1-qubit versus 2-qubit measurement based quantum computing

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Abstract

We present a comprehensive and universal quantum computing model based on 2-qubit measurements (TQC), which embeds trivially in a model based on 1-qubit measurements known as the one-way model (1WQC). One inherits through this embedding a calculus whereby each TQC computing pattern can be put in a standard form where entanglement is done first, then measurement, then local corrections.

1 Introduction

The basic quantum computing toolkit consists of unitary transformations and measurements acting on a finite-dimensional complex inner product space. Up until recently, quantum information processing has focussed on unitary transformations as the key ingredient. The measurements have been regarded

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as secondary or, more precisely, thought of as ways of extracting classical information from a quantum system. The general view was that one would only invoke them at the end of the computation [NC00].

The recent measurement based models - perhaps influenced by the teleportation protocol - where a 2-qubit Bell measurement is performed to propagate information, have taken a radically different stance towards quantum information processing [GC99, Nie03, RB02, RBB03, Leu04, AL04, CLN04, JP04, PJ04, DKP04a]. In these models, measurements become an integral part of the computational process. Among these recent models, the *one-way model* [RB02, RBB03] (1WQC) stands out as particularly interesting, since it is based only on 1-qubit measurements: therefore both mathematically as well as physically it works with the most basic operations. Another class of measurement-based models - more recent, in fact, than 1WQC - uses 2-qubit measurements. These are collectively referred to as *teleportation models* [Leu04] (TQC).

Several papers that are concerned with the relation and possible unification of these models [CLN04, AL04, JP04] have already appeared. And we will discuss closely and compare the one [AL04] that stands the closest to the matter of the present paper, right at the end of the introduction. One aspect of these models that stands in the way of a complete understanding of this relation, is that, whereas in the one-way model one has a clearly identified class of measurements, there is less agreement concerning which measurements are allowed in teleportation models.

In the present paper, we propose to take as our class of 2-qubit measurements a family obtained as the conjugate under the operator $\wedge Z$ (read controlled-Z) of tensors of 1-qubit measurements. Based on recent results [DKP04b], we show that the resulting teleportation model is universal. Moreover, almost by construction, it embeds into the one-way model, and thus exposes completely the relation between the two models.

This buys us two further things. First, we get a workable syntax for handling the dependencies of operators on previous measurement outcomes, just by mimicking the one obtained in the 1-qubit measurement case [DKP04a]. Second - and this is the main point of the present paper - one can use this embedding to transfer the measurement calculus previously developed for the 1-qubit case to obtain a 2-qubit measurement calculus. The resulting calculus consists of a set of equations over 2-qubit measurement patterns such that any pattern can be rewritten into the standard entanglement, measurement, correction sequence. This procedure, called *standardisation*, can be easily automated, and completely solves the question of how corrections should be propagated when patterns are composed, be it in the 1-qubit case, or in the 2-qubit case. Note that this procedure demands to introduce in the model a notion of dependent measurements which is familiar from the one-way model, but was never considered in the context of teleportation.

Might one conclude from this that the 2-qubit measurement-based models are subsidiary to the 1-qubit measurement based models ? Well, not quite. What we show is that a *certain* class of teleportation models is indeed ancillary to the one-way model, and that all the theory of the one-way model transfers seamlessly to these models. The 2-qubit measurements considered here, although they are quite general and universal, are still a fragment of what is possible. And there might be larger or different classes that are not amenable to the treatment proposed here.

Nevertheless, the strong structural tie between our teleportation model and the one-way model clarifies one aspect of the relation between 2-qubit and 1-qubit measurement-based computing models, and exposes the one-way model as the "éminence grise" behind the teleportation model. Finally, we might add that the approach taken here generalizes straightforwardly to n-qubit measurement based models.

Related Work. As said, the idea that there is a strong connection with TQC and 1WQC is not new. In a recent paper [AL04], Aliferis and Leung exhibit a local mapping between TQC and 1WQC. It is perhaps important here to stress the distinctive contributions of our approach. There is one essential difference, namely that their mapping is based on a selected set of patterns, implementing the Pauli rotations R_x, R_z, and $\wedge X$, and needs some craft. Our mapping is trivial and general, because it is based on commands. Specifically, the only thing to do is to replace 2-qubit measurements $M_{ij}^{\alpha\beta}$ in our class, with an entanglement followed by two 1-qubit measurements $M_i^{\alpha} M_j^{\beta} \wedge Z_{ij}$. (We also provide a trivial backward embedding from 1WQC to TQC by introducing dummy qubits).

As a consequence the definition and proof of the pattern based embedding needs some three pages, while the command based needs just the line above, and its obvious justification. One smaller difference is that we get rid of the fact that TQC uses $\wedge X$ as a primitive while 1WQC uses $\wedge Z$, just by rephrasing TQC with $\wedge Z$. This is a minor point, but a one that helps in simplifying further the translation.

Not only is the command based embedding simpler, but it also works for any pattern, be it an implementation of a unitary or not. Conceptually, the pattern based mapping is based on circuits, the command based one is not, it is intrinsic to the measurement based models. One could say the latter considers things locally.

The absence of a local view on measurement patterns can actually be traced back to the original paper on 1WQC [RB02], which uses a pattern based propagation of corrections, whereas we use a command based propagation, given by the measurement calculus. This local treatment also makes the use of stabiliser methods unnecessary.

Finally, another difference, which again can be traced back to the original paper, is the input output overlap question. If one wants to use clusters states as an implementation, then it could be convenient to have all patterns have disjoint sets of inputs and outputs. At the level of our analysis, it does not seem necessary to restrict to cluster states, and using arbitrary graphs allows for input output overlap.

Though this paper is mostly self-contained, some familiarity with our recent paper [DKP04a] developing the measurement calculus for the one-way model might be helpful.

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2 Disentangling 2-qubit measurements

2.1 Notations

We write X, Z and $P(\alpha)$ for the Pauli and phase operators:

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad P(\alpha) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

and also use the commonplace notation:

$$\begin{aligned} |+\rangle &:= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &:= \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ |+_\alpha\rangle &:= \tfrac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle) = P(\alpha)(|+\rangle) \\ |-_\alpha\rangle &:= \tfrac{1}{\sqrt{2}}(|0\rangle - e^{i\alpha}|1\rangle) = P(\alpha)(|-\rangle) \end{aligned}$$

In what follows we will write M_i^α for a 1-qubit xy-measurement on the ith qubit, *i.e.*, a projection of qubit i into the basis $|\pm_\alpha\rangle$. We will also use the following abbreviations:

$$M_i^x := M_i^0$$
$$M_i^y := M_i^{\frac{\pi}{2}}$$

Given U a unitary over $\mathbb{C}^2$, one defines $\wedge U$ (read controlled-U) a unitary on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{array}{rcl}\wedge U|0\rangle|\psi\rangle & := & |0\rangle|\psi\rangle \\ \wedge U|1\rangle|\psi\rangle & := & |1\rangle U(|\psi\rangle)\end{array}$$

2.2 Preliminary discussion

In teleportation models, one uses certain 2-qubit measurements. Such measurements are given by an orthonormal basis $\mathcal{A}$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$, or alternatively by a set of four orthogonal rank 1 projections in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

As an example consider the *Bell basis*, $\mathcal{B}$, given below which defines a complete 2-qubit measurement often called the *Bell measurement*, $M^{\mathcal{B}}$:

$$\begin{array}{l}\Phi_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ \Phi_{01} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ \Phi_{10} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ \Phi_{11} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\end{array}$$

Another example, which plays a central role in this paper, is the *Graph measurement*, $M^{\mathcal{G}}$, which is defined by the *Graph basis*, $\mathcal{G}$:

$$\begin{array}{l}\Gamma_{00} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\ \Gamma_{01} = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle) \\ \Gamma_{10} = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \\ \Gamma_{11} = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle)\end{array}$$

Both the Bell states and the Graph states are families of maximally entangled states, and are actually locally equivalent.

As it happens, both these bases can be obtained from disentangled bases by applying a simple unitary belonging to the Clifford group:

$$\begin{array}{rcl}\mathcal{B} & = & \wedge X\{|\pm\rangle \otimes |0/1\rangle\} \\ \mathcal{G} & = & \wedge Z\{|\pm\rangle \otimes |\pm\rangle\}\end{array}$$

This means that both Bell and Graph measurements, can be seen as 1-qubit measurements if one conjugates them by a well-chosen element of the Clifford group. These 2-qubit measurements are in fact tensors of 1-qubit measurements up to a Clifford operator, and in some sense, this means that they can be completely understood in the 1-qubit world. This situation generalizes well to a family of 2-qubit measurements that we described now, and which will lead naturally to the formulation of our specific teleportation model.

2.3 A family of 2-qubit measurements

Before embarking on the specifics of our family of 2-qubit measurements, we remark that the situation commented above is more general:

Lemma 1 *Let $\mathcal{A}$ be an orthonormal basis in $\otimes^n \mathbb{C}^2$, with associated n-qubit measurement $M^{\mathcal{A}}$, and $\mathcal{A}_i$ with $i = 1, \ldots, n$ be orthonormal bases in $\mathbb{C}^2$, with associated 1-qubit measurements $M^{\mathcal{A}_i}$. Then there exists a unique (up to a permutation) n-qubit unitary operator U such that:*

$$M^{\mathcal{A}}_{1\cdots n} \quad = \quad U_{1\cdots n}(\otimes_i M^{\mathcal{A}_i})U^{\star}_{1\cdots n} \tag{1}$$

Take U to map $\otimes_i \mathcal{A}_i$ to $\mathcal{A}$. $\square$

This simple lemma says that general n-qubit measurements can always be seen as conjugated 1-qubit measurements, provided one uses the appropriate unitary to do so. In particular any 2-qubit measurement can be obtained by conjugating two 1-qubit measurements by a well-chosen unitary operator. We have seen already two instances of this general situation in the preliminary discussion. When $n = 2$, and $U = \wedge Z$, one gets:

$$M^{\mathcal{G}}_{12} \quad = \quad \wedge Z_{12}(M^x_1 \otimes M^x_2) \wedge Z_{12} \tag{2}$$

and when $U = \wedge X$:

$$M^{\mathcal{B}}_{12} \quad = \quad \wedge X_{12}(M^x_1 \otimes M^z_2) \wedge X_{12} \tag{3}$$

where M^z is the 1-qubit measurement in the computational basis $|0\rangle$, $|1\rangle$.
Note how similar these last equations are. This clearly suggests that everything that will be done here can be transferred to the case where X replaces Z, and $\mathcal{B}$ replaces $\mathcal{G}$. However, since the methodology that we adopt is to embed the 2-qubit measurement based model in the one-way model, and the

latter is based on Z and $\mathcal{G}$, we use only the first instance in this paper. Besides, since $\wedge Z$ is symmetric, whereas $\wedge X$ (*a.k.a.* CNOT) is not, the algebra is nicer to work with.

It is now natural to extend our definition of $M_{12}^{\mathcal{G}}$ to obtain the family of 2-qubit measurements of interest:

$$M_{12}^{\alpha,\beta} \;:=\; \wedge Z(M_1^{\alpha} \otimes M_2^{\beta}) \wedge Z \tag{4}$$

corresponding to projections on the basis:

$$\mathcal{G}_{\alpha,\beta} \;:=\; \wedge Z(P(\alpha) \otimes P(\beta))(\{|\pm\rangle \otimes |\pm\rangle\}) \tag{5}$$

We prove below that this family of measurements leads to a universal model, which embeds nicely into the one-way model, but first we need to describe an important notion of dependency over measurements, which was never considered before in the existing teleportation models.

3 A notation for dependent measurements

We introduced in a recent work [DKP04a] a notation for computing patterns in the one-way model. That same notation can be easily extended to to include 2-qubit measurements as well. One has three types of quantum commands reflecting the three kinds of operations supported in the teleportation model: *entanglement operators* E, *measurements* M, and *corrections* C (Pauli operators). The commands of the M and C types may be parameterized by signals, which are expressions depending on the outcomes (or observations) generated by previous measurements. This is where we need to be precise about the notation.

In the following V is a finite set indexing the computation space $\mathfrak{H}_V := \otimes_V \mathbb{C}^2$. *Signals* are $s ::= 0, 1, s(i), s + s$, where $i \in V$. Angles are written α with $\alpha \in [-\pi, \pi]$. Expressions involving angles are always evaluated modulo 2π. As a group, $\mathbb{Z}_2$ defines the following two actions on $[-\pi, \pi]$:

$$\begin{array}{lllllll} 0.\alpha & = & \alpha & \quad & 1.\alpha & = & -\alpha \\ 0 * \alpha & = & \alpha & \quad & 1 * \alpha & = & \alpha + \pi \end{array}$$

which will be taken as the meaning of $i.\alpha$, and $i * \alpha$, for $i \in \mathbb{Z}_2$. These actions commute since $-\alpha + \pi \equiv_{2\pi} -\alpha - \pi$. One might wonder why we take these

two particular actions: we prove later, they correspond to conjugations of measurements under local Pauli corrections.
Commands come in three types:

$$\begin{array}{rcll} A & ::= & E_{ij} & \text{entanglement} \\ & & {}^{(u,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)} & \text{measurement} \\ & & X_i^s, Z_i^s & \text{correction} \end{array}$$

with i, $j \in V$, s, t, u, v signals, α and β angles.
Indices i, j represent the qubits on which the commands apply. A command is said to be a V-command if (1) the qubits to which it applies are in V, and (2) the qubits occurring in its signals are also in V.
We present now the interpretation of each type of command.
The *entanglement* command, written E_{ij}, builds the quantum computing resource on the computation space and we have $E_{ij} = \wedge Z_{ij}$. Note that since the $\wedge Z$ operators are diagonal, all the E_{ij} commands commute with each other. The reason of choosing this particular entangling operator will be justified when we describe the standardisation procedure.
Non-dependent *measurements* $M_{ij}^{\alpha,\beta}$ were described in the preceding section. We now have to deal with measurement outcomes and dependencies. When one applies a measurement at qubits i and j, the underlying system chooses one of the four projections with a certain probability, and one can observe *which* was chosen. This is what is called the *outcome* of the measurement. We write $(s(i), s(j)) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ to represent such an outcome, with the specific convention that $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, correspond respectively to the cases where the state collapses to $\wedge Z|+_\alpha\rangle|+_\alpha\rangle$, $\wedge Z|+_\alpha\rangle|-_\alpha\rangle$, $\wedge Z|-_\alpha\rangle|+_\alpha\rangle$, and $\wedge Z|-_\alpha\rangle|-_\alpha\rangle$. This is the same convention that we used for labeling of the $\mathcal{G}$ basis.
Importantly, measurements are allowed to depend on signals, and therefore indirectly on the outcomes of preceding measurements. We will use two types of dependencies for measurements associated with actions $s.\alpha$ and $s * \alpha$:

$$\begin{array}{rcccl} [M_{ij}^{\alpha,\beta}]^{(s,t)} & = & M_{ij}^{s.\alpha,t.\beta} & = & M_{ij}^{(-1)^s\alpha,(-1)^t\beta} \\ {}^{(u,v)}[M_{ij}^{\alpha,\beta}] & = & M_{ij}^{u*\alpha,v*\beta} & = & M_{ij}^{\alpha+u\pi,\beta+v\pi} \end{array}$$

where s, t, u and v are in $\mathbb{Z}_2$. As noted above, the two actions $s.\alpha$ and $u * \alpha$ commute, so the equations above define unambiguously the full-dependent measurement ${}^{(u,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)}$.

Here are some useful abbreviations:

$$\begin{array}{l}
{}^{(0,0)}[M^{\alpha,\beta}]^{(s,t)} := [M^{\alpha,\beta}]^{(s,t)} \\
{}^{(u,v)}[M^{\alpha,\beta}]^{(0,0)} := {}^{(u,v)}[M^{\alpha,\beta}] \\
{}^{(0,0)}[M^{\alpha,\beta}]^{(0,0)} := M^{\alpha,\beta} \\
M^{\alpha,x} := M^{\alpha,0} \\
M^{\alpha,y} := M^{\alpha,\frac{\pi}{2}}
\end{array}$$

The *corrections* X_i^s and Z_i^s stand for unary Pauli operators depending on the signal s and applied at qubit i. The interpretation is simply given by:

$$\begin{array}{l}
X_i^0 = Z_i^0 = I \\
X_i^1 = X_i \\
Z_i^1 = Z_i
\end{array}$$

4 Patterns

The definition of patterns, pattern combinations, and the various conditions one might require of a pattern are taken from previous work on the one-way model [DKP04a]. The fact that we are using here 2-qubit measurements is not changing anything at this stage.

Definition 2 *Patterns consist of three finite sets V, I, O, together with two injective maps $\iota : I \to V$ and $o : O \to V$ and a finite sequence of commands $A_n \dots A_1$ applying to qubits in V.*

The set V is called the pattern *computation space*, and we write $\mathfrak{H}_V$ for the associated quantum state space $\otimes_{i \in V} \mathbb{C}^2$. To ease notation, we will forget altogether about the maps ι and o, and write simply I, O instead of $\iota(I)$ and $o(O)$. Note however, that these maps are useful to define classical manipulations of the quantum states, such as permutations of the qubits. The sets I, O will be called respectively the pattern *inputs* and *outputs*, and we will write $\mathfrak{H}_I$, and $\mathfrak{H}_O$ for the associated quantum state spaces. The sequence $A_n \dots A_1$ will be called the pattern *command sequence*.

As in the one-way model, one prepares the input qubits in some input state $\psi \in \mathfrak{H}_I$, while the non-input qubits are all set in the $|+\rangle$ state, then the commands are executed in sequence, and finally the result of the pattern computation is some $\phi \in \mathfrak{H}_O$. There might be qubits in the pattern, which are neither inputs nor outputs qubits, and are used as auxiliary qubits during

the computation. As said earlier, one does not require inputs and outputs to be disjoint subsets of V.
Patterns can be combined together, either by plain composition (serial combination), or by tensoring (parallel combination). We refer the reader to our paper on the one-way model for detailed explanations [DKP04a].
Finally, patterns might satisfy various conditions:

(D) no command mentions a signal not yet measured;
(M0) a qubit i is measured iff i is not an output;
(M1) a qubit i is measured at most once;
(M2) a measured qubit i is not entangled thereafter;
(EMC) commands occurs in the order of Es first, then Ms, then Cs;

and it is easily seen that these conditions are preserved under serial and parallel combination. We define condition (M) to be the conjunction of (M0), (M1) and (M2).

Example. Here is a simple example to implement teleportation in our teleportation model (which indeed ought to be simple!):

$$\mathfrak{T} \quad := \quad (\{1,2,3\}, \{1\}, \{3\}, X_3^{s_1} Z_3^{s_2} M_{12}^{x,x} E_{23})$$

What is this pattern doing ? First the last two qubits, prepared in state $|+\rangle$, are entangled to obtain $|\psi\rangle_1 \otimes (|00\rangle + |01\rangle + |10\rangle - |11\rangle)$, where ψ is some input. Then the first two qubits are measured with the plain graph measurement $M_{12}^{x,x}$. Finally an XZ correction is applied on the third qubit, depending on the two outcomes of the measurement. Computation ends in state ψ_3. In the Alice and Bob narrative, Alice's qubits are 1 and 2, and Bob's only qubit is 3. After running the protocol, the state ψ has been 'teleported' from Alice to Bob. This teleportation example satisfies all the conditions given above.
Patterns not respecting the EMC condition will be called *wild*. In the next section we exhibit a simple rewriting procedure to obtain an equivalent standard EMC pattern for any given wild pattern. We call this procedure *standardisation*. The reasons to demand a result like standardisation are the following. The most important one is that one can easily prepare the entire entangled state that we will need during the computation of the pattern, instead of entangling individual qubits on the fly [CAJ04]. As a result one will need to perform fewer corrections: only the final corrections on the output qubits. One other important aspect of standardisation is that it gives the possibility of parallelizing the computation. In a wild pattern one is forced

to compute sequentially and in the order given by the commands sequence. Whereas after standardising the dependency structure will come to the front and one can perform measurements in this new order that might have a shorter depth. We recommend [RBB03, DKP04a] for more detailed discussion on why standardisation is an essential ingredient of the measurement-based model.

5 Pattern rewriting

Now that we have our language in place, and the appropriate notion of pattern, we define a set of local pattern rewriting rules, able to put any given wild pattern in the EMC form.
We first describe the *EC* rewriting rules used to commute corrections and entanglements acting on a same qubit.

Lemma 3 (*EC*) *One has for all i, j, s:*

$$E_{ij} X_i^s = X_i^s Z_j^s E_{ij} \tag{6}$$
$$E_{ij} Z_i^s = Z_i^s E_{ij} \tag{7}$$

The proof is an easy calculation.
Note that, the operators E_{ij} belong to the Clifford group, and therefore map the Pauli group to itself under conjugation. On the other hand if we took any other general entangling operator outside the normalizer of the Pauli group (*i.e.* the Clifford group) then after commuting E_{ij} corrections will no longer belong to the simple Pauli group and may grow exponentially as they propagate. However it is still an open question whether one can change these operators and get as simple an algebra as the one we present here.
We also need a second set of rewriting rules for commuting corrections and measurements acting on a same qubit. There are two cases depending on whether the incoming Pauli correction is of the X type or of the Z type.

Lemma 4 (X correction) *One has for all i, j, α, β, s:*

$$M_{ij}^{\alpha,\beta} X_i^s = X_i^{s(0,s)} [M_{ij}^{\alpha,\beta}]^{(s,0)} \tag{8}$$
$$M_{ij}^{\alpha,\beta} X_j^s = X_j^{s(s,0)} [M_{ij}^{\alpha,\beta}]^{(0,s)} \tag{9}$$

We know from results obtained in the 1-qubit measurement case [DKP04a]:

$$\begin{aligned} M_i^\alpha X_i^s &= X_i^s [M_i^\alpha]^s & (10) \\ M_i^\alpha Z_i^s &= Z_i^s {}^s[M_i^\alpha] & (11) \end{aligned}$$

where $[M_i^\alpha]^s = M_i^{(-1)^s \alpha}$ and ${}^s[M_i^\alpha] = M_i^{s\pi+\alpha}$.
On the other hand from Lemma 1 we have:

$$M_{ij}^{\alpha,\beta} = E_{ij}(M_i^\alpha \otimes M_j^\beta)E_{ij}.$$

Now combining the above equations with the EC equation given above, and using $(Z_j^s)^2 = I$, we obtain:

$$\begin{aligned} E_{ij}M_{ij}^{\alpha,\beta}X_i^s &= (M_i^\alpha \otimes M_j^\beta)E_{ij}X_i^s \\ &= (M_i^\alpha \otimes M_j^\beta)X_i^s Z_j^s E_{ij} \\ &= X_i^s Z_j^s([M_i^\alpha]^s \otimes {}^s[M_j^\beta])E_{ij} \\ &= X_i^s Z_j^s E_{ij} {}^{(0,s)}[M_{ij}^{\alpha,\beta}]^{(s,0)} \\ &= E_{ij} X_i^s {}^{(0,s)}[M_{ij}^{\alpha,\beta}]^{(s,0)} \end{aligned}$$

The other equation is obtained in a similar way. □

Lemma 5 (Z correction) *One has for all i, j, α, β, s:*

$$\begin{aligned} M_{ij}^{\alpha,\beta} Z_i^s &= Z_i^s {}^{(s,0)}[M_{ij}^{\alpha,\beta}] & (12) \\ M_{ij}^{\alpha,\beta} Z_j^s &= Z_j^s {}^{(0,s)}[M_{ij}^{\alpha,\beta}] & (13) \end{aligned}$$

We prove the equations again using Lemma 1 and the same previous results established in the 1-qubit measurement case:

$$\begin{aligned} E_{ij}M_{ij}^{\alpha,\beta}Z_i^s &= (M_i^\alpha \otimes M_j^\beta)E_{ij}Z_i^s \\ &= (M_i^\alpha \otimes M_j^\beta)Z_i^s E_{ij} \\ &= Z_i^s({}^s[M_i^\alpha] \otimes M_j^\beta)E_{ij} \\ &= Z_i^s E_{ij} {}^{(s,0)}[M_{ij}^{\alpha,\beta}] \\ &= E_{ij} Z_i^s {}^{(s,0)}[M_{ij}^{\alpha,\beta}] \end{aligned}$$

□

5.1 The measurement calculus

Under the assumption that the pattern satisfies condition (M), we can further simplify the MX and MZ equations. Indeed once a qubit is measured, we know it will neither be re-entangled (M2), nor will it be re-measured again (M1), nor will it be part of the output (M0).

This is equivalent to requiring destructive measurements; the justification for this condition is that all the auxiliary qubits are there to act only as a communication channel and once they are measured they have performed their task in the computation and one needs to keep only the classical outcomes for the next commands that are dependent on this measurement. In other words, after projection and propagating the information they will act as a classical control over the rest of the computation. This means that the corrections can be dropped without changing the pattern output.

Hence, under condition (M), we obtain the following pattern rewriting rules:

$$\begin{array}{lcll}
E_{ij}X_i^s & \Rightarrow & X_i^s Z_j^s E_{ij} & EX \\
E_{ij}Z_i^s & \Rightarrow & Z_i^s E_{ij} & EZ \\
{}^{(u,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)} X_i^r & \Rightarrow & {}^{(u,v+r)}[M_{ij}^{\alpha,\beta}]^{(s+r,t)} & MX \\
{}^{(u,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)} X_j^r & \Rightarrow & {}^{(u+r,v)}[M_{ij}^{\alpha,\beta}]^{(s,t+r)} & MX \\
{}^{(u,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)} Z_i^r & \Rightarrow & {}^{(u+r,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)} & MZ \\
{}^{(u,v)}[M_{ij}^{\alpha,\beta}]^{(s,t)} Z_j^r & \Rightarrow & {}^{(u,v+r)}[M_{ij}^{\alpha,\beta}]^{(s,t)} & MZ
\end{array}$$

to which we add also the trivial commutation rewriting which are possible between commands that don't overlap (meaning, acting on disjoint sets of qubits). This constitutes the set of directed equations that we call the 2-*qubit measurement calculus*.

It is then routine to prove that (1) all pattern conditions are preserved under these rewriting, and that (2) this system drives in finitely many steps a pattern to a *unique* EMC form, no matter in which order the rewriting rules are applied [1].

6 Universality

In this section we prove universality of wild patterns as well as universality of the smaller class of (EMC) patterns. We provide a simple generating set

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[1] This is called a confluent and strongly standardizing rewriting system in rewriting theory terminology.

for unitary operators, and in so doing justify further the choice of our class of measurements $M^{\alpha,\beta}$ for the teleportation model. (A first justification is given by the existence of a standardisation procedure.)

Lemma 6 *The following patterns are universal:*

$$\begin{aligned} \mathfrak{J}(\alpha) &:= (\{i,j,k\},\{i\},\{k\},X_k^{s_i}M_{ij}^{\alpha,x}E_{ij}E_{ik}) && (14)\\ \wedge\mathfrak{Z} &:= (\{i,j\},\{i,j\},\{i,j\},E_{ij}) && (15) \end{aligned}$$

We know that the set $\{\wedge Z, J(\alpha)\}$ for $\alpha \in [-\pi,\pi]$ is generating all unitaries (under composition and tensorisation) [DKP04b], where $J(\alpha)$ is:

$$J(\alpha) \quad := \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{i\alpha} \\ 1 & -e^{i\alpha} \end{pmatrix}$$

Now we prove the $\mathfrak{J}(\alpha)$ pattern implements the unitary operator $J(\alpha)$. From Lemma 1 we have:

$$X_k^{s_i}M_{ij}^{\alpha,x}E_{ij}E_{ik} = X_k^{s_i}E_{ij}M_i^{\alpha}M_j^{x}E_{ik}$$

where we can drop the E_{ij} command acting after the measurements to obtain the following pattern:

$$X_k^{s_i}M_i^{\alpha}M_j^{x}E_{ik}$$

The M_j^x command has no effect in the pattern since the j qubit is not entangled to the rest and $M_j^x|+\rangle = |+\rangle$ therefore we obtain

$$X_k^{s_i}M_i^{\alpha}M_j^{x}E_{ik} \sim X_k^{s_j}M_i^{\alpha}E_{ik}$$

which is known to implement $J(\alpha)$ [DKP04b]. On the other hand $\wedge\mathfrak{Z}$ pattern implements $\wedge Z$. It then suffices to compose and tensor these two basic patterns to realize all unitaries. □

7 Standardisation and Composing

Having all the required ingredients we now present a few examples to illustrate both the extended notion of module composition and EMC-standardisation.

x-rotation.

We know that $R_x(\alpha) \propto R(\alpha)H$ (where $\propto$ is equality up to a global phase), hence the module $\mathfrak{J}(\alpha)(3,4,5) \circ \mathfrak{H}(1,2,3)$ implements $R_x(\alpha)$ by compositionality. Now we may standardize it:

$$\begin{array}{lll}
\mathfrak{J}(\alpha)(3,4,5) \circ \mathfrak{H}(1,2,3) & = & X_5^{s_3} M_{34}^{-\alpha,x} E_{34} E_{35} X_3^{s_1} M_{12}^{x,x} E_{12} E_{13} \\
& \Rightarrow_{EX} & X_5^{s_3} M_{34}^{-\alpha,x} E_{34} X_3^{s_1} Z_5^{s_1} E_{35} M_{12}^{x,x} E_{12} E_{13} \\
& \Rightarrow_{EX} & X_5^{s_3} M_{34}^{-\alpha,x} X_3^{s_1} Z_4^{s_1} Z_5^{s_1} E_{34} E_{35} M_{12}^{x,x} E_{12} E_{13} \\
& \Rightarrow_{MX} & X_5^{s_3} [M_{34}^{-\alpha,x}]^{(s_1,0)} Z_4^{s_1} Z_5^{s_1} E_{34} E_{35} M_{12}^{x,x} E_{12} E_{13} \\
& \Rightarrow_{MZ} & X_5^{s_3} Z_5^{s_1} {}^{(0,s_1)} [M_{34}^{-\alpha,x}]^{(s_1,0)} M_{12}^{x,x} E_{12} E_{13} E_{34} E_{35} \\
& = & X_5^{s_3} Z_5^{s_1} [M_{34}^{\alpha,x}]^{(s_1+1,0)} M_{12}^{x,x} E_{12} E_{13} E_{34} E_{35}
\end{array}$$

obtaining an EMC pattern for x-rotation. In the next section we show how to obtain exactly the same pattern by embedding the x-rotation pattern of the 1-qubit measurement model in the 2-qubit measurement pattern.

CNOT ($\wedge X$).

This is an example with two inputs and two outputs. We need here the *trivial* module $\mathfrak{I} = (\{1\}, \{1\}, \{1\}, \emptyset)$ with empty command sequence, which implements the identity over $\mathfrak{H}_1$.
One has $\wedge X = (I \otimes H) \wedge Z (I \otimes H)$, so we get a module using 6 qubits over $(\{1,2,3,4,5,6\}, \{1,2\}, \{1,6\})$ (note that inputs and outputs are overlapping on qubit $\{1\}$):

$$\begin{array}{c}
(\mathfrak{I}(1) \otimes \mathfrak{h}(4,5,6)) \wedge \mathfrak{Z}(1,4)(\mathfrak{I}(1) \otimes \mathfrak{h}(2,3,4)) = \\
X_6^{s_4} M_{45}^{x,x} E_{46} E_{45} E_{14} X_4^{s_2} M_{23}^{x,x} E_{24} E_{23}
\end{array}$$

By standardizing:

$$\begin{array}{ll}
X_6^{s_4} M_{45}^{x,x} E_{46} E_{45} E_{14} X_4^{s_2} M_{23}^{x,x} E_{24} E_{23} & \Rightarrow_{EX} \\
X_6^{s_4} M_{45}^{x,x} E_{46} E_{45} X_4^{s_2} Z_1^{s_2} M_{23}^{x,x} E_{14} E_{24} E_{23} & \Rightarrow_{EX} \\
X_6^{s_4} M_{45}^{x,x} E_{46} X_4^{s_2} Z_5^{s_2} Z_1^{s_2} M_{23}^{x,x} E_{14} E_{24} E_{23} E_{45} & \Rightarrow_{EX} \\
X_6^{s_4} M_{45}^{x,x} X_4^{s_2} Z_6^{s_2} Z_5^{s_2} Z_1^{s_2} M_{23}^{x,x} E_{14} E_{24} E_{23} E_{45} E_{46} & \Rightarrow_{MX} \\
X_6^{s_4} Z_6^{s_2} M_{45}^{x,x} Z_5^{s_2} Z_1^{s_2} M_{23}^{x,x} E_{14} E_{24} E_{23} E_{45} E_{46} & \Rightarrow_{MZ} \\
X_6^{s_4} Z_6^{s_2} Z_1^{s_2} M_{45}^{x,x} M_{23}^{x,x} E_{14} E_{24} E_{23} E_{45} E_{46} &
\end{array}$$

In the next section we show how we can simplify the above pattern even more by using a property of the embedding between 1 and 2 qubit measurement.

Note that the above procedures of obtaining bigger patterns from the small building blocks ($\mathfrak{J}(\alpha)$ and $\wedge\mathfrak{Z}$) is automatic *i.e.* one needs only to write the corresponding decomposition of the unitary matrices and then translate it to the corresponding patterns and apply the standardisation which is again an automatic process. This approach simplifies the understanding of the teleportation model and for the first time introduces the notation of the EMC-standard pattern.

In the next section we present a simple embedding map between 1 and 2 qubits measurement models which clarifies where most of the above structures are obtained from.

8 Embedding

We describe how to translate 2-qubit EMC pattern to 1-qubit pattern and vice versa. The following equation which is an special case of Lemma 1 plays the central role in the translation:

$$M_{ij}^{\alpha,\beta} = E_{ij}(M_i^{\alpha} \otimes M_j^{\beta})E_{ij} \tag{16}$$

2-qubit to 1-qubit

We simply use the Equation 16 and then perform a simple optimisation procedure which deletes the unnecessary commands: an entangling command appearing after a measurement commands; a measurement command acting on a qubit which is not entangled to the rest of the qubits; or two consequent entangling commands acting on the same qubits.

Teleportation

Recall the teleportation pattern with 2-qubit measurements command, we perform the above steps:

$$\begin{array}{ll}
X_3^{s_1} Z_3^{s_2} M_{12}^{x,x} E_{23} & \Rightarrow_{Equation} \\
X_3^{s_1} Z_3^{s_2} E_{12} M_1^{x} M_2^{x} E_{12} E_{23} & \Rightarrow_{Optimisation} \\
X_3^{s_1} Z_3^{s_2} M_1^{x} M_2^{x} E_{12} E_{23} &
\end{array}$$

and hence obtain the teleportation pattern with 1-qubit measurement [DKP04a].

CNOT ($\wedge X$)

Consider the standard $\wedge X$ pattern obtained from the compositions:

$$\begin{array}{ll}
X_6^{s_4} Z_6^{s_2} M_{45}^{x,x} M_{23}^{x,x} E_{14} E_{24} E_{23} E_{45} E_{46} & \Rightarrow_{Equation} \\
X_6^{s_4} Z_6^{s_2} E_{45} M_4^x M_5^x E_{45} E_{23} M_2^x M_3^x E_{23} E_{14} E_{24} E_{23} E_{45} E_{46} & \Rightarrow_{Optimisation} \\
X_6^{s_4} Z_6^{s_2} M_4^x M_5^x M_2^x M_3^x E_{14} E_{24} E_{23} E_{23} E_{45} E_{45} E_{46} & \Rightarrow_{Optimisation} \\
X_6^{s_4} Z_6^{s_2} M_4^x M_5^x M_2^x M_3^x E_{14} E_{24} E_{46} & \Rightarrow_{Optimisation} \\
X_6^{s_4} Z_6^{s_2} M_4^x M_2^x E_{14} E_{24} E_{46} &
\end{array}$$

which is again up to a relabeling the same $\wedge X$ pattern with 1-qubit measurement defined in [DKP04a].

1-qubit to 2-qubit

In order to obtain the reverse operation of the above embedding we perform the following steps. First, for each measurement command M_i^α we add a *dummy* measurement command $M_{i_d}^x$ acting on a dummy qubit which has been prepared in the sate $|+\rangle$ and is disentangled from the rest of the qubits. Next we can use Eq. 16 to substitute every measurement command and its corresponding dummy measurement with a 2-qubit measurement command. Finally we run an optimisation procedure.

z-rotation.

We have the following EMC 1-qubit pattern for $R_z(\alpha)$ [DKP04a] which can be embedded to an EMC 2-qubit pattern using the above steps:

$$\begin{array}{ll}
X_3^{s_2} Z_3^{s_1} [M_2^x]^{s_1} M_1^{-\alpha} E_{12} E_{23} & \Rightarrow_{Dummy\ M} \\
X_3^{s_2} Z_3^{s_1} [M_2^x]^{s_1} M_{2_d}^x M_1^{-\alpha} M_{1_d}^x E_{12} E_{23} & \Rightarrow_{Equation} \\
X_3^{s_2} Z_3^{s_1} E_{22_d} [M_{22_d}^{x,x}]^{(s_1,0)} E_{22_d} E_{11_d} M_{11_d}^{-\alpha,x} E_{11_d} E_{12} E_{23} & \Rightarrow_{Optimisation} \\
X_3^{s_2} Z_3^{s_1} [M_{22_d}^{x,x}]^{(s_1,0)} M_{11_d}^{-\alpha,x} E_{11_d} E_{22_d} E_{12} E_{23} &
\end{array}$$

There is no further possible simplification to perform.

x-rotation.

Here we show how the embedding from 1-qubit measurement pattern and composition within 2-qubit measurement model leads to the same pattern for an x-rotation. We start with the following EMC 1-qubit pattern for $R_x(\alpha)$ [DKP04a] and apply again the above steps:

$$\begin{array}{ll}
X_3^{s_2} Z_3^{s_1} [M_2^{\alpha}]^{1+s_1} M_1^{x} E_{23} E_{12} & \Rightarrow_{\mathit{Dummy\ M}} \\
X_3^{s_2} Z_3^{s_1} [M_2^{\alpha}]^{1+s_1} M_{2_d}^{x} M_1^{x} M_{1_d}^{x} E_{12} E_{23} & \Rightarrow_{\mathit{Equation}} \\
X_3^{s_2} Z_3^{s_1} E_{22_d} [M_{22_d}^{\alpha,x}]^{(s_1+1,0)} E_{22_d} E_{11_d} M_{11_d}^{x,x} E_{11_d} E_{12} E_{23} & \Rightarrow_{\mathit{Optimisation}} \\
X_3^{s_2} Z_3^{s_1} [M_{22_d}^{\alpha,x}]^{(s_1+1,0)} M_{11_d}^{x,x} E_{11_d} E_{22_d} E_{12} E_{23} &
\end{array}$$

which up to relabeling of the qubits, is the same pattern of the last section.

CNOT ($\wedge X$)

In the next example we revisit the pattern for $\wedge X$ where through the embedding procedure we can simplify further the obtained pattern. Consider the EMC $\wedge X$ pattern for the 1-qubit measurement [DKP04a]:

$$X_4^{s_3} Z_4^{s_1} Z_2^{s_1} M_3^{x} M_1^{x} E_{34} E_{23} E_{13}$$

By adding the dummy qubits and necessary simplification one obtain the same pattern (up to a relabeling of qubits) which we derived through direct composition in the previous section:

$$X_6^{s_4} Z_6^{s_2} Z_1^{s_2} M_{45}^{x,x} M_{23}^{x,x} E_{14} E_{24} E_{23} E_{45} E_{46}$$

Now using the following relation:

$$M_{kl}^{\gamma,\delta} M_{ij}^{\alpha,\beta} E_{ij} E_{jk} E_{kl} = M_{jk}^{\beta,\delta}$$

which is a simple application of the embedding Equation 16, we obtain a 4 qubits pattern for $\wedge X$ in the 2-qubit measurement model:

$$X_6^{s_4} Z_6^{s_2} Z_1^{s_2} M_{24}^{x,x} E_{14} E_{46}$$

Note that again all the embedding procedures are automatic and can be performed by a classical computer. This is the main advantage of our approach in unifying these two models compared to the extant work. Furthermore the above discussion suggests that we do not gain anything by working in the teleportation model since all the 2-qubit patterns can be re-arranged as simpler 1-qubit patterns. Also note that from the implementation point of view, in order to implement a 2-qubit measurement one needs to first entangle the two qubits and then perform two individual 1-qubit measurements. Hence the 1-qubit measurement model in a sense is a more faithful representation of what actually has to be done.

9 Conclusion

We have constructed a well defined teleportation model, that is a 2-qubit measurement-based quantum computing language. In essence the construction consists in keeping only those measurements that are conjugate of tensors of 1-qubit measurements under the basic entanglement operator controlled-Z.

This construction yields a well-structured model which inherits the properties of the one-way model: clean compositional analysis, a standardisation procedure to reduce composite patterns to the standard entanglement-measurement-correction form, and universality with respect to unitaries. In that sense, it compares well with the traditional approach where one is left wondering why one is working with some specific choice of measurements.

On the other hand, this construction does not results in a smaller than usual teleportation model. Quite the contrary, we have not found in the quantum computing literature any reference to concrete 2-qubit measurements which fall off our particular class. One could draw a quick conclusion, and say that 2-qubit measurement based models considered so far are actually 1-qubit measurement-based models in disguise. Whether this is true of all choices of 2-qubit measurement bases remains to be seen.

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