^{k+1}_{M,\underline{n}}$ (this equation holds because, for all $N \in \mathsf{S}$, one has $\mathsf{Red}_{M,N} = \delta_{N,M'}$; indeed, if $M \xrightarrow{\lambda} N$, then $\lambda = 1$, $M \to_{\mathsf{d}} N$ and $N = M'$).

Assume that $\sigma = (\varphi \Rightarrow \psi)$. Assume first that $f \,\mathcal{R}^\sigma\, M$ and let us show that $f \,\mathcal{R}^\sigma\, M'$. So let $x \in \mathsf{P}([\varphi])$ and let P with $\vdash P : \varphi$ be such that $x \,\mathcal{R}^\varphi\, P$; we must show that $f(x) \,\mathcal{R}^\psi\, (M')\, P$. We have to consider two cases.

- M is an abstraction, that is $M = \lambda \zeta^\varphi\, N$ for some N with $\zeta : \varphi \vdash N : \psi$. Then we have $N \to_{\mathsf{d}} N'$ with $M' = \lambda \zeta\, N'$. We have $(M)\, P \to_{\mathsf{d}} N\, [P/\zeta]$ and, by Lemma 31, we have $N\, [P/\zeta] \to_{\mathsf{d}} N'\, [P/\zeta]$. So, applying twice the inductive hypothesis at type ψ (in the left to right direction of the implication), we get $f(x) \,\mathcal{R}^\psi\, N'\, [P/\zeta]$. We conclude, applying the inductive hypothesis (in the right to left direction of the implication) thanks to the fact that $(M')\, P \to_{\mathsf{d}} N'\, [P/\zeta]$.
- M is not an abstraction. In that case, $(M)\, P \to_{\mathsf{d}} (M')\, P$ and we apply straightforwardly the inductive hypothesis.

Conversely, assume that $f \,\mathcal{R}^\sigma\, M'$ and let us show that $f \,\mathcal{R}^\sigma\, M$. We have the same two cases to consider, the second (M is not an abstraction) being quite easy. So let us assume that $M = \lambda \zeta^\varphi\, N$, as above. With the same notations, since we have assumed that $f \,\mathcal{R}^\sigma\, \lambda \zeta\, N'$, we get $f(x) \,\mathcal{R}^\psi\, (\lambda \zeta\, N')\, P$, but $(\lambda \zeta\, N')\, P \to_{\mathsf{d}} N'\, [P/\zeta]$, so by inductive hypothesis, we have $f(x) \,\mathcal{R}^\psi\, N'\, [P/\zeta]$, and since $N\, [P/\zeta] \to_{\mathsf{d}} N'\, [P/\zeta]$ by Lemma 31, we get $f(x) \,\mathcal{R}^\psi\, N\, [P/\zeta]$, and hence $f(x) \,\mathcal{R}^\psi\, (\lambda \zeta\, N)\, P$ by inductive hypothesis again, since we have $(\lambda \zeta\, N)\, P \to_{\mathsf{d}} N\, [P/\zeta]$. □

Lemma 37 *Assume that $\vdash M : \sigma$. Then $0 \,\mathcal{R}^\sigma M$. And let $(x_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements of $[\sigma]$ such that $x_n \,\mathcal{R}^\sigma\, M$ for all $n \in \mathbb{N}$. Then $\sup_{n \in \mathbb{N}} x_n \,\mathcal{R}^\sigma\, M$.*

Proof. The base case of the induction is clear, and the inductive hypothesis is based on the fact that the order relation in $[\varphi \Rightarrow \psi]$ is the pointwise order on functions. □

Lemma 38 *Let $x, y, z \in [\iota]$ and let M, L, R be closed terms of type ι. Assume that $x \,\mathcal{R}^\iota\, M$, $y \,\mathcal{R}^\iota\, L$ and $z \,\mathcal{R}^\iota\, R$. Then we have*

$$x_0 y + x_{>0} z \,\mathcal{R}^\iota\, \mathsf{if}(M, L, R)\,.$$

Proof. Let $n \in \mathbb{N}$, we must show that

$$x_0 y_n + x_{>0} z_n \leq \sum_{w:\mathsf{if}(M,L,R)\rightsquigarrow \underline{n}} \mathsf{p}(w)\,.$$

Given the transition path $u = (L = L_1 \xrightarrow{\lambda_1} \cdots \xrightarrow{\lambda_{l-1}} L_l = \underline{n})$, we denote by u^0 the transition path $(\mathsf{if}(\underline{0}, L, R) \xrightarrow{1} L = L_1 \xrightarrow{\lambda_1} \cdots \xrightarrow{\lambda_{l-1}} L_l = \underline{n})$. Similarly, given the transition path $v = (R = R_1 \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_{r-1}} R_r = \underline{n})$ and $k \in \mathbb{N}$, we denote by v^{k+1} the transition path $(\mathsf{if}(\underline{k+1}, L, R) \xrightarrow{1} R = R_1 \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_{r-1}} R_r = \underline{n})$. We have of course $\mathsf{p}(u^0) = \mathsf{p}(u)$ and $\mathsf{p}(v^{k+1}) = \mathsf{p}(v)$.

We use the following notation: if $t = (M_1, \ldots, M_q)$ is a transition path, we denote by $\mathsf{if}(t, L, R)$ the sequence $(\mathsf{if}(M_1, L, R), \ldots, \mathsf{if}(M_q, L, R))$. It is clear that $\mathsf{if}(t, L, R)$ is a transition path and that $\mathsf{p}(\mathsf{if}(t, L, R)) = \mathsf{p}(t)$.

Given any transition path $w : \mathsf{if}(M, L, R) \rightsquigarrow \underline{n}$, we can find, in an unique way

- either a transition path $t : M \rightsquigarrow \underline{0}$ and a transition path $u : L \rightsquigarrow \underline{n}$, such that $w = \mathsf{if}(t, L, R)u^0$
- or $k \in \mathbb{N}$ and two transition paths $t : M \rightsquigarrow \underline{k+1}$ and $v : R \rightsquigarrow \underline{n}$, such that $w = \mathsf{if}(t, L, R)v^{k+1}$.

Therefore, we have

$$\begin{aligned}
\sum_{w:\mathsf{if}(M,L,R)\rightsquigarrow \underline{n}} \mathsf{p}(w) &= \sum_{\substack{t:M\rightsquigarrow \underline{0}\\ u:L\rightsquigarrow \underline{n}}} \mathsf{p}(t)\,\mathsf{p}(u) + \sum_{k=0}^{\infty} \sum_{\substack{t:M\rightsquigarrow \underline{k+1}\\ v:R\rightsquigarrow \underline{n}}} \mathsf{p}(t)\,\mathsf{p}(v) \\
&= \left(\sum_{t:M\rightsquigarrow \underline{0}} \mathsf{p}(t)\right)\left(\sum_{u:L\rightsquigarrow \underline{n}} \mathsf{p}(u)\right) + \left(\sum_{k=0}^{\infty} \sum_{t:M\rightsquigarrow \underline{k+1}} \mathsf{p}(t)\right)\left(\sum_{v:R\rightsquigarrow \underline{n}} \mathsf{p}(v)\right).
\end{aligned}$$

and we conclude, applying our hypotheses $x\ \mathcal{R}^\iota\ M$, $y\ \mathcal{R}^\iota\ L$ and $z\ \mathcal{R}^\iota\ R$. □

3.4.3 Adequacy Lemma.

We can prove now the *Adequacy Lemma* for this logical relation, also known as *Logical Relation Lemma*. In the present setting, it reads as follows.

Proposition 39 *Assume that $\Gamma \vdash M : \tau$, where $\Gamma = (\zeta_1 : \sigma_1, \ldots, \zeta_q : \sigma_q)$. Let $P_1, \ldots, P_q$ be closed terms such that $\vdash P_i : \sigma_i$. Let $x_i \in [\sigma_i]$ for $i = 1, \ldots, q$ and assume that $x_i\ \mathcal{R}^{\sigma_i}\ P_i$ for $i = 1, \ldots, q$. Then we have*

$$[M]_\Gamma(\vec{x})\ \mathcal{R}^\tau\ M\left[\vec{P}/\vec{\zeta}\right].$$

Proof. By induction on M. Let us just deal with a few cases, the other ones being straightforward.

The cases $M = \zeta_i$ and $M = \underline{l}$ with $l \in \mathbb{N}$ are left to the reader (for the second case, observe that there is exactly one transition path $\underline{l} \rightsquigarrow \underline{l}$, which is the path of length 0 and probability 1).

Assume that $M = \mathsf{ran}(\vec{\lambda})$, with $\vec{\lambda} \in [0,1]^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \lambda_n = 1$. Let $n \in \mathbb{N}$, we have $[M]_\Gamma(\vec{x})_n = \lambda_n$ and there is exactly one transition path $w : M\left[\vec{P}/\vec{\zeta}\right] = M \rightsquigarrow \underline{n}$; this path consists of one application of the probabilistic rules, and one has $\mathsf{p}(w) = \lambda_n$.

The cases $M = \mathsf{pred}(N)$ and $M = \mathsf{succ}(N)$ are left to the reader.

Assume that $M = \mathsf{if}(N, L, R)$. Let $x = [M]_\Gamma(\vec{x})$, $y = [L]_\Gamma(\vec{x})$ and $z = [R]_\Gamma(\vec{x})$. By inductive hypothesis, we have $x\ \mathcal{R}^\iota\ N\left[\vec{P}/\vec{\zeta}\right]$, $y\ \mathcal{R}^\iota\ L\left[\vec{P}/\vec{\zeta}\right]$ and $z\ \mathcal{R}^\iota\ R\left[\vec{P}/\vec{\zeta}\right]$ and we conclude, applying Lemma 38.

When $M = (N)\,L$, with $\Gamma \vdash N : \sigma \Rightarrow \tau$ and $\Gamma \vdash L : \sigma$, one applies straightforwardly the definition of $\mathcal{R}^{\sigma\Rightarrow\tau}$ and the inductive hypotheses.

Assume that $\sigma = (\varphi \Rightarrow \psi)$, $M = \lambda\zeta^\varphi\, N$ with $\Gamma, \zeta : \varphi \vdash N : \psi$. Given any $x \in \mathsf{P}([\varphi])$ and any term P such that $\vdash P : \varphi$ and $x\ \mathcal{R}^\varphi\ P$, we must show that $[\lambda\zeta\, N]_\Gamma(\vec{x})(x)\ \mathcal{R}^\psi\ \left(\lambda\zeta\, N\left[\vec{P}/\vec{\zeta}\right]\right) P$, that is $[N]_{\Gamma,\zeta:\varphi}(\vec{x}, x)\ \mathcal{R}^\psi\ \left(\lambda\zeta\, N\left[\vec{P}/\vec{\zeta}\right]\right) P$. By inductive hypothesis, we know that $[N]_{\Gamma,\zeta:\varphi}(\vec{x}, x)\ \mathcal{R}^\psi$ $N\left[\vec{P}/\vec{\zeta}, P/\zeta\right]$ and we conclude by Lemma 36 since $\left(\lambda\zeta\, N\left[\vec{P}/\vec{\zeta}\right]\right) P \to_{\mathsf{d}} N\left[\vec{P}/\vec{\zeta}, P/\zeta\right]$ by the very definition of $\to_{\mathsf{d}}$.

Assume last that $M = \mathsf{fix}(N)$, with $\Gamma \vdash N : \tau \Rightarrow \tau$. Let $f = [N]_\Gamma(\vec{x})$, it is an entire function $\mathsf{P}([\tau]) \to \mathsf{P}([\tau])$, and we have $[M]_\Gamma(\vec{x}) = \sup_{k=0}^{\infty} f^k(0)$. By Lemma 37, it suffices to prove that $\forall k \in \mathbb{N}\ f^k(0)\ \mathcal{R}^\tau\ M\left[\vec{P}/\vec{\zeta}\right] = \mathsf{fix}(N\left[\vec{P}/\vec{\zeta}\right])$, and this is done by induction on k. The base case $k = 0$ results from Lemma 37. For the inductive step, let $N' = N\left[\vec{P}/\vec{\zeta}\right]$, we assume that $f^k(0)\ \mathcal{R}^\tau\ \mathsf{fix}(N')$. By inductive hypothesis (in the "external" induction, on terms), we have $f\ \mathcal{R}^{\tau\Rightarrow\tau}\ N'$. Hence $f^{k+1}(0)\ \mathcal{R}^\tau\ (N')\,\mathsf{fix}(N')$ and we conclude by Lemma 36 since we have $\mathsf{fix}(N') \to_{\mathsf{d}} (N')\,\mathsf{fix}(N')$. □

From this, we derive easily the following result.

Theorem 40 *Let M be a closed term of type ι. Then $[M] \in \mathsf{P}\mathbf{N}$ is the sub-probability distribution on $\mathbb{N}$ such that $[M]_n = \mathsf{Red}^{\infty}_{M,\underline{n}} = \sum_{w:M\rightsquigarrow n} \mathsf{p}(w)$.*

In other words, $[M]_n$ is the probability that M reduces to $\underline{n}$ in our leftmost outermost strategy.

4 Conclusion: towards intrinsic PCSs

We consider now the possibility of getting rid of the webs of PCSs. Indeed, PCSs are similar to vector spaces, and from this viewpoint, the webs are like choices of a particular bases. We would like to understand if the idea of PCS can be carried to a geometrical "intrinsic", and therefore mathematically nicer and more flexible setting, where this choice of bases is no more necessary.

The first observation in this direction is that a Banach space can naturally be associated with any PCS.

4.1 Associating a Banach space with a PCS

4.1.1 Preliminaries on normed vector spaces. All the vector spaces that we consider are $\mathbb{R}$-vector spaces.

A subset C of a vector space E is absolutely convex if, whenever $x, y \in C$ and $\lambda, \mu \in \mathbb{R}$ are such that $|\lambda| + |\mu| \le 1$, one has $\lambda x + \mu y \in C$.

A semi-norm on a vector space E is a function $N : E \to \mathbb{R}^+$ such that $N(\lambda x) = |\lambda|\, N(x)$ and $N(x + y) \le N(x) + N(y)$. A semi-norm N is a norm if moreover $N(x) = 0 \Rightarrow x = 0$. The unit ball $B = \{x \in E \mid N(x) \le 1\}$ of a semi-norm N on E is an absolutely convex subset of E.

A normed vector space is a pair $(E, \|\_\|)$ where E is a vector space and $\|\_\|$ is a norm on E. Such a vector space has a topology whose open sets are the subsets U of E such that $\forall x \in E\, \exists \varepsilon > 0\, \forall y \in E\ \|y - x\| < \varepsilon \Rightarrow y \in U$. A sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if $\forall \varepsilon > 0\, \exists n \in \mathbb{N}\, \forall p, q \in \mathbb{N}\ p, q \ge n \Rightarrow \|x_p - x_q\| < \varepsilon$. And one says that E is complete if any Cauchy sequence in E converges. A Banach space is a complete normed vector space.

A subset B of a normed vector space E is bounded if $\exists r \in \mathbb{R}^+\, \forall x \in B\ \|x\| \le r$. This can be restated as follows: B is bounded if, for any neighborhood U of 0, there exists $\varepsilon > 0$ such that $\varepsilon B \subseteq U$: on says that U absorbs B. A subset U of E is absorbing if it absorbs all finite subsets of E (in other words $E = \bigcup_{\lambda>0} \lambda U$).

Given a countable set I, we denote as $l^1(I)$ the Banach space of absolutely summable I-indexed families of real numbers, equipped with the norm $\|x\|_1 = \sum_{i\in I} |x_i|$. By definition $l^1(I)$ is the vector space of all $x \in \mathbb{R}^I$ such that $\|x\|_1 < \infty$.

4.1.2 A normed vector space. Remember that, if $x, y \in \mathsf{P}X$ and $\lambda, \mu \in \mathbb{R}^+$ are such that $\lambda+\mu \le 1$, then $\lambda x + \mu y \in \mathsf{P}X$. In particular, if $x \in \mathsf{P}X$ and $\lambda \in [0, 1]$, then $\lambda x \in \mathsf{P}X$. Also, it is obvious that $0 \in \mathsf{P}X$.

Given a PCS X, let $\mathsf{B}X = \{u \in \mathbb{R}^{|X|} \mid |u| \in \mathsf{P}X\}$. This is an absolutely convex subset of $\mathbb{R}^{|X|}$.

Let
$$\mathsf{e}X = \bigcup_{\lambda>0} \lambda \mathsf{B}X\,.$$
This set is an $\mathbb{R}$-vector space. Observe that
$$\mathsf{e}X = \{u \in \mathbb{R}^{|X|} \mid \exists \lambda > 0\, \forall u' \in \mathsf{P}X^\perp\ \langle |u|, u'\rangle < \lambda\}\,.$$

If $u \in \mathsf{e}X$, we set

$$\|u\|_X = \inf\{\lambda > 0 \mid |u| \in \lambda \mathsf{B}X\} = \sup_{u' \in \mathsf{P}X^\perp} \langle |u|, u' \rangle \in \mathbb{R}^+ .$$

This number is finite by the very definition of $\mathsf{e}X$ ($\mathsf{B}X$ is absorbing in $\mathsf{e}X$). We have $\mathsf{P}X \subseteq \mathsf{e}X$, and for the elements u of $\mathsf{P}X$, the definition above of $\|u\|_X$ coincides with the definition given in 1.1.2.

The function $\|\_\|_X$, also known as the *Minkowski functional* (or gauge) of $\mathsf{B}X$, is a semi-norm, again because $\mathsf{B}X$ is absolutely convex. We have

$$\|e_a\|_X = \mathsf{c}_X(a)^{-1} .$$

Indeed, $\mathsf{c}_X(a) e_a \in \mathsf{P}X$, that is $e_a \in \mathsf{c}_X(a)^{-1} \mathsf{P}X$ and hence $\mathsf{c}_X(a)^{-1} \geq \|e_a\|_X$. Conversely, if $\lambda > \|e_a\|_X$, then $e_a \in \lambda \mathsf{P}X$, that is $\lambda^{-1} e_a \in \mathsf{P}X$, and hence $\lambda^{-1} \leq \mathsf{c}_X(a)$. Therefore $\mathsf{c}_X(a)^{-1} \leq \|e_a\|_X$.

Observe also that $\|\_\|_X$ is a norm on $\mathsf{e}X$. Indeed, let $u \in \mathsf{e}X$ and assume that $\|u\|_X = 0$, that is $\forall \lambda > 0\ |u| \in \lambda \mathsf{P}X$. Let $a \in |X|$. Since $\pi^a(\mathsf{P}X) \subseteq \mathbb{R}^+$ is upper-bounded by $\mathsf{c}_X(a)$, we have $|u_a| \leq \lambda \mathsf{c}_X(a)$ for all $\lambda > 0$. So $u = 0$. Hence $(\mathsf{e}X, \|\_\|_X)$ is a normed space and the unit ball of $\|\_\|_X$ is $\mathsf{B}X$.

Let $u' \in \mathsf{B}X^\perp$. Let $u \in \mathsf{e}X$ and let $\lambda > \|u\|_X$. We have $u \in \lambda \mathsf{B}X$ and hence the sum $\sum_{a \in |X|} |u_a u'_a|$ converges to a value which is $\leq \lambda$. So the sum

$$\langle u, u' \rangle = \sum_{a \in |X|} u_a u'_a$$

is well defined and satisfies $|\langle u, u' \rangle| \leq \|u\|_X$. Moreover, we have

$$\|u\|_X = \sup_{u' \in \mathsf{B}X^\perp} |\langle u, u' \rangle| . \tag{7}$$

More generally, given $u \in \mathsf{e}X$ and $u' \in \mathsf{e}X^\perp$, one has $\sum_{a \in |X|} |u_a u'_a| < \infty$ and so the sum $\langle u, u' \rangle = \sum_{a \in |X|} u_a u'_a$ converges and we have

$$|\langle u, u' \rangle| \leq \|u\|_X \|u'\|_{X^\perp} . \tag{8}$$

Therefore, for any given $u' \in \mathsf{e}X^\perp$, the map $u \mapsto \langle u, u' \rangle$ is a continuous linear map from $\mathsf{e}X$ to $\mathbb{R}$ (and so it is uniformly continuous). The map $u \mapsto |u| = (|u_a|)_{a \in |X|}$ from $\mathsf{e}X$ to $\mathsf{e}X$ is also uniformly continuous, because $\||v| - |u|\|_X \leq \|v - u\|_X$ for any $u, v \in \mathsf{e}X$.

4.1.3 Completeness. We show that the normed vector space $\mathsf{e}X$ is complete. So let $(u(n))_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathsf{e}X$. For any $a \in |X|$, the map π^a is uniformly continuous, and hence the sequence $(u(n)_a)_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$ and converges to some $u_a \in \mathbb{R}$. Let $u = (u_a)_{a \in |X|}$: we have seen that $(u(n))_{n \in \mathbb{N}}$ converges pointwise to u.

Since $v \mapsto \|v\|_X$ is uniformly continuous, the sequence $\|u(n)\|_X$ is Cauchy and is therefore upper-bounded by some $\lambda > 0$.

We prove now that $u \in \mathsf{e}X$. Let $u' \in \mathsf{P}X^\perp$. Since $\|u(n)u'\|_1 \leq \|u(n)\|$, the sequence $u(n)u'$ is Cauchy in the Banach space $l^1(|X|)$ and therefore converges to some $w \in l^1(|X|)$ such that $\|w\|_1 \leq \lambda$. Since $(u(n)u')_{n \in \mathbb{N}}$ converges pointwise to uu', we must have $w = uu'$ and so we have shown that $\langle |u|, u' \rangle = \|uu'\|_1 \leq \lambda$ and since this holds for all $u' \in \mathsf{P}X^\perp$, we have shown that $u \in \mathsf{e}X$.

Finally we have to prove that $u(n) \to u$. Let $w(n) = u(n) - u$, we must check that $\|w(n)\|_X \to 0$. We have

- $\forall a \in |X|\ \lim_{n \to \infty} w(n)_a = 0$
- and $(w(n))_{n \in \mathbb{N}}$ is Cauchy.

So the sequence $(\|w(n)\|_X)_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$ and therefore converges to some $\lambda \in \mathbb{R}^+$. Assume towards a contradiction that $\lambda > 0$. Upon cutting off an initial segment of the sequence $(w(n))_{n \in \mathbb{N}}$, we can assume that $\forall n \in \mathbb{N}\ \|w(n)\|_X \geq \lambda/2$. Therefore

$$\forall n \in \mathbb{N}\, \exists u' \in \mathsf{P}X^\perp \quad \langle |w(n)|, u' \rangle \geq \lambda/3 .$$

Since $(w(n))_{n\in\mathbb{N}}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $\forall n \geq N\, \|w(N) - w(n)\|_X \leq \lambda/6$, that is

$$\forall n \geq N\, \forall u' \in \mathsf{P}X^\perp \quad \langle |w(N) - w(n)|, u'\rangle \leq \lambda/6\,.$$

Let $u' \in \mathsf{P}X^\perp$ be such that $\langle |w(N)|, u'\rangle \geq \lambda/3$. For $n \geq N$, we have

$$\langle |w(N)|, u'\rangle - \langle |w(N) - w(n)|, u'\rangle \geq \lambda/6\,.$$

Let $I \subseteq |X|$ be finite and such that $\sum_{a\in|X|\setminus I} |w(N)_a|\, u'_a \leq \lambda/12$. Then we have

$$\sum_{a\in I} |w(N)_a|\, u'_a - \langle |w(N) - w(n)|, u'\rangle \geq \lambda/12$$

and hence

$$h_n = \sum_{a\in I} (|w(N)_a| - |w(N)_a - w(n)_a|) u'_a \geq \lambda/12\,.$$

But since I is finite and $\forall a \in I\ \lim_{n\to\infty} w(n)_a = 0$, we have $\lim_{n\to\infty} h_n = 0$, contradiction.

To summarize, we have proved the following result.

Theorem 41 *For any PCS X, the normed vector space $(\mathsf{e}X, \|\_\|_X)$ is a Banach space.*

The Banach space $\mathsf{e}(1^{\mathbb{N}^+})$ is l^∞ and $\mathsf{e}(1^{(\mathbb{N}^+)})$ is l^1. Of course, we would expect $\mathsf{e}(X^\perp)$ to be the topological dual of $\mathsf{e}X$, but these two examples show that this is hopeless since the dual of l^∞ is much larger than l^1. The solution to this problem is well known and consists in considering "dual pairs" of Banach spaces instead of Banach spaces. We adopt the presentation of [Gir99], but this idea is already developed in the work of Barr and Chu [Bar79], in a more general setting (without the exponential construction however).

4.2 The associated coherent Banach space

In [Gir99], a *coherent Banach space* (CBS) is defined as a triple $E = (E^+, E^-, \langle\_\,,\_\rangle_E)$ where E^+ and E^- are Banach spaces (each of them is given with an explicit choice of norm, $\|\_\|_E^+$ and $\|\_\|_E^-$ respectively) and $\langle\_\,,\_\rangle_E$ is a bilinear form $E^+ \times E^- \to \mathbb{R}$ which satisfies

$$\forall x \in E^+ \qquad \|x\|_E^+ = \sup_{\|x'\|_E^- \leq 1} |\langle x, x'\rangle_E|$$
$$\forall x' \in E^- \qquad \|x\|_E^- = \sup_{\|x\|_E^+ \leq 1} |\langle x, x'\rangle_E|\,.$$

We have shown that, for any PCS X, the triple $\mathsf{cbs}(X) = (\mathsf{e}X, \mathsf{e}X^\perp, \langle\_\,,\_\rangle)$ is a CBS.

Given two CBSs E and F, a linear morphism from E to F is a bounded linear map $f : E^+ \to F^+$ such that there exists a map $g : F^- \to E^-$ satisfying

$$\forall x \in E^+\, \forall y' \in F^- \quad \langle f(x), y'\rangle_F = \langle x, g(y')\rangle_E\,.$$

The map g is then easily seen to be uniquely determined by this property, and to be a bounded linear map $F^- \to E^-$; it is called the *transpose* of f and denoted as ${}^{\mathrm{t}}f$.

Given two PCSs X and Y and a matrix $w \in \mathsf{e}(X \multimap Y)$, it is clear that the map $\mathsf{fun}(w) : \mathsf{e}X \to \mathsf{e}Y$ is a BCS morphism from $\mathsf{cbs}(X)$ to $\mathsf{cbs}(Y)$. It is clear moreover that the operation $w \mapsto \mathsf{fun}(w)$ is functorial. A natural question is whether this functor is full, and the answer is negative, as shown by the following counter-example derived from [Ehr02].

4.2.1 A counter-example. Let $X = 1^{\mathbb{N}^+}$ and $Y = 1^{(\mathbb{N}^+)} = X^\perp$, so that $\mathsf{e}X = l^\infty(\mathbb{N}^+)$ and $\mathsf{e}Y = l^1(\mathbb{N}^+)$.

For each $p \in \mathbb{N}$, let $(P_j^{(p)})_{j=1,\dots,4^p}$ be an enumeration of $\mathcal{P}(\{1,\dots,2p\})$. Let $\varphi : \mathbb{N} \to \mathbb{N}^+$ be the function defined by $\varphi(p) = 4^0 + \cdots + 4^p = \frac{4^{p+1}-1}{3}$. We extend this function as a function from $\mathbb{N} - 1$ to $\mathbb{N}$ by setting $\varphi(-1) = 0$. Then we have, for all $p \in \mathbb{N}$,

$$\varphi(p) - \varphi(p-1) = \#\mathcal{P}(\{1,\dots,2p\})\,.$$

For any $j \in \mathbb{N}^+$, there is a uniquely determined $p \in \mathbb{N} - 1$ such that $\varphi(p) + 1 \leq j < \varphi(p+1)$. This p will be denoted as $\psi(j)$. We have $\psi(1) = 0$, $\psi(2) = \cdots = \psi(5) = 1$, $\psi(6) = \cdots = \psi(21) = 2$ etc.

Let $A \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$ be the matrix defined by

$$A_{i,j} = \begin{cases} \frac{1}{p^2 4^p} & \text{if } 1 \leq i \leq 2p \text{ and } i \in P_j^{(p)}, \text{ where } p = \psi(j) \\ \frac{-1}{p^2 4^p} & \text{if } 1 \leq i \leq 2p \text{ and } i \notin P_j^{(p)}, \text{ where } p = \psi(j) \\ 0 & \text{otherwise} \end{cases}$$

Then $|A| \notin \mathsf{P}(X \multimap Y)$ since, denoting by u the element of $\mathsf{P}X = \mathsf{P}Y^\perp$ which is defined by $\forall i\, u_i = 1$, we have

$$\begin{aligned} \langle |A| \cdot u, u \rangle &= \sum_{j=1}^{\infty} \frac{1}{\psi(j)^2 4^{\psi(j)}} \sum_{1 \leq i \leq 2\psi(j)} 1 \\ &= \sum_{j=1}^{\infty} \frac{2}{\psi(j) 4^{\psi(j)}} \\ &= \sum_{p=1}^{\infty} \sum_{\psi(j)=p} \frac{2}{\psi(j) 4^{\psi(j)}} \\ &= \sum_{p=1}^{\infty} 4^p \frac{2}{p 4^p} = \infty \end{aligned}$$

since $\#\{j \mid \psi(j) = p\} = 4^p$. This shows that $\varepsilon A \notin \mathsf{e}(X \multimap Y)$, for all $\varepsilon > 0$.

On the other hand, let $u \in \mathsf{B}X$, that is $u \in \mathbb{R}^{\mathbb{N}^+}$ with $\forall i \ \ |u_i| \leq 1$. Then we have

$$\begin{aligned} \|A \cdot u\|_Y &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} A_{i,j} u_i \right| \\ &= \sum_{p=1}^{\infty} \sum_{\psi(j)=p} \left| \sum_{i=1}^{\infty} A_{i,j} u_i \right| \\ &= \sum_{p=1}^{\infty} \frac{1}{p^2 4^p} \sum_{\psi(j)=p} \left| \sum_{i=1}^{2p} \eta(i,j,p) u_i \right| \end{aligned}$$

where $\eta(i,j,p) = 1$ if $i \in P_j^{(p)}$ and $\eta(i,j,p) = -1$ if $i \notin P_j^{(p)}$. Let $I_p = \{1, \ldots, 2p\}$ and let

$$\begin{aligned} S_p &= \sum_{\psi(j)=p} \left| \sum_{i=1}^{2p} \eta(i,j,p) u_i \right| \\ &= \sum_{J \subseteq I_p} \left| \sum_{i \in J} u_i - \sum_{i \in I_p \setminus J} u_i \right| . \end{aligned}$$

Using the fact that, for any $a \in \mathbb{R}$, the function $x \mapsto |x - a| + |x + a|$ is monotone on $\mathbb{R}^+$, we get

$$\begin{aligned} S_p &\leq \sum_{J \subseteq I_p} |\#J - \#I_p \setminus J| \\ &= \sum_{k=0}^{2p} \binom{2p}{k} |2k - 2p| = 4 \sum_{k=0}^{p} (p-k) \binom{2p}{k} = 2p \binom{2p}{p} . \end{aligned}$$

The last equality can be proved using e.g. the fact that $k\binom{2p}{k} = (2p - k + 1)\binom{2p}{k-1}$ for $k = 1, \ldots, 2p$. By Stirling formula, there is a constant $C > 0$ such that $p\binom{2p}{p} \sim C 4^p \sqrt{p}$ and hence there is a constant $C > 0$ such that $\forall p \ p\binom{2p}{p} \leq C 4^p \sqrt{p}$. Let $D = \sum_{p=1}^{\infty} \frac{1}{p\sqrt{p}} < \infty$. We have $\|A \cdot u\|_Y \leq CD$ for all $u \in \mathsf{B}X$. Let

$f : \mathsf{e}X \to \mathsf{e}Y$ be the map defined by $f(u) = A \cdot u$, we have shown that f is a bounded linear map, and that f is a morphism from $\mathsf{cbs}(X)$ to $\mathsf{cbs}(Y)$.

If there were some $w \in \mathsf{e}(X \multimap Y)$ such that $f = \mathsf{fun}(w)$, we would necessarily have $w = A$, so such a w cannot exist, and we have shown that the mapping $w \mapsto \mathsf{fun}(w)$ is not surjective onto the space of morphisms from $\mathsf{cbs}(X)$ to $\mathsf{cbs}(Y)$.

4.2.2 Using partially ordered Banach spaces? This counter-example shows that positivity must play an essential role if we want to have a more abstract characterization of morphisms of PCSs, as bounded linear maps between Banach spaces. Fortunately, there are fairly standard notions of *partially ordered Banach space* that seem quite suitable to this goal. Another approach could consist in using the notion of *Riesz space*, which are partially ordered real vector spaces where any two elements have a lub.

A partially ordered Banach space is a Banach space E equipped with a *positive cone*, that is, a set $C \subseteq E$ such that

- $0 \in C$;
- $\lambda x + \mu y \in C$ as soon as $x, y \in C$ and $\lambda, \mu \in \mathbb{R}^+$;
- and if $x, y \in C$ and $x + y = 0$, then $x = y = 0$.

The reason for the terminology is that one defines a partial order relation on E by setting $x \leq y$ iff $y - x \in C$. Of course, a *positive* linear map from a partially ordered Banach space E to a partially ordered Banach space F is a linear map which sends the positive cone of E in the positive cone of F.

It is clear that, for any PCS X, the Banach space $\mathsf{e}X$ is equipped with such a cone C (the elements x of $\mathsf{e}X$ such that $\forall a \in |X|\, x_a \geq 0$), which is moreover closed and generating (that is $C - C = \mathsf{e}X$), and additional properties are satisfied, relating the norm of $\mathsf{e}X$ and the cone.

The next step would be now to introduce *partially ordered CBSs* (CBSs where both Banach spaces are equipped with positive cones, satisfying suitable axioms, still to be discovered) so that the obtained category be a model of linear logic, and so that the PCS morphisms from a PCS X to a PCS Y be in bijective correspondence with the positive continuous linear maps between the associated partially ordered CBSs. This is of course very reminiscent of Peter Selinger's idea of using positive cones for modelling quantum computations [Sel04], with the difference that we can use the already developed theory of probabilistic coherence spaces for developing this new theory (we think especially of the interpretation of the exponentials which is not addressed in Selinger's work, as far as we know).

These investigations are postponed to further work.

Conclusion

We have developed a model of linear logic suggested by Girard and based on probabilistic coherence spaces. This model provides an interpretation of types by structures which give rise to continuous domains. The morphisms between these domains in the associated cartesian closed category are Scott-continuous, but not all Scott-continuous maps are morphisms: the morphisms are "analytical" in a precise sense. In sharp contrast with other models based on analytic maps such as the finiteness space and Köhte space models of the second author [Ehr02, Ehr05], the probabilistic coherence space model admits fixpoint operators and hosts models of the pure lambda-calculus. We have also provided a probabilistic account of the interpretation of terms in this model, considering an extension of the PCF purely functional language by a probabilistic choice construction: we proved that the denotational semantics of a closed term of base type is the sub-probability distribution describing its probability to reduce to a given value.

We plan to generalize this result for understanding the meaning of the denotation of closed terms of higher types. We also would like to describe more abstractly the objects of our model, using the fact that they naturally give rise to Banach spaces.

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