A model-oriented introduction to differential linear logic

Thomas Ehrhard
CNRS, PPS, UMR 7126
Univ Paris Diderot, Sorbonne Paris Cité
F-75205 Paris, France

August 13, 2011

Introduction

The idea of extending Linear Logic (LL) with differential constructs has been considered by Girard at a very early stage of the design of this system and this option appears at various places in the conclusion of [Gir86], entitled *Two years of linear logic: selection from the garbage collector*. In Section *V.2 The quantitative attempt* of that conclusion, the idea of a syntactic Taylor expansion is explicitly mentioned as a syntactic counterpart of the quantitative semantics of the lambda-calculus [Gir88]. However it is contemplated there as a reduction process rather than as a transformation on terms. In Section *V.5 The exponentials*, the idea of reducing lambda-calculus substitution to a more elementary linear operation explicitly viewed as differentiation is presented as one of the basic intuitions behind the exponential of LL. A possible connection of this idea with Krivine's Machine [Kri85, Kri07] and its *linear head reduction* mechanism [DR99] is explicitly mentioned. In this mechanism, also considered by De Bruijn and called *mini-reduction* in [DB87], it is only the head occurrence of a variable which is substituted during reduction.

LL is based on the distinction of particular proofs among all proofs, that are linear wrt. their hypotheses. The word *linear* has here two deeply related meanings that LL identifies.

- An algebraic meaning: a linear morphism is a function which preserves sums, linear combinations, joins, unions, ... (depending on the context). In most denotational models of LL, linear proofs are interpreted as functions which are linear in that sense.

- An operational meaning: a proof is linear wrt. an hypothesis if the corresponding argument is used exactly once (neither erased nor replicated) during cut-elimination.

LL has an essential logical rule, called *dereliction*. It allows to turn a linear proof into a non linear one, or, more precisely, to forget the linearity of a proof, exactly as one can forget that a function is linear in order to consider it as a polynomial map, or as a smooth map and apply to it non-linear operations. Differentiation is in some sense the converse of dereliction, as it turns a non linear morphism (proof) into a linear one. *A posteriori*, we can think that there were at least two deep reasons not to take this operation into account at an early stage of the development of LL.

- First, differentiation seems incompatible with *totality* which is a denotational analogue of normalization, a property usually considered as an essential feature of any reasonable logical system. Indeed, turning a non-linear proof into a linear one necessarily leads to a loss of information and to the production of partial linear proofs. This is typically what happens when one takes the derivative of a constant proof, which must yield the zero proof: the whole constant value of the proof is lost in this process and cannot be retrieved from its result.

- Second, it seems incompatible with determinism because, when one linearizes a proof obtained by contracting two linear inputs of a proof, one has to choose between these two inputs, and there is no canonical way of doing so: we take the non-deterministic superposition of the two possibilities. Syntactically, this means that one must accept the possibility of adding proofs of the same formula, which is standard in mathematics, but hard to accept as a primitive logical operation on proofs, and incompatible with many denotational models, as the coherence space semantics[1].

The failure of totality is compatible with most mathematical interpretations of proofs and with most denotational models of LL: Scott domains (or more precisely, prime algebraic complete lattices, see [Hut93, Win04]), coherence spaces, games, hypercoherence spaces etc. Moreover, computer scientists are acquainted with the use of syntactic partial objects (fix-point operators in programming languages, Böhm trees of the lambda-calculus etc.) and various modern proof formalisms, such as Girard's Ludics, also incorporate partiality for enlarging the world of "proof-objects" so as to allow the simultaneous existence of "proofs" and "counter-proofs" in order to obtain a rich duality theory on top of which a notion of totality discriminating "real" proof-objects from partial ones can be developed.

The observation that the differential extension of LL is the mirror image of the structural (and dereliction) rules of linear logic encouraged us to study this extension more deeply. It became necessary to admit an intrinsic non-determinism and partiality in logic (these two extensions being related: failure is the neutral element of non-determinism), but the benefit was a new viewpoint on the exponentials, related to the Taylor Formula of Calculus.

In LL, the exponential is usually thought of as the modality of replicable informations. Linear functions are not allowed to copy their arguments and are

[1] Although it is present, in a tamed version, in the additive rules of linear logic.

therefore very limited in terms of computational expressive power, the exponential allows to define non linear functions which can replicate and erase their arguments and are therefore much more powerful. This replication and erasure capabilities seems to result from the presence of the contraction and weakening rules LL, but this is not the case: the really infinite rule of LL is promotion which makes a proof replicable an arbitrary number of times. This fact could not be observed in LL because promotion is the only rule of LL which allows to introduce the "!" modality: without promotion, it is impossible to build a proof object that can be cut *eg.* on a contraction rule.

Differential LL (DiLL) has the same connectives as ordinary LL, and two new rules to introduce the "!" modality: *co-weakening* and *co-dereliction*. The first of these rules allows to introduce an empty proof of type $!A$ and the second one allows to turn a proof of type A into a proof of type $!A$, *without making it replicable* in sharp contrast with the promotion rule. The last new rule, called *co-contraction*, allows to merge two proofs of type $!A$ for creating a new proof of type $!A$. It is similar to the *tensor* rule of ordinary LL with the difference that the two proofs glued together by a co-contraction must have the same type and cannot be separated anymore deterministically whereas the two proofs glued by a tensor can be separated again by cutting the resulting proof against a *par* rule. This changes the viewpoint on the exponential, which become the connectives of *communication* and not of replication: this new viewpoint is illustrated in [EL10b, EL10a].

DiLL has therefore a *finite* fragment which contains the standard "?" rules (weakening, contraction and dereliction) as well as the new "!" ones (co-weakening, co-contraction and co-dereliction), but not the promotion rule. Cut elimination in this system generates sums of proofs, and therefore it is natural to endow proofs with a vector space (or module) structure over a field (or more generally over a semi-ring[2]). This fragment has the following pleasant properties:

- It enjoys strong normalization, even in the untyped case, as long as one considers only proof-nets which satisfy a correctness criterion similar to the standard Danos-Regnier criterion for multiplicative LL (MLL).
- All proofs are linear combinations of "simple proofs" which do not contain linear combinations: this is possible because all the syntactic constructions of this fragment are multilinear. So proofs are similar to polynomials, and simple proofs to monomials. This algebraic analogy is strongly suggested by the denotational models of DiLL.

Moreover, it is possible to transform any instance of the promotion rule, applied to a sub-proof π, into an infinite linear combination of proofs containing copies of π: this is the *Taylor expansion* of promotion. This operation can be applied hereditarily to all instances of the promotion rule in a proof, giving rise to an infinite linear combinations of promotion-free DiLL simple proofs with positive rational coefficients.

[2] This general setting covers also "qualitative" situations sum of proofs is idempotent.

Content. We start with a short syntactic presentation of DiLL, first in a proof-net formalism and then in a lambda-calculus formalism, and we summarize some results, giving bibliographical references.

The paper is essentially devoted to the denotational semantics of DiLL, because this system arose from denotational models and they remain our main source of inspiration. We present a general categorical setting which is a semantic framework for the finitary fragment of DiLL, that we call an *exponential structure*. It consists of an additive $*$-autonomous category $\mathcal{C}$ together with an operation which maps any object X to an object $!X$ equipped with a structure of $\otimes$-bialgebra (representing the structural and co-structural rules) as well as a "dereliction" morphism in $\mathcal{C}(!X, X)$ and a "co-dereliction" morphism $\mathcal{C}(X, !X)$. The important point here is that the operations $X \mapsto !X$ is not assumed to be functorial. Using this simple structure, we define in particular morphisms $\overline{\partial}_X \in \mathcal{C}(!X \otimes X, !X)$ and $\partial_X \in \mathcal{C}(!X, !X \otimes X)$.

An element of $\mathcal{C}(!X, Y)$ can be considered as a non-linear morphism from X to Y (some kind of analytical or smooth functions), but these morphisms cannot be composed. It is nevertheless possible to say when such a morphism is polynomial, and these particular morphisms can be composed, giving rise to a cartesian category.

By composition in $\mathcal{C}$ with $\overline{\partial}_X \in \mathcal{C}(!X \otimes X, !X)$, any element f of $\mathcal{C}(!X, Y)$ can be differentiated, giving rise to an element f' of $\mathcal{C}(!X \otimes X, Y)$ which is its derivative[3]. This operation can be performed again, giving rise to $f'' \in \mathcal{C}(!X \otimes X \otimes X, Y)$ and, due to the commutativity of co-contraction, this morphism is symmetric in its two last linear parameters. Conversely, on can wonder under which circumstances a morphism $g \in \mathcal{C}(!X \otimes X, Y)$ whose derivative $g' \in \mathcal{C}(!X \otimes X \otimes X, Y)$ is symmetric, is the derivative of a morphism $f \in \mathcal{C}(!X, Y)$. Inspired by the usual proof of *Poincaré's Lemma*, we show that such an anti-derivative is available as soon as $\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X) \in \mathcal{C}(!X, !X)$ is an isomorphism. We briefly describe a syntactic version of anti-derivatives in a promotion-free differential lambda-calculus.

To conclude with these general categorical considerations, we consider the case where the operation $X \mapsto !X$ is functorial. We recall the standard notion of *Seely category* (sometimes called *new-Seely*) and explain in which case such a category is a model of differential linear logic. The requirement is twofold: the underlying cartesian $*$-autonomous category has to be additive, and there must be a natural co-dereliction morphism from X to $!X$ (where $!\_$ is the exponential comonad associated with the Seely category structure). We list the axioms that this structure must satisfy, whose categorical phrasing is mainly borrowed to [BCS06, Fio07] and describe the induced exponential structure.

The end of the paper is devoted to concrete models of DiLL. We briefly review the relational model, which is based on the $*$-autonomous category of sets and relations (with the usual cartesian product of sets as tensor product) because it underlies most denotational models of (differential) linear logic. Then

[3]Or differential, or jacobian: it can be seen as an element of $\mathcal{C}(!X, X \multimap Y)$ where $X \multimap Y$ is the object of morphisms from X to Y in $\mathcal{C}$, that is, of linear morphisms from X to Y, and the operation $f \mapsto f'$ satisfies all the properties of differentiation.

we describe the *finiteness space* model which was one of our main motivations for introducing DiLL. We provide a thorough description of this model, insisting on various aspects which were not covered by our initial presentation in [Ehr05] such as *linear boundedness* (whose relevance in this semantic setting has been pointed out by Tasson in [Tas09b, Tas09a]), or the fact that function spaces in the Kleisli category admit an intrinsic description. We conclude with a short section which explains the two approaches to compute the interpretation of a DiLL net in a model.

Some aspects of DiLL are only alluded to in this presentation, the most significant one being certainly the Taylor expansion formula and its connection with linear head reduction. On this topic, we refer to [ER08, ER06, Ehr10].

1 DiLL

This section is devoted to a compact presentation of the logical system DiLL and of associated lambda-calculi.

1.1 Coefficients

In this paper, a set of coefficients $\mathbf{k}$ is needed. This set must be a commutative semi-ring possessing a multiplicative unit 1 (we say that $\mathbf{k}$ is a unitary semi-ring). In Section 6, $\mathbf{k}$ will be assumed to be a field but this assumption is not needed before. At some point, we'll need the property that, in $\mathbf{k}$, each natural number has an inverse. Let us be more precise about this point.

We say that $a \in \mathbf{k}$ has an inverse in $\mathbf{k}$ if there exists $a' \in \mathbf{k}$ such that $aa' = a'a = 1$. When it exists, this inverse is unique and will be denoted as $\frac{1}{a}$ or a^{-1}. Given $n \in \mathbb{N}$ and $a \in \mathbf{k}$, we can define $n \cdot a$ as $\overbrace{a + \cdots + a}^{n\times} \in \mathbf{k}$. We say that *natural numbers are invertible in* $\mathbf{k}$ if $n \cdot 1$ is invertible in $\mathbf{k}$, for each $n \in \mathbb{N}^+$. When $\mathbf{k}$ is a field, this means that it is a field of characteristic 0. When the inverse of $n \cdot 1$ exists in $\mathbf{k}$, it will be simply denoted as $\frac{1}{n}$.

The set of non-negative rational numbers and the (completed) set of non-negative real numbers are typical examples of commutative unitary semi-rings where natural numbers are invertible.

Another extremely simple though quite important example is the following: let $\mathbb{B} = \{0, 1\}$ with addition defined by $1+1 = 1$ and multiplication defined in the obvious way. Then, in $\mathbb{B}$, $n \cdot 1 = 1$ for each $n \in \mathbb{N}^+$ and hence natural numbers are invertible. This particular semi-ring underlies the *qualitative* approach, where linear combinations are sets.

$\mathbb{N}$ is a typical example of a commutative unitary semi-ring where natural numbers do not have inverses.

We recall that a $\mathbf{k}$-module is a set E equipped with a structure of commutative monoid (denoted additively) together with a scalar multiplication (denoted by simple juxtaposition) $\mathbf{k} \times E \to E$ such that the following equations hold

$$\begin{aligned} 0u &= 0 & (a+b)u &= au + bu \\ 1u &= u & a(bu) &= (ab)u \\ a0 &= 0 & a(u+v) &= au + av \end{aligned}$$

When $\mathbf{k}$ is a field, a $\mathbf{k}$-module is called a $\mathbf{k}$-vector space.

An $\mathbb{N}$-module is a commutative monoid, a $\mathbb{Z}$-module is an abelian group, a $\mathbb{B}$-module is a semi-lattice.

1.2 Formulas

The formulas of differential linear logic are those of usual linear logic, obeying the following syntax, given a set of atomic formulas $\alpha, \beta, \ldots$ with or without subscripts or superscripts. For each atomic formula α, we also introduce the negated atomic formula $\overline{\alpha}$.

- If α is an atom, then α and $\overline{\alpha}$ are formulas.
- If A and B are formulas, then so is $A \otimes B$ and $A \parr B$.
- If A and B are formulas, then so is $A \oplus B$ and $A \mathbin{\&} B$.
- If A is a formula, then so are $!A$ and $?A$.

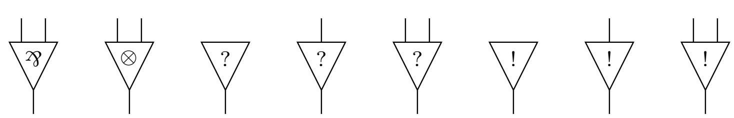
As usual, one defines the linear negation of a formula by induction: $\alpha^\perp = \overline{\alpha}$, $\overline{\alpha}^\perp = \alpha$, $(A \otimes B)^\perp = A^\perp \parr B^\perp$, $(A \parr B)^\perp = A^\perp \otimes B^\perp$, $(A \oplus B)^\perp = A^\perp \mathbin{\&} B^\perp$, $(A \mathbin{\&} B)^\perp = A^\perp \oplus B^\perp$, $(!A)^\perp = ?A^\perp$ and $(?A)^\perp = !A^\perp$. So that $A^{\perp\perp} = A$ for any formula A.

We assume to be given a countable set of names $\mathcal{N}$.

1.3 Inductive definition of pre-nets

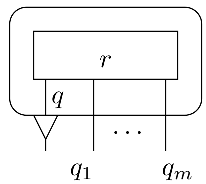
We use the standard formalism of *interaction nets*: see [Laf90] for basic definitions and see [dF09] for a completely rigorous presentation of the corresponding notions and operations. Our presentation here is semi-formal, in order to avoid endless discussions on minor details. Our definition is by induction because of promotion boxes.

1.3.1 Simple pre-nets. A *simple pre-net* is an interaction nets made of the following cells:



called (from left to right) *par*, *tensor*, *weakening*, *dereliction*, *contraction*, *co-weakening*, *co-dereliction* and *co-contraction*. Such a cell has one *main port* (located downward on the picture above) and from 0 to 2 auxiliary ports (located upward).

The last ingredient for building simple pre-nets is the *promotion box*:

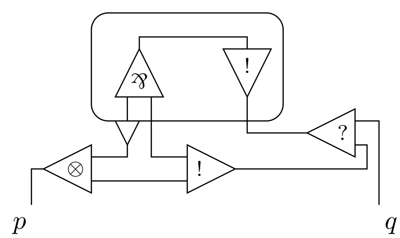


where r is a pre-net (which cannot be assumed to be simple) whose free ports are $q, q_1, \ldots, q_m$. The port indicated by the small triangle is the principal port of the box, the other ones are the auxiliary ports. Such a box will also be called a promotion r-box to insist on the pre-net that it contains.

Remark 1 For many purposes, it is sound to consider promotion boxes as ordinary cells (the principal port of the box becomes the principal port of the corresponding cell), with the peculiarity that they are labeled by pre-nets instead as mere logical symbols such as $\otimes$, $\parr$ etc. This viewpoint is compatible with all typing and logical rules given in Section 1.4 and Section 1.5 as well as with all rewriting rules of Section 1.6, but the two last ones.

A simple pre-nets is a graph made of such cells as well as of *free ports* (which are pairwise distinct elements of $\mathcal{N}$) related through wires. In our pictures, the free ports will always be located downward.

Here is an example of a simple pre-net whose interface is $\{p, q\}$:



In this example, the box contains a simple pre-net made of 2 cells and has 3 free ports.

1.3.2 Pre-nets. Let I be a finite subsets of $\mathcal{N}$. A *pre-net* with interface I is a formal linear combination, with coefficients in $\mathbf{k}$, of simple nets which have all I as interface. In particular, for each I, there is a pre-net 0 with interface I.

This ends the inductive definition of simple pre-nets and of pre-nets.

1.3.3 Extended syntax.

Given two simple pre-net r and s, we denote as $[r, s]$ the pre-net obtained by juxtaposing r and s, relating by wires the identical names of their interface (so that the interface of $[r, s]$ is the symmetric difference of the interfaces of r and s). This operation is commutative, but not associative in general. It is obviously associative when the pre-nets involved have pairwise disjoint interfaces.

We generalize this operation to the case where r and s are not simple, by bilinearity: if $r = \sum_{i=1}^{n} a_i r_i$ and $s = \sum_{j=1}^{m} b_j s_j$ (where the r_i's and the s_j's are simple), then $[r, s]$ denotes the pre-net $\sum_{i,j} a_i b_j [r_i, s_j]$.

Remark 2 Do well notice that a promotion box which contains a pre-net $r = \sum_{i=1}^{n} a_i r_i$ cannot be identified with the linear combination of the boxes which contain the simple pre-nets r_i (with coefficients a_i). Promotion is not a linear operation, and actually, it is the only non-linear operation of LL.

1.4 Typing rules

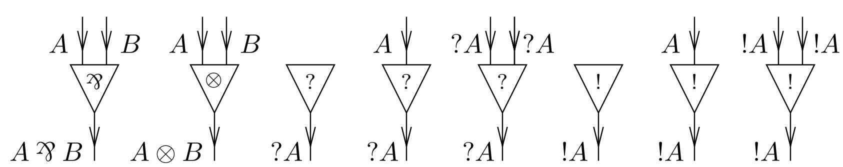
1.4.1 Typing simple pre-nets.

Let r be a simple pre-net.

Formally, a wire of r is a set made of two distinct ports (which can be principal or auxiliary ports of cells or of promotion boxes, or free ports of r). A wire $\{p, q\}$ has two possible orientations: (p, q) and (q, p), and an *oriented wire* is a wire equipped with one of its two orientations. If w is an oriented wire, we use $w^\perp$ for the same wire equipped with the opposite orientation.

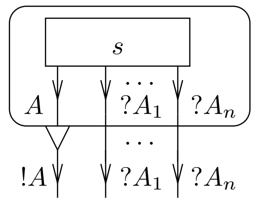
A *type assignment* of r is a mapping τ from the oriented wires of r to LL formulas such that $\tau(w^\perp) = \tau(w)^\perp$. Let I be the interface of r (the set of all free ports of r), then τ induces a mapping from I to LL formulas since each element p of I is the endpoint of exactly one oriented wire w of r: we map p to $\tau(w)$. We denote this function as τ as well.

We say that τ is a *correct type assignment for r* if it satisfies the following constraints, for each cell occurring in r:



meaning for instance that, if the auxiliary wires of a *par* cell are typed by A and B, then its main wire must be types by $A ⅋ B$ (with orientations stipulated by the arrows above).

As to boxes, the typing constraint is as follows:



meaning that the pre-net s must be typed, with a typed interface as pictured above (with types A, $?A_1$,… $?A_n$) and then the main and auxiliary oriented wires of the box are typed as pictured above.

We write a typed pre-net as a pair (r, τ) if we want to insists on the associated type assignment τ, or simply as r if we prefer to keep it implicit.

1.4.2 Typing general pre-nets.

A pre-net $r = \sum_{i=1}^{n} a_i r_i$ with interface $I = \{p_1, \dots, p_n\}$ (where the r_i's are simple and have the same interface I) is *typed with a typed interface* $p_1 : A_1, \dots, p_n : A_n$ if each r_i is a typed pre-net which has this same typed interface I.

1.5 From the sequent calculus to nets: correspondence seen as a proof system

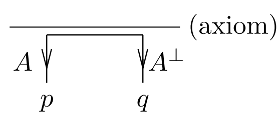
We give now deduction rules for building logically correct typed pre-nets, also called *nets* in the sequel.

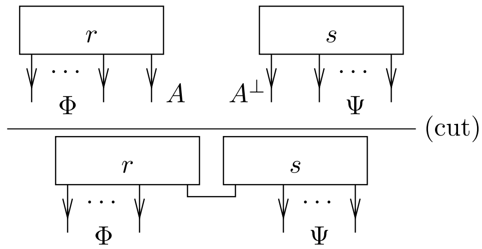
Observe that, from each of the following rules, one can extract a sequent calculus rule. In the rules below, Φ and Ψ stand for finite maps from names to formulas (which are assumed to have disjoint domains); such a function is just a family of formulas, that is, the main ingredient of an LL sequent. As an example, with the rule (cocontr) below, we associate the following sequent calculus rule of co-contraction:

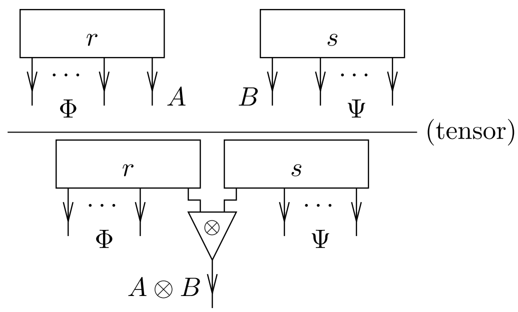
$$\frac{\vdash \Phi, !A \qquad \vdash !A, \Psi}{\vdash \Phi, !A, \Psi}$$

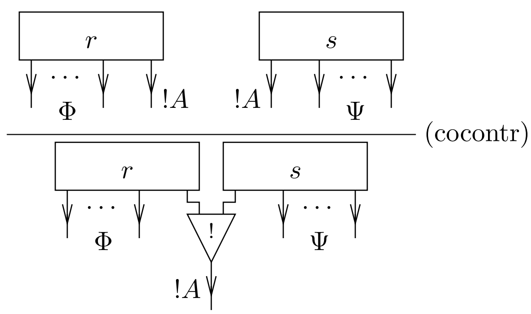
and we can do the same for all the rules below. Therefore, we can also consider this list of rules as a presentation of the DiLL sequent calculus.

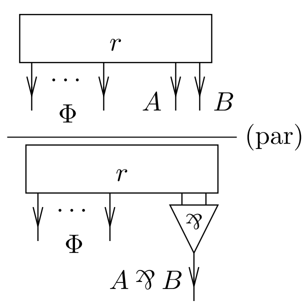
1.5.1 Correctness of simple pre-nets.

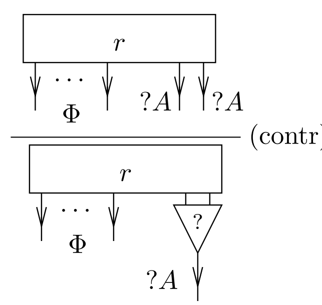


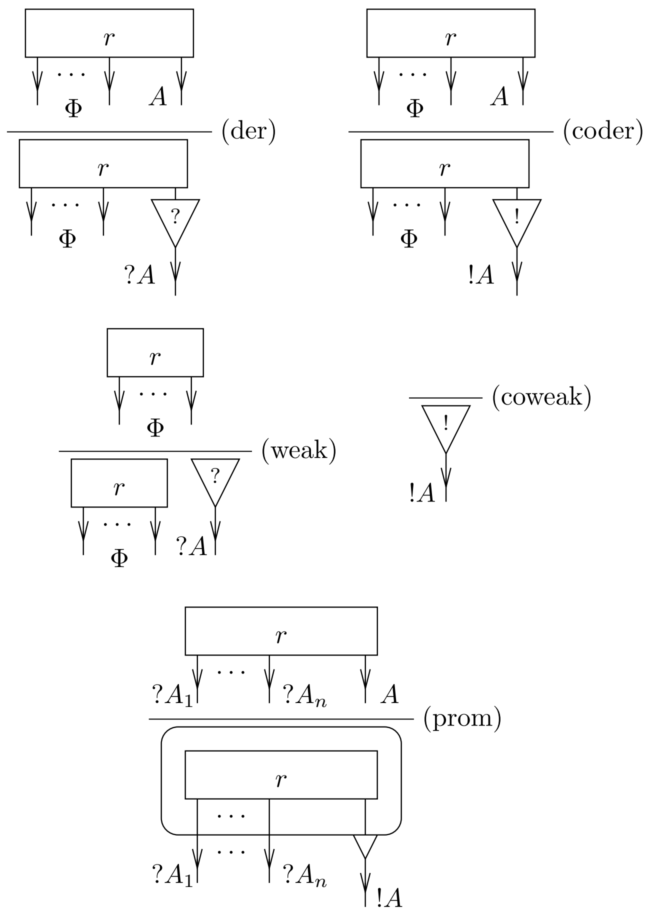

r
s
Φ
A
A⊥
Ψ
(cut)
r
s
Φ
Ψ


r
s
Φ
A
B
Ψ
(tensor)
r
s
⊗
Φ
Ψ
A ⊗ B


r
s
Φ
!A
!A
Ψ
(cocontr)
r
s
!
Φ
Ψ
!A


r
Φ
A
B
(par)
r
⅋
Φ
A ⅋ B


r
Φ
?A
?A
(contr)
r
?
Φ
?A

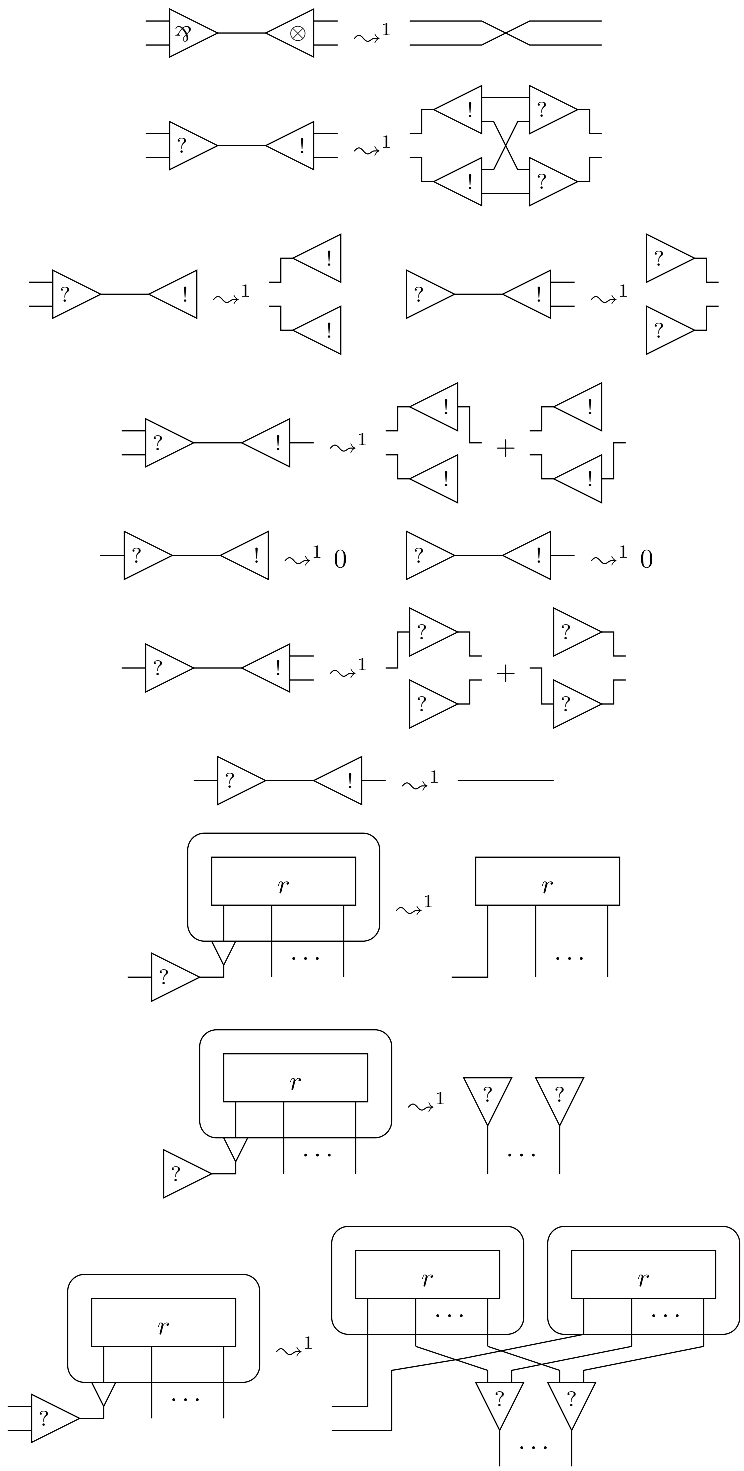


where, in this last rule, the premise means that r is a correct typed pre-net with typed interface $(p_1 : ?A_1, \dots, p_n : ?A_n, p : A)$.

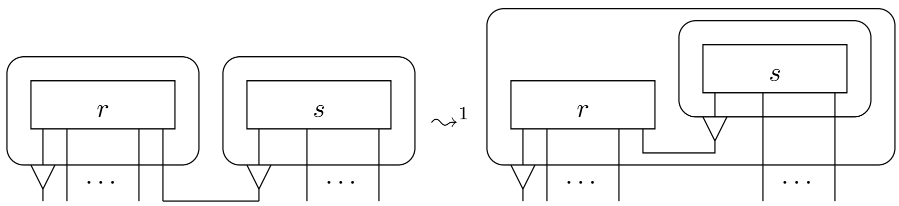
1.5.2 Correctness of pre-nets. If $r_1, \dots, r_k$ are correct simple typed pre-nets with typed interface $(p_1 : A_1, \dots, p_n : A_n)$ and if $a_1, \dots, a_k \in \mathbf{k}$, then $a_1 r_1 + \cdots + a_k r_k$ is a correct typed pre-net with typed interface $(p_1 : A_1, \dots, p_n : A_n)$.

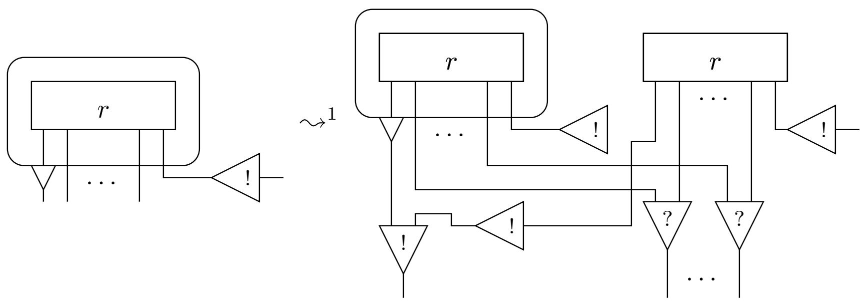
1.6 Reduction rules

We present now the cut-elimination rules of DiLL. The basic reduction relation is denoted as $\rightsquigarrow^1$ and it is a relation from simple pre-nets to pre-nets. When $r \rightsquigarrow^1 r'$, in most cases r is a simple pre-net made of two cells whose principal ports are connected by a wire: this is the general notion of redex in interaction nets, see [Laf90]. To be more precise, considering boxes as cells as explained in Remark 1, this true for all the rules presented here, but the two last ones.



The two next reduction rules are of a different nature as they involve non principal ports of cells and boxes. They are necessary to interpret the β-reduction and the differential β-reduction of the differential λ-calculus in DiLL.





This last rule is nothing but a syntactic rephrasing of the *chain rule* of calculus, in a setting where composition of maps is defined using boxes, according to Girard's encoding of the lambda-calculus in proof-nets [Gir87], which is the same as the definition of composition in the Kleisli category of the comonad "!", see [Mel09].

These rules can be applied anywhere in a simple pre-net, defining a reduction relation $\rightsquigarrow^1$ from simple pre-nets to pre-nets (we use systematically the extended syntax of Section 1.3.3, so that when $r \rightsquigarrow^1 r'$, r is a simple pre-net and r' is a sum of a finite number of simple pre-nets). This means more precisely that if $r \rightsquigarrow^1 r'$ (and hence r and r' have the same interface), and if s is an arbitrary simple pre-net, we stipulate that $[r, s] \rightsquigarrow^1 [r', s]$.

Since boxes can contain non simple pre-nets, the relation $\rightsquigarrow^1$ does not allow to reduce inside boxes. For this purpose, we define a relation $\rightsquigarrow$ from pre-nets to pre-nets by the following rules.

- If $r \rightsquigarrow^1 r'$ or $r = r'$, then $r \rightsquigarrow r'$.
- If $r = \sum_{i=1}^{n} a_i r_i$ and $r' = \sum_{i=1}^{n} a_i r'_i$ where the r_i's and the the r'_i's are not necessarily simple, and if $r_i \rightsquigarrow r'_i$ for all i, then $r \rightsquigarrow r'$.
- If r and r' are pre-nets such that $r \rightsquigarrow r'$ (with same interface $q, q_1, \dots, q_n$) and s is a promotion r-box (see Section 1.3.1) of principal port q and auxiliary ports $q_1, \dots, q_n$, and similarly for s', then $s \rightsquigarrow s'$.

One checks easily that, if $r \rightsquigarrow r'$ and r is typead in the sense of Section 1.4, then so is r', with the same typed interface. And if r is a typed logically correct pre-net (in the sense of Section 1.5), then so is r'.

This reduction relation has good confluence[4] and termination properties, possibly under some assumptions on the semi-ring of coefficients $\mathbf{k}$. For confluence, we can refer to [Tra09] and for normalization we refer to [Pag09, PT11] for a combinatorial approach, and to [Gim11] for an approach based on reducibility which can be generalized to higher order LL. We also refer to [Vau07] for considerations on the influence of the properties of $\mathbf{k}$ on the confluence and normalization of such calculi.

1.7 Differential lambda-calculi

Various lambda-calculi have been proposed, as possible extensions of the ordinary lambda-calculus with constructions corresponding to the above differential and co-structural rules of differential linear logic. We present here briefly our original syntax of [ER03], simplified by Vaux in [Vau05].

We assume given a countable set of variables $x, y, \dots$ with or without subscripts or superscripts. A *simple term* is either

- a variable x
- or an abstraction $\lambda x\, s$ where x is a variable and s is a simple term
- or an ordinary application $(s)\, u$ where s is a simple terms and u is a term
- or a differential application $\mathsf{D}s \cdot t$ where s and t are simple terms.

A *term* is a finite linear combination of simple terms with coefficients in $\mathbf{k}$. Substitution of a term u for a variable x in a simple term s, denoted as $s\,[u/x]$ is defined as usual, whereas differential (or linear) substitution of a simple term for a variable in another simple term, denoted as $\frac{\partial s}{\partial x} \cdot t$, is defined as follows:

$$\frac{\partial y}{\partial x} \cdot t = \begin{cases} t & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial \lambda y\, s}{\partial x} \cdot t = \lambda y\, \frac{\partial s}{\partial x} \cdot t$$

$$\frac{\partial \mathsf{D}u \cdot s}{\partial x} \cdot t = \mathsf{D}\left(\frac{\partial u}{\partial x} \cdot t\right) \cdot s + \mathsf{D}u \cdot \left(\frac{\partial s}{\partial x} \cdot t\right)$$

$$\frac{\partial\, (s)\, u}{\partial x} \cdot t = \left(\frac{\partial s}{\partial x} \cdot t\right) u + \left(\mathsf{D}s \cdot \frac{\partial u}{\partial x} \cdot t\right) u$$

In order to extend the syntax to arbirary terms, we stipulate that constructions are linear, but ordinary application which is not linear in the argument. This means that when we write e.g. $(s_1 + s_2)\, u$, what we actually intend is $(s_1)\, u + (s_2)\, u$. Similarly, substitution $s\,[u/x]$ is linear in s and not in u, whereas differential substitution $\frac{\partial s}{\partial x} \cdot t$ is linear in both s and t. There are two reduction rules:

$$(\lambda x\, s)\, u \;\beta\; s\,[u/x] \qquad \mathsf{D}(\lambda x\, s) \cdot t \;\delta\; \lambda x \left(\frac{\partial s}{\partial x} \cdot t\right)$$

[4] To be precise, this requires to consider pre-nets up to a natural and semantically perfectly justified equivalence relation, see [Tra09].

which have of course to be closed under arbitrary contexts. The resulting calculus can be proved to be Church-Rosser using fairly standard techniques (Tait - Martin-Löf), to have good normalization properties in the typed case etc, see [ER03, Vau05]. To be more precise, Church-Rosser holds only up to the least congruence on terms which identifies $\mathsf{D}(\mathsf{D}s \cdot t_1) \cdot t_2$ and $\mathsf{D}(\mathsf{D}s \cdot t_2) \cdot t_1$, a syntactic version of Schwarz Lemma: terms are always considered up to this congruence that we can call *symmetry of derivatives*.

1.7.1 Resource calculus. Differential application can be iterated: given simple terms $s, t_1, \dots, t_n$, we define $\mathsf{D}^n s \cdot (t_1, \dots, t_n) = \mathsf{D}(\cdots \mathsf{D}s \cdot t_1 \cdots) \cdot t_n$; the order on the terms $t_1, \dots, t_n$ does not matter, by symmetry of derivatives. The resource calculus is another syntax for the differential lambda-calculus, in which the combination $(\mathsf{D}^n s \cdot (t_1, \dots, t_n))\, u$ is considered as one single operation denoted e.g. as $s[t_1, \dots, t_n, u^\infty]$ where the superscript ∞ is here to remind that u can be arbitrarily replicated during reduction, unlike the t_i's (in this syntactic construct, s as well as the t_i's are simple whereas u is not).

This version of the calculus, studied in particular by Tranquilli in [Tra11] and Pagani [PT09], and also used for instance in [BCEM11], has very good properties as well. It is formally close to Boudol's lambda-calculus with resources [Bou93, BCL99], with the difference that the operational semantics of Boudol's calculus is given as a rewriting strategy whereas in the differential version of the resource lambda-calculus, redexes can be reduced everywhere in terms. This means in particular that the differential resource lambda-calculus contains the ordinary lambda-calculus as a subsystem.

1.7.2 The finite calculus. If, in the resource calculus above, one restricts one's attention to the terms where, in all ordinary applications the arguments are null term, that is, null combinations of simple terms, then one gets a calculus which is stable under reduction and where all terms are strongly normalizing. This calculus can be presented as follows. Any variable x is a term. If x is a variable and s is a simple term then $\lambda x\, s$ is a simple term. If S is a finite multiset (also called *bunch* in the sequel) of simple terms then $\langle s \rangle\, S$ is a simple term. In that calculus it is natural to perform several beta-reductions in one step, and one gets

$$\langle \lambda x\, s \rangle\, S\ \delta \begin{cases} \sum_{f \in \mathfrak{S}_n} s\,[s_1/x_1, \dots, s_n/x_n] & \text{if } \deg_s x = n \\ 0 & \text{otherwise} \end{cases}$$

where $d = \deg_x s$ is the number of occurrences of x in s (which is a simple term), $x_1, \dots, x_d$ are the occurrences of x in s and the multiset S is $[s_1, \dots, s_n]$.

Again, this calculus enjoys confluence. It can be used as target language for hereditarily Taylor expanding lambda-terms as explained in [ER08, ER06, Ehr10].

This operation consists in replacing, in a differential lambda-term, any ordinary application $(s)\, t$ by the infinite sum $\sum_{n=0}^{\infty} \frac{1}{n!} (\mathsf{D}^n s \cdot (t, \dots, t))\, 0$ (one has to assume that natural numbers are invertible in $\mathbf{k}$, see Section 1.1). More precisely, it is a transformation $t \mapsto t^*$ from resource terms to resource terms which is defined as follows:

$$
\begin{aligned}
x^* &= x \\
(\lambda x\, t)^* &= \lambda x\, (t^*) \\
((\mathsf{D}^n t \cdot (s_1, \dots, s_n))\, s)^* &= \sum_{p=0}^{\infty} \frac{1}{p!} \left(\mathsf{D}^{n+p} s^* \cdot ({s_1}^*, \dots, {s_n}^*, s^*, \dots, s^*)\right) 0
\end{aligned}
$$

so that the natural target language of this translation is the finite resource calculus described above: resource terms are translated to (generally infinite) linear combinations of finite resource terms with coefficients in the considered semi-ring where division by positive natural numbers must be possible.

In [ER08, ER06] we studied the behaviour of this expansion with respect to differential β-reduction in the case where the expanded terms come from the lambda-calculus (that is, do not contain differential applications; this is the *uniform* case), and we exhibited tight connections between this operation and Krivine's machine, an implementation of linear head reduction. In [Ehr10] we introduced a setting for dealing with the Taylor expansion in the non uniform case, endowing finite resource terms with a finiteness space structure (see Section 6 for a definition of this concept) whose purpose is to prevent coefficients to become infinite during the reduction.

2 Categorical models of promotion-free differential LL

The notion of categorical model for this logical system is quite easy to define. A model consists of a $*$-autonomous category $\mathcal{C}$ together with a further exponential structure.

2.1 $*$-autonomous category

A monoidal category is a category $\mathcal{C}$ (where we denote the composition of morphisms by simple juxtaposition: if $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, then $g\, f \in \mathcal{C}(X, Z)$) together with a bifunctor $\otimes : \mathcal{C}^2 \to \mathcal{C}$, an object $1 \in \mathcal{C}$ and natural isomorphisms $\lambda_X : 1 \otimes X \to X$, $\rho_X : X \otimes 1 \to X$, $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ and these isomorphisms are required to satisfy coherence commutative diagrams that we do not recall here (see [Mac71]).

The monoidal category $\mathcal{C}$ is enriched over $\mathbf{k}$-modules if each homset $\mathcal{C}(X, Y)$ has a structure of $\mathbf{k}$-module (see Section 1.1) and composition, as well as the bifunctor $\otimes$, are bilinear with respect to this structure.

A symmetric monoidal category is a monoidal category together with a natural isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ which has also to satisfy commutations (again, see [Mac71]).

A symmetric monoidal closed category (SMCC for short) is a symmetric monoidal category $\mathcal{C}$ such that, for each object X, the functor $Y \mapsto X \otimes Y$ has a right adjoint $Y \mapsto (X \multimap Y)$. Let X, Y and Z be objects of $\mathcal{C}$, we have a linear evaluation morphism $\mathrm{ev} \in \mathcal{C}((X \multimap Z) \otimes X, Z)$, and, given a morphism $f \in \mathcal{C}(Y \otimes X, Z)$, we have a morphism $\lambda(f) \in \mathcal{C}(Y, X \multimap Z)$. Monoidal closeness boils down to the following three equations:

$$\begin{aligned} \mathrm{ev}\ (\lambda(f) \otimes \mathrm{Id}_X) &= f \\ \lambda(f)\ h &= \lambda(f\ (h \otimes \mathrm{Id}_X)) \quad \text{where } h \in \mathcal{C}(Y', Y) \\ \lambda(\mathrm{ev}) &= \mathrm{Id}_{X \multimap Z} \end{aligned}$$

In particular, we have a morphism $\eta_X = \lambda(\mathrm{ev}\,\sigma) \in \mathcal{C}(X, (X \multimap Z) \multimap Z)$ which is natural in X.

Last, a $*$-autonomous category is an SMCC $\mathcal{C}$ together with an object $\perp$ such that the canonical natural morphism $\eta_X : X \to ((X \multimap \perp) \multimap \perp)$ is an isomorphism.

Therefore, in a $*$-autonomous category $\mathcal{C}$, there is a contravariant functor $X \mapsto X^\perp = (X \multimap \perp)$ which is actually an equivalence of categories between $\mathcal{C}$ and $\mathcal{C}^{\mathsf{op}}$. Through this isomorphism, we can define another symmetric monoidal category structure on $\mathcal{C}$ whose binary operation (the "co-tensor product" or *par*) is defined by $X \parr Y = (X^\perp \otimes Y^\perp)^\perp$ so that we have in particular $X \multimap Y = X^\perp \parr Y$ up to a natural isomorphism).

In a cartesian $*$-autonomous category $\mathcal{C}$, we denote a terminal object as $\top$ and a cartesian product (also known as direct product) as $\&$. Then $\mathcal{C}$ is cocartesian with initial object $0 = \top^\perp$ and cocartesian product (also known as direct sum) $X \oplus Y = (X^\perp \,\&\, Y^\perp)^\perp$. There is then a natural morphism $\iota_{X,Y} : X \oplus Y \to X \,\&\, Y$ which results from the universal properties of the cartesian and of the cocartesian products; there is also a canonical morphism $\iota_0 : 0 \to \top$.

If $\mathcal{C}$ is a cartesian $*$-autonomous category which is enriched over $\mathbf{k}$-modules, then the morphisms $\iota_{X,Y}$ and ι_0 are all isomorphisms, so that 0 and $\top$ coincide, as well as $\oplus$ and $\&$. In that case, we use the notation 0 for the terminal object and $\oplus$ for the cartesian product.

Remark 3 Models of differential linear logic can be axiomatized without assuming the $*$-autonomous to be cartesian, but this assumption makes things much easier and does not seem really problematic: it is fulfilled for free in most concrete situations.

2.2 Exponential structure

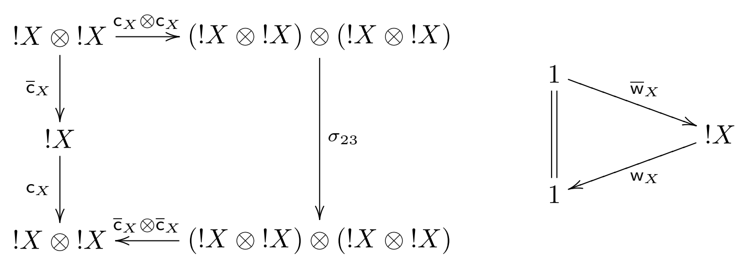
An *exponential structure* is a cartesian $*$-autonomous category $\mathcal{C}$ which is enriched over $\mathbf{k}$-modules and equipped with an operation

$$X \mapsto (!X, \mathsf{c}_X, \mathsf{w}_X, \mathsf{d}_X, \overline{\mathsf{c}}_X, \overline{\mathsf{w}}_X, \overline{\mathsf{d}}_X)$$

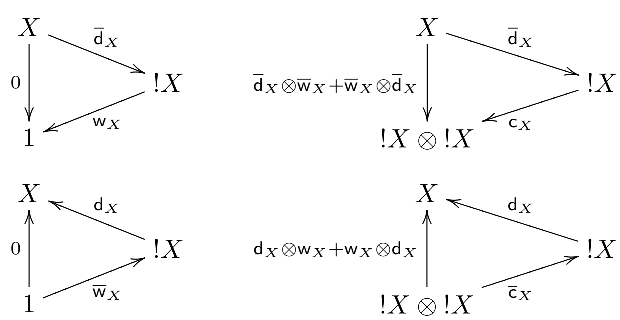
where $!X$ is an object of $\mathcal{C}$ and

$$
\begin{array}{ll}
\mathsf{c}_X : !X \to !X \otimes !X & \bar{\mathsf{c}}_X : !X \otimes !X \to !X \\
\mathsf{w}_X : !X \to 1 & \overline{\mathsf{w}}_X : 1 \to !X \\
\mathsf{d}_X : !X \to X & \bar{\mathsf{d}}_X : X \to !X
\end{array}
$$

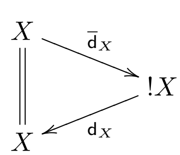
are morphisms called contraction, co-contraction, weakening, co-weakening, dereliction and co-dereliction (from left to right and top to bottom) such that $(!X, \mathsf{c}_X, \mathsf{w}_X, \bar{\mathsf{c}}_X, \overline{\mathsf{w}}_X)$ is a $\otimes$-bialgebra in $\mathcal{C}$. This first condition means that $(!X, \mathsf{c}_X, \mathsf{w}_X)$ is a $\otimes$-comonoid, that $(!X, \bar{\mathsf{c}}_X, \overline{\mathsf{w}}_X)$ is a $\otimes$-monoid and that the following diagrams are commutative



where σ_{23} is the isomorphism which transposes the two central components of the 4-fold tensor product. Concerning d_X (which could be called "creation") and $\bar{\mathsf{d}}_X$ ("annihilation"), we require the following diagrams to commute



where we identify X with $1 \otimes X$ and with $X \otimes 1$ (the isomorphisms λ_X and ρ_X are kept implicit). The last commutation that we require is



This simple set of categorical axioms is sufficient to interpret the reduction rules of the (promotion-free) differential linear logic and lambda-calculus.

2.2.1 Commutativity. We say that the exponential structure $\mathcal{C}$ is cocommutative if the diagram

$$\begin{array}{ccc} !X & \xrightarrow{\mathsf{c}_X} & !X \otimes !X \\ & \searrow^{\mathsf{c}_X} & \downarrow \sigma \\ & & !X \otimes !X \end{array}$$

commutes for each object X, and that it is commutative if the diagram

$$\begin{array}{ccc} !X \otimes !X & \xrightarrow{\overline{\mathsf{c}}_X} & !X \\ \sigma \downarrow & \nearrow_{\overline{\mathsf{c}}_X} & \\ !X \otimes !X & & \end{array}$$

commutes for each object X. If both diagrams are commutative for each object X, we say that $\mathcal{C}$ is bi-commutative.

2.2.2 Notations. We can compose c_X maps in order to define a map $\mathsf{c}_X^n : !X \to (!X)^{\otimes n}$ for each $n \in \mathbb{N}$, for instance we can set $\mathsf{c}_X^0 = \mathsf{w}_X$ and $\mathsf{c}_X^{n+1} = (\mathsf{c}_X^n \otimes !X)\ \mathsf{c}_X$; the resulting morphisms does not depend on the choices that we make in this definition, because of the associativity of c_X. Dually one defines $\overline{\mathsf{c}}_X^n : (!X)^{\otimes n} \to !X$.

We also introduce the contraction-dereliction map $\partial_X = (\mathrm{Id}_{!X} \otimes \mathsf{d}_X)\ \mathsf{c}_X : !X \to !X \otimes X$ as well as the dual co-contraction-co-dereliction map $\overline{\partial}_X = \overline{\mathsf{c}}_X\ (\mathrm{Id}_{!X} \otimes \overline{\mathsf{d}}_X) : !X \otimes X \to !X$. The intended meaning of this morphism is that, if $f : !X \to Y$ is a regular morphism from X to Y, then $f\ \overline{\partial}_X : !X \otimes X \to Y$ represents the derivative of f, which can also be seen as a morphism $f' : !X \to (X \multimap Y)$: take $f' = \lambda(f\ \overline{\partial}_X)$.

These two morphisms are related by the following commutation.

Lemma 4 *We have* $\partial_X\ \overline{\partial}_X = \mathrm{Id}_{!X \otimes X} + (\overline{\partial}_X \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X)$.

The proof is a simple computation based on the above categorical axioms.

We introduce next iterated versions of these morphisms: $\partial_X^n : !X \to !X \otimes X^{\otimes n}$ by $\partial_X^0 = \mathrm{Id}_{!X}$ and $\partial_X^{n+1} = (\partial_X^n \otimes \mathrm{Id}_X)\ \partial_X$. One defines similarly $\overline{\partial}_X^n : !X \otimes X^{\otimes n} \to !X$. We also introduce the morphisms $\mathsf{d}_X^n : !X \to X^{\otimes n}$ and $\overline{\mathsf{d}}_X^n : X^{\otimes n} \to !X$ defined by $\mathsf{d}_X^n = (\mathsf{w}_X \otimes \mathrm{Id}_{X^{\otimes n}})\ \partial_X^n$ and $\overline{\mathsf{d}}_X^n = \overline{\partial}_X^n\ (\overline{\mathsf{w}}_X \otimes \mathrm{Id}_{X^{\otimes n}})$ so that in particular $\mathsf{d}_X^0 = \mathsf{w}_X$, $\mathsf{d}_X^1 = \mathsf{d}_X$, $\overline{\mathsf{d}}_X^0 = \overline{\mathsf{w}}_X$ and $\overline{\mathsf{d}}_X^1 = \overline{\mathsf{d}}_X$

2.2.3 Interpretation of the untyped finite resource lambda-calculus. A *reflexive object* in an exponential structure $\mathcal{C}$ is a triple $(U, \mathsf{app}, \mathsf{lam})$ where $U \in \mathcal{C}$, $\mathsf{app} \in \mathcal{C}(U, !U \multimap U)$ and $\mathsf{lam} \in \mathcal{C}(!U \multimap U, U)$ satisfy $\mathsf{app}\ \mathsf{lam} = \mathrm{Id}$: this is just the standard definition of a reflexive object in a cartesian closed category, with the restriction however that app and lam are *linear* (they are morphism in the linear category $\mathcal{C}$)... and that there is no cartesian closed category around.

By induction on the pure resource term t and a sequence of variables $\vec{x}$ adapted to t (meaning that $\vec{x}$ is repetition-free and contains all the free variables of t), we define $[t]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, U)$ by induction on t as follows.

Assume first that t is the variable x_i (for $i \in \{1, \ldots, n\}$), then we set $[t]^{\vec{x}} = \mathsf{w}_U{}^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U{}^{\otimes(n-i)} : !U \to 1^{\otimes(i-1)} \otimes U \otimes 1^{\otimes(n-i)} \simeq U$.

Next assume that $t = \lambda x\, s$, then we have $[s]^{\vec{x},x} \in \mathcal{C}((!U)^{\otimes n} \otimes !U, U)$. Hence $\lambda([s]^{\vec{x},x}) \in \mathcal{C}((!U)^{\otimes n}, !U \multimap U)$ and we set $[t]^{\vec{x}} = \mathsf{lam}\ \lambda([s]^{\vec{x},x}) \in \mathcal{C}((!U)^{\otimes n}, U)$.

Last assume that $t = \langle s \rangle\, S$ where $S = s_1 \cdots s_k$ is a bunch. We have $[s_i]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, U)$ for $i = 1, \ldots, n$ and hence $\mathsf{d}_U^k\ ([s_1]^{\vec{x}} \otimes \cdots \otimes [s_k]^{\vec{x}}) : ((!U)^{\otimes n})^{\otimes k} \to U$ and we set $[S]^{\vec{x}} = \mathsf{d}_U^k\ ([s_1]^{\vec{x}} \otimes \cdots \otimes [s_k]^{\vec{x}})\ \sigma\ (\mathsf{c}_X^k)^{\otimes n}$ where σ is the obvious isomorphism resulting from the symmetric monoidal structure of $\mathcal{C}$. On the other hand we have $\mathsf{app}\ [s]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, !U \multimap U)$ and hence we set $[t]^{\vec{x}} = \mathsf{ev}\ ((\mathsf{app}\ [s]^{\vec{x}}) \otimes [S]^{\vec{x}})\ \sigma'\ \mathsf{c}_U{}^{\otimes n} \in \mathcal{C}((!U)^{\otimes n}, U)$. Categorical computations show that the interpretation of terms is invariant under differential beta-reduction.

The interpretation of the simply typed finite resource calculus is completely similar, and involves several objects instead of only one reflexive object.

2.2.4 Taylor exponential structures.

Let $\mathcal{C}$ be an exponential structure and let $f \in \mathcal{C}(!X, Y)$. The condition $f\ \overline{\partial}_X = 0$ means intuitively that the derivative of f is uniformly equal to 0, and hence, according to standard intuitions on differentiation, f should be a constant map. In other words we should have $f = f\ \overline{\mathsf{w}}_X\ \mathsf{w}_X$ (the converse implication is clearly true). This property can be stated in a more general way as follows: let $f_1, f_2 : !X \to Y$, then

$$f_1\ \overline{\partial}_X = f_2\ \overline{\partial}_X \Rightarrow f_1 + (f_2\ \overline{\mathsf{w}}_X\ \mathsf{w}_X) = f_2 + (f_1\ \overline{\mathsf{w}}_X\ \mathsf{w}_X)$$

and does not seem to be derivable from the other axioms of bi-commutative exponential structures.

We say that the exponential structure $\mathcal{C}$ *is Taylor* if it satisfies this condition.

Remark 5 There is a dual condition which reads as follows: if $f_1, f_2 : Y \to !X$, then

$$\partial_X\ f_1 = \partial_X\ f_2 \Rightarrow f_1 + (\overline{\mathsf{w}}_X\ \mathsf{w}_X\ f_2) = f_2 + (\overline{\mathsf{w}}_X\ \mathsf{w}_X\ f_1)$$

Its intuitive meaning is that, given $f : Y \to !X$, to be considered as a generalized point of $!X$, if $\partial_X\ f = 0$, then f is the constant function whose value is the unit of the bia-algebra $!X$. In other words, the kernel of ∂_X is generated by this unit.

From now on and until the end of Section 2.2, we assume that, in $\mathbf{k}$, natural numbers are invertible, see Section 1.1 for the definition of this notion.

For $n \in \mathbb{N}$, let $\mathrm{T}_X^n \in \mathcal{C}(!X, !X)$ be defined by

$$\mathrm{T}_X^n = \sum_{i=0}^{n} \frac{1}{i!}\, \overline{\mathsf{d}}_X^i\ \mathsf{d}_X^i\,.$$

Lemma 6 *Assume that $\mathcal{C}$ is bi-commutative and Taylor. For any $n > 0$, we have*

$$\mathrm{T}_X^n\, \overline{\partial}_X = \overline{\partial}_X\, (\mathrm{T}_X^{n-1} \otimes \mathrm{Id}_X)\,.$$

Proof. This results from

$$\overline{\mathsf{d}}_X^n\, \mathsf{d}_X^n\, \overline{\partial}_X = n\overline{\partial}_X\, ((\overline{\mathsf{d}}_X^{n-1}\, \mathsf{d}_X^{n-1}) \otimes \mathrm{Id}_X)$$

which comes from the basic equations of exponential structures. □

Remember that a commutative monoid M is cancellative if, in M, one has $u + v = u' + v \Rightarrow u = u'$.

Proposition 7 *Assume that $\mathcal{C}$ is bi-commutative and Taylor and that each homset $\mathcal{C}(X, Y)$ is a cancellative monoid. Let $n \in \mathbb{N}$ and let $f_1, f_2 : !X \to Y$. If $f_1\, \overline{\partial}_X^{n+1} = f_2\, \overline{\partial}_X^{n+1}$ then $f_1 + (f_2\, \mathrm{T}_X^n) = f_2 + (f_1\, \mathrm{T}_X^n)$.*

In particular, if $f\, \partial_X^{n+1} = 0$ (that is, the $(n+1)$-th derivative of f is uniformly equal to 0), then $f = f\, \mathrm{T}_X^n$, meaning that f is equal to its Taylor expansion of rank n.

Proof. By induction on n. For $n = 0$, this is simply the hypothesis that $\mathcal{C}$ is Taylor. Assume now that $f_1\, \overline{\partial}_X^{n+2} = f_2\, \overline{\partial}_X^{n+2}$ and let us prove that $f_1 + (f_2\, \mathrm{T}_X^{n+1}) = f_2 + (f_1\, \mathrm{T}_X^{n+1})$.

We have $f_1\, \overline{\partial}_X\, (\overline{\partial}_X^{n+1} \otimes \mathrm{Id}_X) = f_2\, \overline{\partial}_X\, (\overline{\partial}_X^{n+1} \otimes \mathrm{Id}_X)$. By monoidal closeness, we have $\lambda(f_1\, \overline{\partial}_X)\, \overline{\partial}_X^{n+1} = \lambda(f_2\, \overline{\partial}_X)\, \overline{\partial}_X^{n+1}$ and hence, by inductive hypothesis, we have

$$\lambda(f_1\, \overline{\partial}_X) + (\lambda(f_2\, \overline{\partial}_X)\, \mathrm{T}_X^n) = \lambda(f_2\, \overline{\partial}_X) + (\lambda(f_1\, \overline{\partial}_X)\, \mathrm{T}_X^n)$$

that is

$$\lambda(f_1\, \overline{\partial}_X) + (\lambda(f_2\, \overline{\partial}_X\, (\mathrm{T}_X^n \otimes \mathrm{Id}_X))) = \lambda(f_2\, \overline{\partial}_X) + (\lambda(f_1\, \overline{\partial}_X\, (\mathrm{T}_X^n \otimes \mathrm{Id}_X)))$$

and hence

$$(f_1\, \overline{\partial}_X) + (f_2\, \overline{\partial}_X\, (\mathrm{T}_X^n \otimes \mathrm{Id}_X)) = (f_2\, \overline{\partial}_X) + (f_1\, \overline{\partial}_X\, (\mathrm{T}_X^n \otimes \mathrm{Id}_X))$$

so applying Lemma 6, we get

$$(f_1 + f_2\mathrm{T}_X^{n+1})\, \overline{\partial}_X = (f_2 + f_1\mathrm{T}_X^{n+1})\, \overline{\partial}_X\,.$$

Applying the hypothesis that $\mathcal{C}$ is Taylor, we get

$$f_1 + f_2\mathrm{T}_X^{n+1} + (f_2 + f_1\mathrm{T}_X^{n+1})\, \overline{\mathsf{w}}_X\, \mathsf{w}_X = f_2 + f_1\mathrm{T}_X^{n+1} + (f_1 + f_2\mathrm{T}_X^{n+1})\, \overline{\mathsf{w}}_X\, \mathsf{w}_X$$

and since $\mathrm{T}_X^{n+1}\, \overline{\mathsf{w}}_X\, \mathsf{w}_X = \overline{\mathsf{w}}_X\, \mathsf{w}_X$ we get $f_1 + f_2\mathrm{T}_X^{n+1} + (f_1 + f_2)\, \overline{\mathsf{w}}_X\, \mathsf{w}_X = f_2 + f_1\mathrm{T}_X^{n+1} + (f_2 + f_1)\, \overline{\mathsf{w}}_X\, \mathsf{w}_X$ and so, applying the cancellativeness hypothesis, we get finally

$$f_1 + f_2\mathrm{T}_X^{n+1} = f_2 + f_1\mathrm{T}_X^{n+1}$$

as required. □

Remark 8 The cancellativeness hypothesis does not apply in cases where $\mathbf{k} = \mathbb{B}$. In these cases, we can obtain the same property under the hypothesis that $\mathrm{Id}_{!X} + \overline{\mathsf{w}}_X\ \mathsf{w}_X = \mathrm{Id}_{!X}$ which holds, for instance, in **Rel**.

2.2.5 The category of polynomials. We assume that $\mathcal{C}$ is such that Proposition 7 holds. We say that $f \in \mathcal{C}(!X, Y)$ is *polynomial* if there exists $n \in \mathbb{N}$ such that $f\ \overline{\partial}_X^{n+1} = 0$, and we call *degree* of f the least such n. The morphism $\mathsf{d}_X \in \mathcal{C}(!X, X)$ is polynomial of degree 1.

Let $f \in \mathcal{C}(!X, Y)$ and $g \in \mathcal{C}(!Y, Z)$ be polynomial of degree m and n respectively. We define the composite $g \circ f \in \mathcal{C}(!X, Z)$ as follows

$$g \circ f = \sum_{i=0}^{n} \frac{1}{i!}\, g\ \overline{\mathsf{d}}_Y^i\ f^{\otimes i}\ \mathsf{c}_X^i\,.$$

Since $\mathsf{d}_X\ \overline{\mathsf{d}}_X^i = 0$ for $i \neq 1$, we get $\mathsf{d}_X \circ g = g$. Next observe that $g \circ \mathsf{d}_X = g\ \mathrm{T}_X^n = g$ by Proposition 7. One can prove that $g \circ f$ is polynomial of degree $\leq mn$, that is $(g \circ f)\ \overline{\partial}_X^{mn} = 0$ by a straightforward (though boring) categorical computation using the basic axioms of exponential structures. Using the same axioms, one shows that this notion of composition is associative, so that we have defined a category of polynomial morphisms.

2.2.6 Weak functoriality of the exponential. We do not require the operation $X \mapsto\ !X$ to be functorial, but some weak form of functoriality can be derived from the above categorical axioms. Let $f \in \mathcal{C}(X, Y)$. By induction on n, we define a family of morphisms $f^n : !X \to !X$ as follows: $f^0 = \overline{\mathsf{w}}_X\ \mathsf{w}_X$ and

$$f^{n+1} = \overline{\partial}_X\ (f^n \otimes f)\ \partial_X\,.$$

Proposition 9 *Let $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ and let $n, p \in \mathbb{N}$. Then*

$$g^p\ f^n = \begin{cases} n!(g\ f)^n & \text{if } n = p \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Simple calculation using the diagram commutations which define an exponential structure. □

So for each $n \in \mathbb{N}$ we can define $!_n f = \sum_{q=0}^{n} \frac{1}{q!} f^q : !X \to !X$, and we have $!_n g\ !_n f = !_n(g\ f)$. So $f \mapsto !_n f$ is a quasifunctor, but not a functor as it does not map Id_X to $\mathrm{Id}_{!X}$, but to the idempotent morphism $\mathrm{T}_X^n : !X \to !X$.

In many concrete models, this sequence $(!_n f)_{n \in \mathbb{N}}$ converges in a sense which depends of course on the model. The limit is then denoted as $!f$ and the operation defined in that way turns out often to be a true functor equipped with a comonad structure.

2.3 Computing anti-derivatives.

We say that an exponential structure $\mathcal{C}$ *has anti-derivatives* if it is bi-commutative and if the morphism $J_X = \mathrm{Id}_X + (\overline{\partial}_X\ \partial_X) : !X \to !X$ is an isomorphism, and in that case we set $I_X = J_X^{-1}$. We assume to be given an exponential structure $\mathcal{C}$ which has anti-derivatives.

In the sequel, we use the following notation

$$\psi_X = (\overline{\partial}_X \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X) : !X \otimes X \to !X \otimes X$$

because this morphism will show up quite often.

Lemma 10 *The following commutation holds*

$$(I_X \otimes \mathrm{Id}_X)\ \psi_X = \psi_X\ (I_X \otimes \mathrm{Id}_X) \tag{1}$$

Proof. Since $I_X = (\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X))^{-1}$, we have $I_X \otimes \mathrm{Id}_X = \varphi^{-1}$ where $\varphi = \mathrm{Id}_{!X\otimes X} + ((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X)$ by functoriality of $\otimes$. To prove (1), it suffices therefore to prove that φ commutes with ψ_X. For this, it suffices to show that $(\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X$ commutes with ψ_X. We have

$$((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X)\ \psi_X = ((\overline{\partial}_X\ \partial_X\ \overline{\partial}_X) \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X)$$

but remember that $\partial_X\ \overline{\partial}_X = \mathrm{Id}_{!X\otimes X} + \psi_X$ by Lemma 4, and hence

$$((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X)\ \psi_X = \psi_X + ((\overline{\partial}_X\ \psi_X) \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X)$$

But $\overline{\partial}_X\ (\overline{\partial}_X \otimes \mathrm{Id}_X)\ \sigma_{23} = \overline{\partial}_X\ (\overline{\partial}_X \otimes \mathrm{Id}_X)$ by commutativity of the bialgebra $!X$ and by definition of $\overline{\partial}_X$. Therefore $\overline{\partial}_X\ \psi_X = \overline{\partial}_X\ ((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X)$. So we can write

$$\begin{aligned}
&((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X)\ \psi_X \\
&\quad = \psi_X + (\overline{\partial}_X \otimes \mathrm{Id}_X)\ ((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X) \\
&\quad = \psi_X + (\overline{\partial}_X \otimes \mathrm{Id}_X)\ ((\overline{\partial}_X\ \partial_X) \otimes \sigma)\ (\partial_X \otimes \mathrm{Id}_X)
\end{aligned}$$

A similar, and completely symmetric computation, using this time the cocommutativity of the bialgebra $!X$, leads to

$$\psi_X\ ((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X) = \psi_X + (\overline{\partial}_X \otimes \mathrm{Id}_X)\ ((\overline{\partial}_X\ \partial_X) \otimes \sigma)\ (\partial_X \otimes \mathrm{Id}_X)$$

and we are done. □

We can now prove a completely categorical version of Poincaré's Lemma.

Proposition 11 *Let $f : !X \otimes X \to Y$ be such that the differential $f\ (\partial_X \otimes \mathrm{Id}_X) : !X \otimes X \otimes X \to Y$ satisfies*

$$f\ (\partial_X \otimes \mathrm{Id}_X)\ \sigma_{23} = f\ (\partial_X \otimes \mathrm{Id}_X)\,.$$

Then there exists $g : !X \to Y$ such that $g\ \overline{\partial}_X = f$; in other words, g is an "anti-derivative" of f.

Proof. One sets

$$g = f\ (I_X \otimes \mathrm{Id}_X)\ \partial_X\,. \tag{2}$$

Then we have

$$\begin{aligned} g\ \overline{\partial}_X &= f\ (I_X \otimes \mathrm{Id}_X)\ \partial_X\ \overline{\partial}_X \\ &= f\ (I_X \otimes \mathrm{Id}_X)\ (\mathrm{Id}_{!X \otimes X} + \psi_X) \\ &= f\ (I_X \otimes \mathrm{Id}_X) + f\ (I_X \otimes \mathrm{Id}_X)\ \psi_X \\ &= f\ (I_X \otimes \mathrm{Id}_X) + f\ \psi_X\ (I_X \otimes \mathrm{Id}_X) \end{aligned}$$

by Lemma 10. But

$$\begin{aligned} f\ \psi_X &= f\ (\overline{\partial}_X \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X) \\ &= f\ (\overline{\partial}_X \otimes \mathrm{Id}_X)\ (\partial_X \otimes \mathrm{Id}_X) \quad \text{by our hypothesis on } f \\ &= f\ ((\overline{\partial}_X\ \partial_X) \otimes \mathrm{Id}_X)\,. \end{aligned}$$

So we get

$$\begin{aligned} g\ \overline{\partial}_X &= f\ (I_X \otimes \mathrm{Id}_X) + f\ ((\overline{\partial}_X\ \partial_X\ I_X) \otimes \mathrm{Id}_X) \\ &= f\ (((\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X))\ I_X) \otimes \mathrm{Id}_X) \\ &= f \end{aligned}$$

since I_X is the inverse of $\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X)$. □

2.3.1 Comments. Let us give some intuition about our axiom on I_X. Given $f : !X \to Y$ seen as a "regular function" (analytical, smooth...) from X to Y, the basic idea is that the morphism $f\ I_X : !X \to Y$ stands for the regular function g defined by

$$g(x) = \int_0^1 f(tx)\,dt$$

assuming of course that this integral makes sense. Then $g\ \overline{\partial}_X : !X \otimes X \to Y$ represents the differential $\mathsf{D}g$ of g, a regular function $X \times X \to Y$ which maps (x, y) to $\mathsf{D}g(x) \cdot y$ and is linear in y. Applying the ordinary rules of differential calculus, and the fact that differentiation commutes with integration, we get

$$\mathsf{D}g(x) \cdot y = \int_0^1 t(\mathsf{D}f(tx) \cdot y)dt\,.$$

The morphism $h = g\ \overline{\partial}_X\ \partial_X : !X \to Y$ corresponds therefore to the regular function from X to Y such that

$$h(x) = \int_0^1 t(\mathsf{D}f(tx) \cdot x)dt = \int_0^1 t\frac{df(tx)}{dt}dt = f(x) - \int_0^1 f(tx)\,dt\,,$$

integrating by parts. In other words, we have seen that

$$f\ I_X\ \overline{\partial}_X\ \partial_X = f - (f\ I_X)$$

that is

$$f\ I_X\ (\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X)) = f$$

this is why our first axiom on I_X is that $I_X\ (\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X)) = \mathrm{Id}_{!X}$. To explain why we also require $(\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X))\ I_X = \mathrm{Id}_{!X}$, observe that $l = f\ \overline{\partial}_X\ \partial_X : {!X} \to Y$ corresponds to the regular function $X \to Y$ defined by $l(x) = \mathsf{D}f(x) \cdot x$ and hence $l\ I_X$ corresponds to the regular function $m : X \to Y$ given by

$$m(x) = \int_0^1 (\mathsf{D}f(tx) \cdot (tx))dt = h(x)$$

by linearity of the differential. So we have

$$f\ \overline{\partial}_X\ \partial_X\ I_X = f - (f\ I_X)$$

that is

$$f\ (\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X))\ I_X = f$$

and this is why we also assume that $(\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X))\ I_X = \mathrm{Id}_{!X}$.

A quite natural feature of this axiomatization of anti-derivatives is the fact that it is a mere *property* of $\mathcal{C}$ and not an additional structure: to have anti-derivatives, the exponential structure must be bi-commutative and such that $\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X)$ has an inverse.

2.3.2 The fundamental theorem of calculus.

This is the statement according to which one can use anti-derivatives for computing integrals: if $f, g : \mathbf{R} \to \mathbf{R}$ are such that $g' = f$, then $\int_a^b f(t)\,dt = g(b) - g(a)$. In the present setting, it boils down to a simple categorical equation.

Proposition 12 *Let $\mathcal{C}$ be an exponential structure which has anti-derivatives and is Taylor. Then*

$$\overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X)\ \partial_X + \overline{\mathsf{w}}_X\ \mathsf{w}_X = \mathrm{Id}_{!X}\ .$$

Proof. Let $f_1 = \overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X)\ \partial_X : {!X} \to {!X}$ and let $f_2 = \mathrm{Id}_{!X}$. We have

$$\begin{aligned}
f_1\ \overline{\partial}_X &= \overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X)\ \partial_X\ \overline{\partial}_X \\
&= \overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X) + \overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X)\ (\overline{\partial}_X \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X) \\
&= \overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X) + \overline{\partial}_X\ (\overline{\partial}_X \otimes \mathrm{Id}_X)\ \sigma_{23}\ (\partial_X \otimes \mathrm{Id}_X)\ (I_X \otimes \mathrm{Id}_X) \\
&\qquad \text{by Lemma 10} \\
&= \overline{\partial}_X\ (I_X \otimes \mathrm{Id}_X) + \overline{\partial}_X\ (\overline{\partial}_X \otimes \mathrm{Id}_X)\ (\partial_X \otimes \mathrm{Id}_X)\ (I_X \otimes \mathrm{Id}_X) \\
&\qquad \text{by commutativity of co-contraction} \\
&= \overline{\partial}_X\ ((\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X)) \otimes \mathrm{Id}_X)\ (I_X \otimes \mathrm{Id}_X) \\
&= \overline{\partial}_X \qquad \text{since } I_X = (\mathrm{Id}_{!X} + (\overline{\partial}_X\ \partial_X))^{-1} \\
&= f_2\ \overline{\partial}_X\ .
\end{aligned}$$

Since $\mathcal{C}$ is Taylor, we have therefore $f_1 + (f_2\ \overline{\mathsf{w}}_X\ \mathsf{w}_X) = f_2 + (f_1\ \overline{\mathsf{w}}_X\ \mathsf{w}_X)$, which is exactly the announced equation since $f_1\ \overline{\mathsf{w}}_X = 0$. □

Remark 13 We give now an intuitive interpretation of this property. Let $f : !X \to Y$, considered as a regular function $X \to Y$. Then $f\,\overline{\partial}_X\,(I_X \otimes \mathrm{Id}_X)\,\partial_X : !X \to Y$ represents the regular function $g : X \to Y$ given by

$$g(x) = \int_0^1 \mathsf{D}f(tx) \cdot x\, dt = \int_0^1 \frac{df(tx)}{dt}\, dt$$

so that $g(x) = f(x) - f(0)$ by the Fundamental Theorem of Calculus. In other words $g(x) + f(0) = f(x)$, that is $f\,(\overline{\partial}_X\,(I_X \otimes \mathrm{Id}_X)\,\partial_X + \overline{\mathsf{w}}_X\,\mathsf{w}_X) = f$.

3 Computing anti-derivatives in the resource calculus

We make now a first attempt at introducing anti-derivatives in the syntax. For this purpose, we consider finite linear combinations of finite resource terms (see Section 1.7.2) as polynomials, and with this respect, it seems natural to formally compute the anti-derivative of such a term, as one does for polynomials. This is the purpose of this short section.

We assume that, in $\mathbf{k}$, natural numbers have inverses.

As with ordinary polynomials, we define first the anti-derivative of a monomial, that is, of a simple resource term. Remember that, for ordinary one variable polynomials, the anti-derivative of X^d is $\frac{1}{d+1}X^{d+1}$; the definition is completely similar here. Let $t \in \Delta^!$ be a simple resource term and let x be a variable. We set

$$I_x(t) = \frac{1}{\deg_x t + 1} t\,.$$

We extend this operation by linearity to all elements $u \in \mathbf{k}\langle\Delta^!\rangle$, that is we set $I_x(u) = \sum_{t\in\Delta} u_t I_x(t)$.

For $d \in \mathbb{N}$, let $\Delta_x^{(d)} = \{t \in \Delta \mid \deg_x t = d\}$ be the set of all simple resource terms of degree d in the variable x. The elements of $\mathbf{k}\langle\Delta_x^{(d)}\rangle$ are said to be homogeneous of degree d in x.

With these notations, we can write

$$I_x(u) = \sum_{d=0}^{\infty} \frac{1}{d+1} \sum_{t\in\Delta_x^{(d)}} u_t t$$

Intuitively, $I_x(u)$ stands for the integral $\int_0^1 u(\tau x)\, d\tau$ which is the basic ingredient in the proof above of Poincaré's Lemma.

Let $u \in \mathbf{k}\langle\Delta\rangle$ which is linear in the variable h, in other words $u \in \mathbf{k}\langle\Delta_h^{(1)}\rangle$. Let h' be a variable which does not occur free in u, we assume that

$$\frac{\partial u}{\partial x} \cdot h' = \frac{\partial u\,[h'/h]}{\partial x} \cdot h$$

which is our symmetry hypothesis on u (see the hypothesis in the statement of Proposition 11). In other words, for any $d \in \mathbb{N}$, we have

$$\sum_{t \in \Delta_x^{(d)}} u_t \frac{\partial t}{\partial x} \cdot h' = \sum_{t \in \Delta_x^{(d)}} u_t \frac{\partial t\,[h'/h]}{\partial x} \cdot h\,. \tag{3}$$

Mimicking (2), we set

$$v = I_x(u)\,[x/h]$$

and we prove that

$$\frac{\partial v}{\partial x} \cdot h = u\,. \tag{4}$$

Choose h' as above, we contend that $\frac{\partial v}{\partial x} \cdot h' = u\,[h'/h]$ which of course implies (4). We have indeed

$$\begin{aligned}
\frac{\partial v}{\partial x} \cdot h' &= \sum_{d=0}^{\infty} \frac{1}{d+1} \sum_{t \in \Delta_x^{(d)}} u_t \frac{\partial t\,[x/h]}{\partial x} \cdot h' \\
&= \sum_{d=0}^{\infty} \frac{1}{d+1} \sum_{t \in \Delta_x^{(d)}} u_t \left(\left(\frac{\partial t}{\partial x} \cdot h' \right) [x/h] + t\,[h'/h] \right) \\
&= \sum_{d=0}^{\infty} \frac{1}{d+1} \sum_{t \in \Delta_x^{(d)}} u_t \left(\left(\frac{\partial t\,[h'/h]}{\partial x} \cdot h \right) [x/h] + t\,[h'/h] \right) \quad \text{by (3)} \\
&= \sum_{d=0}^{\infty} \frac{1}{d+1} \sum_{t \in \Delta_x^{(d)}} u_t \left(\frac{\partial t\,[h'/h]}{\partial x} \cdot x + t\,[h'/h] \right) \\
&= \sum_{d=0}^{\infty} \frac{1}{d+1} \sum_{t \in \Delta_x^{(d)}} u_t \left(dt\,[h'/h] + t\,[h'/h] \right) \quad \text{since } \frac{\partial s}{\partial x} \cdot x = ds \text{ for all } s \in \Delta_x^{(d)} \\
&= u\,[h'/h]
\end{aligned}$$

and we are done. This result can easily be extended to infinite linear combinations of finite resource terms and one can check that anti-derivatives preserve the finiteness properties developed in [Ehr10]. The problem of defining anti-derivatives in a setting where promotion is available as a syntactic construction remains open.

4 Models of full differential linear logic

Until now, we have considered situations where the "!" operation is not required to be functorial. Such "finitary" models allow to interpret the promotion-free fragment of differential linear logic, or the finite resource lambda-calculus. In the present section, we address the categorical structure required to interpret full differential linear logic, including the promotion operation. The main references for this section are [BCS06, Fio07], though many of the constructions and conditions presented here were already introduced in [Ehr02, Ehr05] where we presented models of LL which were at the origine of the discovery of DiLL.

Such a model consists of a Seely category (in the sense of [Mel09], a notion originally called “new-Seely category” in [Bie95]), that is

- a cartesian $*$-autonomous category $\mathcal{C}$ (the cartesian product of two objects X and Y is denoted as $X \mathbin{\&} Y$ and the terminal object is denoted as $\top$);
- a comonad $!\_ : \mathcal{C} \to \mathcal{C}$ which is monoidal from $(\mathcal{C}, \top, \&)$ to $(\mathcal{C}, \otimes, 1)$ (counit denoted as $\mathsf{d}_X : !X \to X$ and called *dereliction*, comultiplication denoted as $\mathsf{p}_X : !X \to !!X$ and called *digging*, monoidality isomorphisms $\mathsf{m}_{X,Y} : !X \otimes !Y \to !(X \mathbin{\&} Y)$, $\mathsf{m}_1 : 1 \to !\top$, often called *Seely isomorphisms* though they were noticed first by Girard, see [Gir87]) such that the following diagram commutes (it expresses a coherence condition relating the isomorphism m and the natural transformation p)

$$\begin{array}{ccc}
!X \otimes !Y & \xrightarrow{\mathsf{m}_{X,Y}} & !(X \mathbin{\&} Y) \\
\Big\downarrow & & \Big\downarrow \mathsf{p}_{X \& Y} \\
\mathsf{p}_X \otimes \mathsf{p}_Y & & !!(X \mathbin{\&} Y) \\
\Big\downarrow & & \Big\downarrow !\langle !\pi_1, !\pi_2 \rangle \\
!!X \otimes !!Y & \xrightarrow{\mathsf{m}_{!X,!Y}} & !(!X \mathbin{\&} !Y)
\end{array} \tag{5}$$

This monoidal structure allows to define a lax monoidal structure on the functor $!\_$ from the monoidal category $(\mathcal{C}, \otimes, 1)$ to itself: this monoidal structure consists of a morphism $\mu_1 : 1 \to !1$ and of a natural transformation $\mu_{X,Y} : !X \otimes !Y \to !(X \otimes Y)$ that we give now explicitly. We define μ_1 as the following composition of morphisms:

$$1 \xrightarrow{\mathsf{m}_1} !\top \xrightarrow{\mathsf{p}_\top} !!\top \xrightarrow{!(\mathsf{m}_1^{-1})} !1$$

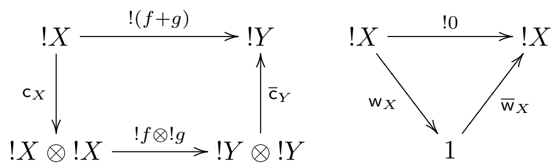
and $\mu_{X,Y}$ as

$$\begin{array}{c}
!X \otimes !Y \xrightarrow{\mathsf{m}_{X,Y}} !(X \mathbin{\&} Y) \xrightarrow{\mathsf{p}_{X \& Y}} !!(X \mathbin{\&} Y) \xrightarrow{!(\mathsf{m}_{X,Y}^{-1})} !(!X \otimes !Y) \\
\Big\downarrow !(\mathsf{d}_X \otimes \mathsf{d}_Y) \\
!(X \otimes Y)
\end{array}$$

We assume moreover that the $*$-autonomous cartesian category $\mathcal{C}$ is enriched over $\mathbf{k}$-modules as explained in Section 2.1, but the functor $!\_$ is not assumed to preserve this structure. From now on we use only the notation $\oplus$ for denoting the binary bi-product and 0 for denoting its unit.

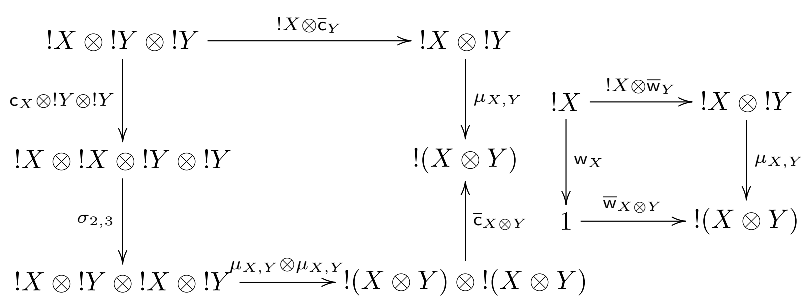
Using these ingredients, we can endow each object $!X$ with a canonical structure of commutative bi-algebra. For the coalgebraic part, let $\Delta_X \in \mathcal{C}(X, X \oplus X)$ be the diagonal morphism associated with the cartesian product of X with itself. Then we set $\mathsf{c}_X = \mathsf{m}^{-1}_{X,X}\ !\Delta_X : !X \to !X \otimes !X$, and similarly we set $\mathsf{w}_X = \mathsf{m}_1\ !\mathsf{t}_X$ where $\mathsf{t}_X : X \to 0$ results from the fact that 0 is the terminal object. The algebraic structure is defined similarly, using the codiagonal $\overline{\Delta}_X : X \oplus X \to X$ and the morphism $\bar{\mathsf{t}}_X : 0 \to X$. Checking that the required commutation hold is a straightforward verification.

Let $f, g \in \mathcal{C}(X, Y)$. It results from these definitions of c_X, $\bar{\mathsf{c}}_Y$, w_X and $\overline{\mathsf{w}}_X$ that the following diagrams commute



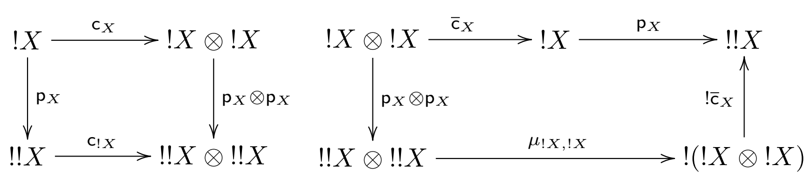
enforcing the idea that $!\_$ really behaves like an exponential operation.

Due to these definitions, we have the following very useful commutations:



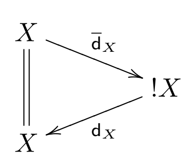
(6)

Using the definitions of c_X and $\bar{\mathsf{c}}_X$ and the above axioms, we can also prove the two following commutations.



(7)

Observe that, until now, we are in a completely standard linear logic setting, the only addition wrt. the usual situations being the $\mathcal{C}$ is enriched over **k**-modules. We introduce now the only differential structure needed to interpret differential linear logic: we assume to be given a *co-dereliction* natural transformation $\bar{\mathsf{d}}_X : X \to !X$ such that



(8)

Using simply the naturality of d_X and $\overline{\mathsf{d}}_X$ as well as the monoidality commutation involving $\mathsf{m}_{X,Y}$ and m_1, it is then easy to prove that

$$X \mapsto (!X, \mathsf{c}_X, \mathsf{w}_X, \mathsf{d}_X, \overline{\mathsf{c}}_X, \overline{\mathsf{w}}_X, \overline{\mathsf{d}}_X)$$

defines an exponential structure on $\mathcal{C}$, in the sense of Section 2.2.

Further commutations are required, involving the codereliction morphism. They give rise in particular to a categorical version of the chain rule of calculus. As observed by Fiore [Fio07], we need to have

$$\begin{array}{ccc} X \otimes !Y & \xrightarrow{\overline{\mathsf{d}}_X \otimes !Y} & !X \otimes !Y \\ \Big\downarrow {\scriptstyle X \otimes \mathsf{d}_Y} & & \Big\downarrow {\scriptstyle \mu_{X,Y}} \\ X \otimes Y & \xrightarrow{\overline{\mathsf{d}}_{X \otimes Y}} & !(X \otimes Y) \end{array} \tag{9}$$

It would be interesting to know if this condition can be reduced to a more primitive one, involving $\overline{\mathsf{d}}_X$ and the isomorphism m (of course, one can replace μ by its expression in terms of m in the diagram above, be we would like to find a simpler and more elegant commuting diagram).

Last we have to provide a commutation relating $\overline{\mathsf{d}}_X$ and p_X. We have of course $\overline{\mathsf{d}}_{!X}\ \overline{\mathsf{d}}_X : X \to !!X$. Also, $\mu_1 : 1 \to !1$ and therefore $!\overline{\mathsf{w}}_X\ \mu_1 : 1 \to !!X$. Keeping implicit the isomorphism $X \otimes 1 \simeq X$, we get $(\overline{\mathsf{d}}_{!X}\ \overline{\mathsf{d}}_X) \otimes (!\overline{\mathsf{w}}_X\ \mu_1) : X \to !!X \otimes !!X$, and we require the following diagram to commute:

$$\begin{array}{ccc} X & \xrightarrow{\overline{\mathsf{d}}_X} & !X \\ {\scriptstyle (\overline{\mathsf{d}}_{!X}\ \overline{\mathsf{d}}_X) \otimes (!\overline{\mathsf{w}}_X\ \mu_1)} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathsf{p}_X} \\ !!X \otimes !!X & \xrightarrow{\overline{\mathsf{c}}_{!X}} & !!X \end{array} \tag{10}$$

As an illustration of these axioms, let us argue that the chain rule holds. Consider two morphisms $g \in \mathcal{C}(!Y, Z)$ and $f \in \mathcal{C}(!X, Y)$ (so that $g : Y \to Z$ and $f : X \to Y$ in the Kleisli category of the comonad $!\_$, which is cartesian closed, see [Mel09]). Their composition in the Klesili category, that we denote as $h = f \circ g = f\ g^!$ where $g^! = !g\ \mathsf{p}_Y : !Y \to !Z$, that we consider as a map of one parameter (of type X) with values in type Z. The derivative of h is $h' = h\ \overline{\partial}_X : !X \otimes X \to Z$ and, according to the usual chain rule of Calculus, we should have $h' = f'\ (g^! \otimes g')\ (\mathsf{c}_X \otimes X)$. As easily seen, this boils down to the commutation of the following diagram (given in [Fio07] as an axiom):

$$\begin{array}{ccccc} !X \otimes X & \xrightarrow{\overline{\partial}_X} & !X & \xrightarrow{\mathsf{p}_X} & !!X \\ {\scriptstyle \mathsf{c}_X \otimes X} \Big\downarrow & & & & \Big\uparrow {\scriptstyle \overline{\partial}_{!X}} \\ !X \otimes !X \otimes X & \xrightarrow{!X \otimes \overline{\partial}_X} & !X \otimes !X & \xrightarrow{\mathsf{p}_X \otimes !X} & !!X \otimes !X \end{array}$$

The proof of this commutation is a rather tedious categorical computation based essentially on Diagram (10) and Diagrams (7).

5 Relational semantics

We introduce now the simplest $*$-autonomous category equipped with an exponential structure: the category of sets and relations.

Let **Rel** be the category whose objects are sets and where $\mathbf{Rel}(X,Y) = \mathcal{P}(X \times Y)$, identities being the diagonal relations and composition being defined as follows: if $R \in \mathbf{Rel}(X,Y)$ and $S \in \mathbf{Rel}(Y,Z)$ then

$$S\,R = \{(a,c) \in X \times Z \mid \exists b \in Y\ (a,b) \in R \text{ and } (b,c) \in S\}\,.$$

Let $x \subseteq X$, we set $Rx = \{b \in Y \mid (a,b) \in R\} \subseteq Y$ which is the direct image of x by R. We also define ${}^{\mathrm{t}}R = \{(b,a) \in Y \times X \mid (a,b) \in Y\}$ which is the transpose of R. Given $x \subseteq X$ and $y' \subseteq Y$, we have

$$(Rx) \cap y' = \pi_2(R \cap (x \times y')) \quad \text{and} \quad ({}^{\mathrm{t}}Ry') \cap x = \pi_1(R \cap (x \times y'))\,. \tag{11}$$

Observe that an isomorphism in **Rel** is a relation which is a bijection.

The symmetric monoidal structure is given by the tensor product $X \otimes Y = X \times Y$ and the unit 1 is an arbitrary singleton. The neutrality, associativity and symmetry isomorphisms are defined as the obvious corresponding bijections (for instance, the symmetry isomorphism $\sigma_{X,Y} \in \mathbf{Rel}(X \otimes Y, Y \otimes X)$ is given by $\sigma(a,b) = (b,a)$). This symmetric monoidal category is enriched over $\mathbb{B}$-modules (addition of morphisms being simply their union) and is closed, with linear function space given by $X \multimap Y = X \times Y$, the natural bijection between $\mathbf{Rel}(Z \otimes X, Y)$ and $\mathbf{Rel}(Z, X \multimap Y)$ being induced by the cartesian product associativity isomorphism. Last, one takes for $\perp$ an arbitrary singleton, and this turns **Rel** into a $*$-autonomous category. One denotes as $\star$ the unique element of 1 and $\perp$.

This category is cartesian, with cartesian product of X and Y defined as their disjoint union, and terminal object being the empty set.

Rel is also a Seely category (see Section 4), for a comonad $!\_$ defined as follows:

- $!X$ is the set of all finite multisets of elements of X;
- if $R \in \mathbf{Rel}(X,Y)$, then we set $!R = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid n \in \mathbb{N} \text{ and } \forall i\,(a_i,b_i) \in R\}$;
- $\mathsf{d}_X \in \mathbf{Rel}(!X,X)$ is $\mathsf{d}_X = \{([a],a) \mid a \in X\}$;
- $\mathsf{p}_X = \{(m_1 + \cdots + m_n, [m_1,\dots,m_n]) \mid n \in \mathbb{N} \text{ and } m_1,\dots,m_n \in !X\}$.

The monoidality isomorphism $\mathsf{m}_{X,Y} \in \mathbf{Rel}(!X \otimes !Y, !(X \mathbin{\&} Y))$ is the bijection which maps $([a_1,\dots,a_l],[b_1,\dots,b_r])$ to $[(1,a_1),\dots,(1,a_l),(2,b_1),\dots,(2,b_r)]$. Last, we also provide a co-dereliction natural transformation $\overline{\mathsf{d}}_X \in \mathbf{Rel}(X, !X)$ which is simply given by $\overline{\mathsf{d}}_X = \{(a, [a]) \mid a \in X\}$.

With these definitions, it is easy to see that $\mathsf{c}_X = \{(l + r, (l, r)) \mid l, r \in !X\}$, $\mathsf{w}_X = \{([], \star)\}$, $\overline{\mathsf{c}}_X = \{((l, r), l + r) \mid l, r \in !X\}$ and $\overline{\mathsf{w}}_X = \{(\star, [])\}$. The required diagrams (8), (9) and (10) are easily seen to commute.

Anti-derivatives. This exponential structure is bi-commutative and can easily seen to be Taylor in the sense of 2.2.4. Moreover, it has anti-derivatives in the sense of 2.3, simply because the morphism $J_X = \mathrm{Id}_X + (\overline{\partial}_X\ \partial_X) : !X \to !X$ coincides here with the identity. Indeed $\partial_X = \{(l + [a], (l, a)) \mid l \in !X \text{ and } a \in X\}$, $\overline{\partial}_X = \{((l, a), l + [a]) \mid l \in !X \text{ and } a \in X\}$ and therefore $\overline{\partial}_X\ \partial_X = \{(l, l) \mid l \in !X \text{ and } \#l > 0\}$.

Concretely, saying that a morphism $f \in \mathbf{Rel}(!X \otimes X, Y)$ satisfies the symmetry condition of Proposition 11 simply means that, given $m \in \mathcal{M}_{\mathrm{fin}}(X)$, $a, a' \in X$ and $b \in Y$, one has $((m + [a], a'), b) \in f \Leftrightarrow ((m + [a'], a), b) \in f$. In that case, the anti-derivative $g \in \mathbf{Rel}(!X, Y)$ given by that proposition is simply

$$g = \{(m + [a], b) \mid ((m, a), b) \in f\}\,.$$

Other exponential structures. As observed by De Carvalho [dC07], other exponential structures are possible in this relational setting, for instance we can define $!X$ as the set of all finite sequences $\langle a_1, \ldots, a_n\rangle$ of elements of X. The (co)weakening and (co)dereliction can be defined as above (for instance $\mathsf{d}_X = \{(\langle a\rangle, a) \mid a \in X\}$). Given two finite sequences $l, r \in !X$, let $\mathsf{Shuffle}(l, r)$ be the set of all shuffles of the sequences l and r (a shuffle of $\langle a_1, \ldots, a_m\rangle$ and $\langle b_1, \ldots, b_n\rangle$ is any sequence $\langle c_1, \ldots, c_{m+n}\rangle$ such that there are strictly increasing functions $\varphi : \{1, \ldots, m\} \to \{1, \ldots, n + m\}$ and $\psi : \{1, \ldots, n\} \to \{1, \ldots, n + m\}$ with disjoint ranges and such that $c_{\varphi(i)} = a_i$ and $c_{\psi(j)} = b_j$ for each i, j). Then contraction and co-contraction can be defined by

$$\overline{\mathsf{c}}_X = \{(l, r), lr) \mid l, r \in !X\} \quad \text{and} \quad \mathsf{c}_X = \{(q, (l, r)) \mid q \in \mathsf{Shuffle}(l, r)\}$$

where we denote as lr the concatenation of the sequences l and r. In that case, the bialgebra $!X$ is cocomutative, but not commutative. This relational bialgebra structure underlies the standard shuffle bialgebra, see eg. [Reu93].

6 Finiteness spaces

This model can be seen as an enrichment of the model of sets and relations of Section 5. But it can also be described as a category of topological vector spaces and linear continuous maps. From now on, $\mathbf{k}$ denotes an arbitrary field which is always endowed with the discrete topology.

6.1 Linearly topologized vector spaces (ltvs)

We start with general definitions and properties about ltvs. The main references for these topics are [Lef42] (where the notion is introduced) and [Köt69], chapters 10-13.

Let E be a $\mathbf{k}$-vector space. A *linear topology* on E is a topology λ such that there is a filter $\mathcal{L}$ of linear subspaces[5] of E with the following property: a subset U of E is λ-open iff for any $x \in U$ there exists $V \in \mathcal{L}$ such that $x + V \subseteq U$. One says that such a filter $\mathcal{L}$ *generates* the topology λ. A $\mathbf{k}$-ltvs is a $\mathbf{k}$-vector space equipped with a linear topology. Observe that E is Hausdorff iff $\bigcap \mathcal{L} = \{0\}$ (for one, and hence any, generating filter $\mathcal{L}$); from now on we assume always that this is the case.

It is easy to check that addition $E \times E \to E$ and scalar multiplication $\mathbf{k} \times E \to E$ are continuous, whatever be the linear topology λ on E (remember that $\mathbf{k}$ is endowed with the discrete topology, and as such, it is an ltvs).

Proposition 14 *Let E be a $\mathbf{k}$-ltvs. Any linear subspace U of E which is a neighborhood of 0 is both open and closed. So E is totally disconnected (the only subsets of E which are connected are the empty set and the one point sets).*

Proof. Let $\mathcal{L}$ be a generating filter for the topology of E. First, let $x \in U$ and let $V \in \mathcal{L}$ be such that $V \subseteq U$, then we have $x + V \subseteq U$ since U is a linear subspace and hence U is open. Next let $x \in E \setminus U$. If $y \in U \cap (x + U)$ then we have $y - x \in U$ and hence $x \in U$ since $y \in U$ and U is a linear subspace: contradiction. Therefore $U \cap (x + U) = \emptyset$ and U is closed since U open. □

Observe also that a linear subspace U of an ltvs E is open iff it is a neighborhood of 0 (if U is a neighborhood of 0, one can find an open linear subspace V of E such that $V \subseteq U$ but then $x + V \subseteq U$ for all $x \in U$ since U is a linear subspace). In other words, any linear subspace which contains an open linear subspace is open.

6.1.1 Cauchy completeness.

A *net* in E is a family $(x_d)_{d \in D}$ of elements of E indexed by a directed poset D. The net $(x_d)_{d \in D}$ *converges* to $x \in E$ if, for any neighborhood U of 0, there exists $d \in D$ such that $\forall e \in D\ e \geq d \Rightarrow x_e - x \in U$. Because E is Hausdorff, a net converges to at most one point. As usual, one can check that a subset U of E is open iff, for any net $(x_d)_{d \in D}$ which converges to a point $x \in U$, there exists $d \in D$ such that $\forall e \in D\ e \geq d \Rightarrow x_e \in U$.

A net $(x_d)_{d \in D}$ is *Cauchy* if, for any neighborhood U of 0, there exists $d \in D$ such that $\forall e, e' \in D\ e, e' \geq d \Rightarrow x_e - x_{e'} \in U$. This latter statement is equivalent to $\forall e \in D\ e \geq d \Rightarrow x_e - x_d \in U$.

One says that E is *complete* if any Cauchy net in E converges.

6.1.2 Linear boundedness.

Let E be an ltvs and let U be an open linear subspace of E. Let $\pi_U : E \to E/U$ be the canonical projection. This map is of course linear, and its kernel is U which is a neighborhood of 0. This means

[5] This is the main feature of this notion: basic neighborhoods are linear subspaces, and this makes these topological vector spaces quite different from the ones we are used to, which are locally convex and based on the usual topology of $\mathbb{R}$ or $\mathbb{C}$. They are nevertheless very well behaved.

that, endowing E/U with the discrete topology, π_U is continuous. Hence the quotient topology on E/U is the discrete topology.

We say that a subspace B of E is linearly bounded if $\pi_U(B)$ is finite dimensional, for all linear open subspace or E. In other words, for any linear open subspace U, there is a finite dimensional subspace A of E such that $B \subseteq U + A$.

Proposition 15 *Any finite dimensional subspace of an ltvs E is linearly bounded. Let B_1 and B_2 be subspaces of E. If $B_1 \subseteq B_2$ and B_2 is linearly bounded, so is B_1. If B_1 and B_2 are linearly bounded, so is $B_1 + B_2$.*

A collection of subspaces of a vector space F having these properties is called a *linear bornology* on F.

An ltvs E is *locally linearly bounded* if it has a linear open subspace which is linearly bounded.

6.1.3 Linear and multilinear maps.

Let $E_1, \dots, E_n$ and F be $\mathbf{k}$-ltvs's. An n-multilinear function $f : E_1 \times \cdots \times E_n \to F$ is *hypocontinuous* if, for any $i \in \{1, \dots, n\}$, any linear open subspace $V \subseteq F$ and any linearly bounded subspaces $B_1 \subseteq E_1, \dots, B_{i-1} \subseteq E_{i-1}$, $B_{i+1} \subseteq E_{i+1}, \dots, B_n \subseteq E_n$, there exists an open linear subspace $U \subseteq E_i$ such that $f(B_1 \times \cdots \times B_{i-1} \times U \times B_{i+1} \times \cdots \times B_n) \subseteq V$.

We denote by $(E_1, \dots, E_n) \multimap F$ the $\mathbf{k}$-vector space of all such multilinear maps. Given linearly bounded subspaces $B_1, \dots, B_n$ of $E_1, \dots, E_n$ respectively and given a linear open subspace V of F, we define

$$\mathsf{ann}(B_1, \dots, B_n, V) = \{f \in (E_1, \dots, E_n) \multimap F \mid f(B_1 \times \cdots \times B_n) \subseteq V\}.$$

This is a linear subspace of $(E_1, \dots, E_n) \multimap F$ and by Proposition 15 these subspaces form a filter which defines a linear topology on $(E_1, \dots, E_n) \multimap F$ and this topology is Hausdorff. Indeed, if $f \in (E_1, \dots, E_n) \multimap F$ is $\neq 0$, then take $x_i \in E_i$ such that $f(x_1, \dots, x_n) \neq 0$. Since F is Hausdorff, there is a linear neighborhood V of 0 in F such that $f(x_1, \dots, x_n) \notin V$. Let $B_i = \mathbf{k}x_i$; this is a linearly bounded subspace of E_i and $f(B_1 \times \cdots \times B_n) \not\subseteq V$.

In the case $n = 1$ (and $E = E_1$), the corresponding maps $f : E \to F$ are simply called linear, and they are continuous. The corresponding function space is denoted as $E \multimap F$. If $F = \mathbf{k}$, then the corresponding maps are called (multi)linear (hypo)continuous forms. If furthermore $n = 1$ the corresponding function space is denoted as E' and is called *topological dual* of E.

Proposition 16 *Let $f : E_1 \times \cdots \times E_n \to F$ be multilinear and hypocontinuous and let $B_i \subseteq E_i$ be linearly bounded subspaces for $i = 1, \dots, n$. Then $f(B_1 \times \cdots \times B_n)$ is a linearly bounded subspace of F.*

Proof. Let V be an open linear subspace of F. Let U_1 be an open linear subspace of E_1 such that $f(U_1 \times B_1 \times \cdots \times B_n) \subseteq V$. Let A_1 be a finite dimensional subspace of E_1 such that $B_1 \subseteq U_1 + A_1$, we have $f(B_1 \times \cdots \times B_n) \subseteq V + f(A_1 \times B_2 \times \cdots \times B_n)$. Since A_1 is bounded, one can find similarly a finite dimensional subspace A_2 of E_2 such that $f(A_1 \times B_2 \times \cdots \times B_n) \subseteq V + f(A_1 \times A_2 \times B_3 \times \cdots \times B_n)$ and hence (since $V + V = V$) we get $f(B_1 \times \cdots \times B_n) \subseteq V + f(A_1 \times A_2 \times B_3 \times \cdots \times B_n)$. Continuing this process, we find finite dimensional subspaces A_i of E_i for $i = 1, \dots, n$ such that $f(B_1 \times \cdots \times B_n) \subseteq V + f(A_1 \times \cdots \times A_n)$ and we conclude that $f(B_1 \times \cdots \times B_n)$ is linearly bounded since $f(A_1 \times \cdots \times A_n)$ is finite dimensional. □

It is tempting to think that (multi)linear (hypo)continuous maps could be characterized as those which preserve linear boundedness. This cannot be the case: think of a linear map $f : E \to F$ where F is finite dimensional. Such a map preserves linear boundedness (any subspace of F is linearly bounded) but has no reason to be continuous.

It should also be noticed that multilinear maps which are continuous wrt. the product topology are hypocontinuous. The converse holds when the source ltvs's are locally linearly bounded, but not when they are more general spaces.

6.2 Finiteness spaces and the related ltvs's

We restrict now our attention to particular ltvs's which can be described in a simple combinatorial way. This material is mainly borrowed to [Ehr05], with a viewpoint improved thanks to the work of Tasson [Tas09b, Tas09a], and we provide also additional considerations and results.

6.2.1 Basic definitions. Let I be a set. Given $\mathcal{F} \subseteq \mathcal{P}(I)$, we define $\mathcal{F}^\perp \subseteq \mathcal{P}(I)$ by

$$\mathcal{F}^\perp = \{u' \subseteq I \mid \forall u \in \mathcal{F}\ u \cap u' \text{ is finite}\}.$$

We have $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G}^\perp \subseteq \mathcal{F}^\perp$, $\mathcal{F} \subseteq \mathcal{F}^{\perp\perp}$ and therefore $\mathcal{F}^{\perp\perp\perp} = \mathcal{F}^\perp$.

A finiteness space is a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is a set and $\mathsf{F}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$. The following properties follow easily from the definition

- if $u \subseteq |X|$ is finite then $u \in \mathsf{F}(X)$
- if $u, v \in \mathsf{F}(X)$ then $u \cup v \in \mathsf{F}(X)$
- if $u \subseteq v \in \mathsf{F}(X)$, then $u \in \mathsf{F}(X)$.

Let us prove for instance the second statement. Let $u' \in \mathsf{F}(X)^\perp$, then $(u \cup v) \cap u' = (u \cap u') \cup (v \cap u')$ is finite since both sets $u \cap u'$ and $v \cap u'$ are finite by our hypothesis that $u, v \in \mathsf{F}(X)$. Since this holds for all $u' \in \mathsf{F}(X)^\perp$, we have $u \cup v \in \mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$.

A *strong isomorphism*[6] between two finiteness spaces X and Y is a bijection $\varphi : |X| \to |Y|$ such that, for all $u \subseteq |X|$, one has $u \in \mathsf{F}(X)$ iff $\varphi(u) \in \mathsf{F}(Y)$.

[6] This would coincide with the categorical notion of isomorphism if we were using morphisms which are defined as relations. With linear continuous maps (between the associated ltvs's) as morphisms, the present notion of isomorphism is a particular case of the standard categorical one: we can have more linear homeomorphisms from $\mathbf{k}\langle X \rangle$ to $\mathbf{k}\langle Y \rangle$ than those which are generated by such finiteness-preserving bijections between webs.

Let X be a finiteness space. We define a $\mathbf{k}$-vector space $\mathbf{k}\langle X\rangle$ as the set of all families $x \in \mathbf{k}^{|X|}$ such that the set $|x| = \{a \in |X| \mid x_a \neq 0\}$ belongs to $\mathsf{F}(X)$.

Given $u' \in \mathsf{F}(X)^{\perp}$, we define a linear subspace of $\mathbf{k}\langle X\rangle$ by

$$\mathsf{V}(u') = \{x \in \mathbf{k}\langle X\rangle \mid |x| \cap u' = \emptyset\}\,.$$

Observe first that $\forall u', v' \in \mathsf{F}(X)^{\perp} \quad u' \subseteq v' \Leftrightarrow \mathsf{V}(v') \subseteq \mathsf{V}(u')$.

Since, given $u', v' \in \mathsf{F}(X)^{\perp}$, we have $\mathsf{V}(u' \cup v') = \mathsf{V}(u') \cap \mathsf{V}(v')$, the set $\{\mathsf{V}(u') \mid u' \in \mathsf{F}(X)^{\perp}\}$ is a filter of linear subspaces of $\mathbf{k}\langle X\rangle$. Moreover observe that $\bigcap_{u' \in \mathsf{F}(X)^{\perp}} \mathsf{V}(u') = \{0\}$ (because $\forall a \in |X|\ \{a\} \in \mathsf{F}(X)^{\perp}$), and therefore this filter defines an Hausdorff linear topology on $\mathbf{k}\langle X\rangle$, that we call the *canonical topology* of $\mathbf{k}\langle X\rangle$.

Proposition 17 *For any finiteness space X, the ltvs $\mathbf{k}\langle X\rangle$ is Cauchy-complete.*

Proof. Let $(x(d))_{d \in D}$ be a Cauchy net. Let $a \in |X|$. By taking $u' = \{a\}$ in the definition of a Cauchy net, we see that there exist $x_a \in \mathbf{k}$ and $d_a \in D$ such that $\forall e \geq d_a\ x(e)_a = x_a$.

We prove first that

$$\forall u' \in \mathsf{F}(X)^{\perp}\, \exists d \in D\, \forall e \geq d\, \forall a \in u' \quad x(e)_a = x_a\,. \tag{12}$$

Let $u' \in \mathsf{F}(X)^{\perp}$. Let $d^0 \in D$ be such that $x(e) - x(d^0) \in \mathsf{V}(u')$ for all $e \geq d^0$. Let $a \in u'$ and let $d_a \geq d^0$ be such that $x(e)_a = x_a$ for all $e \geq d_a$. Let $e \geq d^0$. Let $e' \geq e, d_a$. We have $x_a = x(e')_a$ since $e' \geq d_a$ and $x(e')_a = x(e)_a$ since $e, e' \geq d^0$ and $a \in u'$. It follows that $\forall a \in u'\ x_a = x(e)_a$.

From this we deduce now that $x \in \mathbf{k}\langle X\rangle$. Let $u' \in \mathsf{F}(X)^{\perp}$. Let $d \in D$ be such that $\forall e \geq d\, \forall a \in u'\ x(e)_a = x_a$. Then $|x| \cap u' = |x(d)| \cap u'$ is finite, so $|x| \in \mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$, that is $x \in \mathbf{k}\langle X\rangle$.

Now Condition (12) expresses exactly that $\lim_{d \in D} x(d) = x$ and hence the net $(x(d))_{d \in D}$ converges. □

A natural question is whether $\mathsf{F}(X)$ is always metrizable. We provide a necessary and sufficient condition under which this is the case.

Proposition 18 *A finiteness space X is metrizable iff there exists a sequence $(u'_n)_{n \in \mathbb{N}}$ of elements of $\mathsf{F}(X)^{\perp}$ which is monotone ($n \leq m \Rightarrow u'_n \subseteq u'_m$) and such that $\forall u' \in \mathsf{F}(X)^{\perp}\, \exists n \in \mathbb{N}\ u' \subseteq u'_n$.*

Proof. Let first $(u'_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathsf{F}(X)^{\perp}$ which satisfies the condition stated above. Given $x, y \in \mathbf{k}\langle X\rangle$, we define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is the least integer} \\ & \text{such that } u'_n \cap |x-y| \neq \emptyset\,. \end{cases}$$

Indeed, if $x \neq y$, then $|x-y| \neq \emptyset$ and hence, taking $a \in |x-y|$, we can find $n \in \mathbb{N}$ such that $\{a\} \subseteq u'_n$. This function d is easily seen to be an ultrametric distance (that is $d(x,z) \leq \max(d(x,y), d(y,z))$) and it generates the canonical topology of $\mathbf{k}\langle X\rangle$. Indeed we have

$$d(x,y) < 2^{-n} \quad \text{iff} \quad x-y \in \mathsf{V}(u'_n)\,.$$

and hence $B_{2^{-n}} = \mathsf{V}(u'_n)$, where B_ε is be the open ball centered at 0 and of radius ε.

Conversely, assume that $\mathbf{k}\langle X\rangle$ is metrizable and let d be a distance defining the canonical topology of $\mathbf{k}\langle X\rangle$. For each $n \in \mathbb{N}$, $B_{2^{-n}}$ is a neighborhood of 0 and hence there exist $v'_n \in \mathsf{F}(X)^\perp$ such that $\mathsf{V}(v'_n) \subseteq B_{2^{-n}}$. Let $u'_n = v'_0 \cup \cdots \cup v'_n \in \mathsf{F}(X)^\perp$. Then $\mathsf{V}(u'_n) \subseteq \mathsf{V}(v'_n) \subseteq B_{2^{-n}}$. Now let $u' \in \mathsf{F}(X)^\perp$, then $\mathsf{V}(u')$ is a neighborhood of 0 and hence there exists n such that $B_{2^{-n}} \subseteq \mathsf{V}(u')$, which implies $\mathsf{V}(u'_n) \subseteq \mathsf{V}(u')$ and hence $u' \subseteq u'_n$. □

It follows that there are non-metrizable ltvs's associated with finiteness spaces. We give an example of this situation in Proposition 19, using exponential constructions that will be introduced in Section 6.2.3 (but the proof can be understood without reading this section before).

Proposition 19 *The ltvs $\mathbf{k}\langle !?1\rangle$ is not metrizable*

Proof. Let $X = !?1$, so that $|X| = \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ and a subset u for $|X|$ belongs to $\mathsf{F}(X)$ iff $\exists n \in \mathbb{N}\ u \subseteq \mathcal{M}_{\mathrm{fin}}(\{0,\dots,n\})$. The proof is a typical Cantor diagonal reasoning.

Let $(u'_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathsf{F}(X)^\perp$. Let $n \in \mathbb{N}$, we have $\{p[n] \mid p \in \mathbb{N}\} \in \mathcal{M}_{\mathrm{fin}}(\{0,\dots,n\})$ where $p[n] = [\overbrace{n,\dots,n}^{p\times}]$ and hence $u'_n \cap \{p[n] \mid p \in \mathbb{N}\}$ is finite. Therefore we can find a function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}\ f(n)[n] \notin u'_n$. Let $u' = \{f(n)[n] \mid n \in \mathbb{N}\}$. Then $u' \in \mathsf{F}(X)^\perp$ since, for any $n \in \mathbb{N}$, $u' \cap \mathcal{M}_{\mathrm{fin}}(\{0,\dots,n\}) = \{f(i)[i] \mid i \in [0,n]\}$ is finite. But for all $n \in \mathbb{N}$ we have $f(n)[n] \in u' \setminus u'_n$ and so $u' \not\subseteq u'_n$.

By Proposition 18, this shows that $\mathbf{k}\langle !?1\rangle$ is not metrizable. □

This is a very interesting phenomenon which reveals a relation between the topological complexity of the interpretation of a type and its functional complexity (alternation of exponentials).

6.2.2 Linearly bounded subspaces. Let X be a finiteness space. We are interested in characterizing the linearly bounded subspaces of $\mathbf{k}\langle X\rangle$.

Given $u \subseteq |X|$, let $\mathsf{D}(u) = \{x \in \mathbf{k}\langle X\rangle \mid \mathsf{Dom}\,x \subseteq u\}$. This is a linear subspace of $\mathbf{k}\langle X\rangle$.

Let $u \in \mathsf{F}(X)$. We prove that $\mathsf{D}(u)$ is linearly bounded. Let u' in $\mathsf{F}(X)^\perp$. Observe that $\mathsf{V}(u') = \mathsf{D}(|X| \setminus u')$. We have therefore $\mathsf{D}(u) \subseteq \mathsf{V}(u') + \mathsf{D}(u \cap u')$, and since $u \cap u'$ is finite, the space $\mathsf{D}(u \cap u')$ is finite dimensional. Let U be an open subspace of U, let $u' \subseteq \mathsf{F}(X)^\perp$ be such that $\mathsf{V}(u') \subseteq U$. Then

$\mathsf{D}(u) \subseteq U + \mathsf{D}(u \cap u')$. Hence $\mathsf{D}(u)$ is linearly bounded. We show now that this condition is actually sufficient.

Proposition 20 *A linear subspace B of $\mathbf{k}\langle X\rangle$ is linearly bounded iff there exists $u \in \mathsf{F}(X)$ such that $B \subseteq \mathsf{D}(u)$.*

Proof. Assume that B is linearly bounded. Let $u = \bigcup_{x\in B} |x|$, so that $B \subseteq \mathsf{D}(u)$, we prove that $u \in \mathsf{F}(X)$. Let $u' \in \mathsf{F}(X)^\perp$. Let A be a finite dimensional subspace of E such that $B \subset \mathsf{V}(u') + A$. Let A_0 be a finite generating subset of A and let $u_0 = \bigcup_{y\in A_0} |y| \in \mathsf{F}(X)$. Then $x \in A \Rightarrow |x| \subseteq u_0$ (that is $A \subseteq \mathsf{D}(u_0)$).

Let $x \in B$, we write $x = x_1 + x_2$ where $x_1 \in \mathsf{V}(u')$ and $x_2 \in A$. We have $|x| \subseteq |x_1| \cup |x_2|$ and hence $u' \cap |x| \subseteq (u' \cap |x_1|) \cup (u' \cap |x_2|) \subseteq u' \cap u_0$ since $u' \cap |x_1| = \emptyset$. Since this holds for all $x \in B$, we have $u' \cap u \subseteq u' \cap u_0$ so $u' \cap u$ is finite and hence $u \in \mathsf{F}(X)$ ☐

From this, we deduce easily a very simple characterization of locally linearly bounded spaces.

Proposition 21 *The ltvs $\mathbf{k}\langle X\rangle$ is locally linearly bounded iff there exist $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X)^\perp$ such that $u \cup u' = |X|$.*

Proof. Assume first that we have $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X)^\perp$ such that $u \cup u' = |X|$. Then $\mathsf{V}(u')$ is a neighborhood of 0 and we have $\mathsf{V}(u') \subseteq \mathsf{D}(u)$ since $x \in \mathsf{V}(u') \Rightarrow |x| \cap u' = \emptyset \Rightarrow |x| \subseteq u$.

Conversely, let U be a linearly bounded neighborhood of 0. Let $u' \in \mathsf{F}(X)^\perp$ be such that $\mathsf{V}(u') \subseteq U$ and let $u \in \mathsf{F}(X)$ be such that $U \subseteq \mathsf{D}(u)$. Let $x \in \mathbf{k}\langle X\rangle$ be such that $|x| = \{a\}$. If $x \in U$, then $a \in u$. If $x \notin U$ then $x \notin \mathsf{V}(u')$, and this means that $a \in u'$. ☐

When this condition holds, $\mathbf{k}\langle X\rangle$ is metrizable: given an enumeration $a_1, a_2, \dots$ of $|X|$, set $u'_n = u' \cup \{a_1, \dots, a_n\} \in \mathsf{F}(X)^\perp$. Then, given any $v' \in \mathsf{F}(X)^\perp$, we know that $v' \cup u$ is finite and hence we can find n such that $v' \subseteq u'_n$ since $u \cup u' = |X|$.

Let $\mathbf{Fin}(\mathbf{k})$ be the category whose objects are the finiteness spaces and such that $\mathbf{Fin}(\mathbf{k})(X, Y)$ is the set of all continuous linear maps $\mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$.

6.2.3 Constructions of finiteness spaces.

We give a number of constructions on finiteness spaces which allow to interpret differential linear logic, starting with the most important one, which is the linear function space.

The most striking features of these constructions can be summarized by the two following statements.

- In spite of the fact that these constructions are algebraic in nature (tensor product, linear function space, topological dual etc), they are entirely performed on the webs of the finiteness spaces and do not involve the scalar coefficients in their definition. This means in particular that they do not depend on the choice of the field, and this is quite surprising.

- On the other hand, though these constructions are performed on the webs, they do not really depend on them, in the following sense. Defining an *intrinsic finiteness space* as a $\mathbf{k}$-ltvs which is linearly homeomorphic to $\mathbf{k}\langle X\rangle$ for *some* finiteness space X, all these constructions can be transferred to the category of intrinsic finiteness spaces and continuous and linear maps.

Let X and Y be finiteness spaces. Let $X \multimap Y$ be the finiteness space such that $|X \multimap Y| = |X| \times |Y|$ and

$$\begin{aligned}\mathsf{F}(X \multimap Y) &= \{u \times v' \mid u \in \mathsf{F}(X) \text{ and } v' \in \mathsf{F}(Y^\perp)\}^\perp \\ &= \{w \subseteq |X| \times |Y| \mid \forall u \in \mathsf{F}(X)\, \forall v' \in \mathsf{F}(Y)^\perp \ w \cap (u \times v') \text{ is finite}\}\end{aligned}$$

Let $w \in \mathsf{F}(X \multimap Y)$, $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(X)^\perp$. It follows from (11) that $wu \in \mathsf{F}(Y)$ and that ${}^{\mathsf{t}}wv' \in \mathsf{F}(X)^\perp$.

Let $M \in \mathbf{k}\langle X \multimap Y\rangle$. If $x \in \mathbf{k}\langle X\rangle$ and $b \in |Y|$, then ${}^{\mathsf{t}}|M|\{b\} \in \mathsf{F}(X)^\perp$ and hence the sum $\sum_{a\in|X|} M_{a,b}x_a$ is finite. Therefore we can define $Mx \in \mathbf{k}^{|Y|}$ by $Mx = (\sum_{a\in|X|} M_{a,b}x_a)_{b\in|Y|}$. Since $|Mx| \subseteq |M||x|$, we have $Mx \in \mathbf{k}\langle Y\rangle$ and hence the function $\widehat{M}$ defined by $\widehat{M}(x) = Mx$ is a linear map $\mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$. Moreover, $\widehat{M}$ is continuous. Indeed, for any $v' \in \mathsf{F}(Y)^\perp$ we have $\mathsf{V}({}^{\mathsf{t}}|M|v') \subseteq \widehat{M}^{-1}(\mathsf{V}(v'))$ and hence $\widehat{M}^{-1}(\mathsf{V}(v'))$ is open since ${}^{\mathsf{t}}|M|v' \in \mathsf{F}(X)^\perp$.

Given finiteness spaces Z_1, Z_2, we define immediately the finiteness space $Z_1 \otimes Z_2$ as $Z_1 \otimes Z_2 = (Z_1 \multimap {Z_2}^\perp)^\perp$, so that $|Z_1 \otimes Z_2| = |Z_1| \times |Z_2|$. One of the most pleasant features of the theory of finiteness spaces is the following property (see [Ehr05]) which has been considerably generalized in [TV10].

Proposition 22 *Let $w \subseteq |Z_1|\times|Z_2|$. One has $w \in \mathsf{F}(Z_1 \otimes Z_2)$ iff $\pi_i(w) \in \mathsf{F}(Z_i)$ for $i = 1, 2$.*

Coming back to linear function spaces, this means in particular that, given $w \subseteq |X| \times |Y|$, one has $w \in \mathsf{F}(X \multimap Y)^\perp$ iff there are $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(Y)^\perp$ such that $w \subseteq u \times v'$, from which we derive a simple characterization of the topology of linear function spaces.

Proposition 23 *The function $\theta_{X,Y} : M \mapsto \widehat{M}$ is a linear homeomorphism from $\mathbf{k}\langle X \multimap Y\rangle$ to $\mathbf{k}\langle X\rangle \multimap \mathbf{k}\langle Y\rangle$, equipped with the topology of uniform convergence on linearly bounded subspaces.*

Proof. The proof that $\theta_{X,Y}$ is a linear isomorphism can be found in [Ehr05]. We prove that this linear isomorphism is an homeomorphism. Let $B \subseteq \mathbf{k}\langle X\rangle$ be a bounded subspace and $V \subseteq \mathbf{k}\langle Y\rangle$ is an open subspace. Let $u \in \mathsf{F}(X)$ be such that $B \subseteq \mathsf{D}(u)$ and let $v' \in \mathsf{F}(Y)^\perp$ be such that $\mathsf{V}(v') \subseteq V$. Then $u \times v' \in \mathsf{F}(X \multimap Y)^\perp$ and hence $\mathsf{V}(u \times v') \subseteq \mathbf{k}\langle X \multimap Y\rangle$ is an open subspace. Let $M \in \mathsf{V}(u \times v')$, $x \in B$ and $b \in v'$, we have $(Mx)_b = 0$ since $|x| \subseteq u$, which shows that $\theta_{X,Y}(M)(B) \subseteq V$ and hence $\theta_{X,Y}$ is continuous. Let now $W \subseteq \mathbf{k}\langle X \multimap Y\rangle$ be an open subspace. Let $w \in \mathsf{F}(X \multimap Y)^\perp$ be such that

$\mathsf{V}(w) \subseteq W$. By Proposition 22, there are $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(Y)^\perp$ such that $w \subseteq u \times v'$, and hence $\mathsf{V}(u \times v') \subseteq W$. Then, given $M \in \mathsf{V}(u \times v')$, we have $\theta_{X,Y}(M)(\mathsf{D}(U)) \subseteq \mathsf{V}(V')$, which shows that $\theta_{X,Y}(W)$ is an open linear subspace of $\mathbf{k}\langle X\rangle \multimap \mathbf{k}\langle Y\rangle$. □

The tensor product $X \otimes Y$ defined above is characterized by a standard universal property. Given vectors $x \in \mathbf{k}\langle X\rangle$ and $y \in \mathbf{k}\langle Y\rangle$, then $x \otimes y \in \mathbf{k}^{|X\otimes Y|}$ defined by $(x \otimes y)_{(a,b)} = x_a y_b$ is clearly an element of $\mathbf{k}\langle X \otimes Y\rangle$ since $|x \otimes y| = |x| \times |y| \in \mathsf{F}(X \otimes Y)$. The map $\tau : \mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle X \otimes Y\rangle$ is obviously bilinear, let us check that it is hypocontinuous. Let W be an open linear subspace of $\mathbf{k}\langle X \otimes Y\rangle$ and let $w' \in \mathsf{F}(X \otimes Y)^\perp$ be such that $\mathsf{V}(w') \subseteq W$. Let $B \subseteq \mathbf{k}\langle X\rangle$ be a linearly bounded subspace and let $u \in \mathsf{F}(X)$ be such that $B \subseteq \mathsf{D}(u)$. Since $w \in \mathsf{F}(X \multimap Y^\perp)$ we have $v' = wu \in \mathsf{F}(Y)^\perp$. Let $x \in B$ and $y \in \mathsf{V}(v')$, we have $|x| \subseteq u$ and hence $\pi_2(|x \otimes y| \cap w) \subseteq \pi_2((u \times |y|) \cap w) = (wu) \cap |y| = \emptyset$ by definition of $\mathsf{V}(v')$. Therefore $x \otimes y \in \mathsf{V}(w') \subseteq W$. Similarly, taking a linearly bounded subspace C of $\mathbf{k}\langle Y\rangle$, we show that there is an open linear subspace U of $\mathbf{k}\langle X\rangle$ such that $\tau(U \times C) \subseteq W$. So the map τ is bilinear and hypocontinuous.

Proposition 24 *Let Z be a finiteness space and let $f : \mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Z\rangle$ be bilinear and hypocontinuous. There exists exactly one continuous linear map $\tilde{f} : \mathbf{k}\langle X \otimes Y\rangle \to \mathbf{k}\langle Z\rangle$ such that $f = \tilde{f} \circ \tau$.*

Proof. We define a matrix $M \in \mathbf{k}^{|X|\times|Y|\times|Z|}$ by $M_{a,b,c} = f(e_a, e_b)_c$ and we show first that $|M| \in \mathsf{F}(X \otimes Y \multimap Z)$. So let $u \in \mathsf{F}(X)$, $v \in \mathsf{F}(Y)$ and $w' \in \mathsf{F}(Z)^\perp$; we must show that $|M| \cap (u \times v \times w')$ is finite. Let $v' \in \mathsf{F}(Y)^\perp$ be such that $f(\mathsf{D}(u) \times \mathsf{V}(v')) \subseteq \mathsf{V}(w')$ and let $u' \in \mathsf{F}(X)^\perp$ be such that $f(\mathsf{V}(u') \times \mathsf{D}(v)) \subseteq \mathsf{V}(w')$. Let $(a, b, c) \in |M| \cap (u \times v \times w')$, since $f(e_a, e_b)_c \neq 0$ we have $f(e_a, e_b) \notin \mathsf{V}(w')$, and hence we must have $(e_a, e_b) \notin \mathsf{D}(u) \times \mathsf{V}(v')$ and $(e_a, e_b) \notin \mathsf{V}(u') \times \mathsf{D}(v)$ and therefore $e_a \notin \mathsf{V}(u')$, that is $a \in u'$, and similarly $b \in v'$ (since we know that $a \in u$ and $b \in v$, that is $e_a \in \mathsf{D}(u)$ and $e_b \in \mathsf{D}(v)$). Since $\mathsf{D}(u)$ and $\mathsf{D}(v)$ are linearly bounded, so is $f(\mathsf{D}(u) \times \mathsf{D}(v))$ by Proposition 16 and hence there exists $w \in \mathsf{F}(Z)$ such that $f(\mathsf{D}(u) \times \mathsf{D}(v)) \subseteq \mathsf{D}(w)$. Therefore $f(e_a, e_b) \in \mathsf{D}(w)$ and hence $c \in w$. We have shown that $|M| \cap (u \times v \times w') \subseteq (u \cap u') \times (v \cap v') \times (w \cap w')$ and hence $|M| \cap (u \times v \times w')$ is finite, so $M \in \mathbf{k}\langle X \otimes Y \multimap Z\rangle$. Let $\tilde{f} = \widehat{M}$, it is a linear and continuous map from $\mathbf{k}\langle X \otimes Y\rangle$ to $\mathbf{k}\langle Z\rangle$. We have $\tilde{f}(e_a \otimes e_b) = f(e_a, e_b)$ for each $(a, b) \in |X| \times |Y|$. Let $x \in \mathsf{F}(X)$ and $y \in \mathbf{k}\langle Y\rangle$, by separate continuity of f we have $f(x, y) = f(\sum_{a\in|X|} x_a e_a, \sum_{b\in|Y|} y_b e_b) = \sum_{(a,b)\in|X|\times|Y|} x_a y_b f(e_a, e_b) = \sum_{(a,b)\in|X|\times|Y|} x_a y_b \tilde{f}(e_a \otimes e_b) = \tilde{f}(x \otimes y)$, by continuity of $\tilde{f}$. Uniqueness of the continuous linear map $\tilde{f}$ results from the fact that necessarily $\tilde{f}(e_a \otimes e_b) = f(e_a, e_b)$. □

Then one proves easily that the category $\mathbf{Fin}(\mathbf{k})$ equipped with this tensor product (whose neutral object is 1, which satisfies obviously $\mathbf{k}\langle 1\rangle = \mathbf{k}$) is $*$-autonomous, the object of morphisms from X to Y being $X \multimap Y$ and the dualizing object being $\perp = 1$ (indeed, the finiteness spaces $X \multimap \perp$ and $X^\perp$ are obviously strongly isomorphic). This monoidal category is also enriched over $\mathbf{k}$-vector spaces.

Countable products and coproducts are available as well. Let $(X_i)_{i\in I}$ be a countable family of finiteness spaces. The finiteness space $X = \&_{i\in I} X_i$ is given by $|X| = \bigcup_{i\in I} |X_i|$ and $\mathsf{F}(X) = \{w \subseteq |X| \mid \forall i \in I\ w_i \in \mathsf{F}(X_i)\}$ where $w_i = \{a \in |X_i| \mid (i,a) \in w\}$. It is easy to check that $\mathsf{F}(X)^{\perp} = \{w' \subseteq |X| \mid \forall i \in I\ w'_i \in \mathsf{F}(X_i)^{\perp}$ and $w'_i = \emptyset$ for almost all $i\}$ and it follows that $\mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$. It is clear that $\mathbf{k}\langle \&_{i\in I} X_i\rangle = \prod_{i\in I} \mathbf{k}\langle X_i\rangle$ up to a straightforward strong isomorphism and that $\&_{i\in I} X_i$ together with projections $\pi_j : \mathbf{k}\langle \&_{i\in I} X_i\rangle \to \mathbf{k}\langle X_j\rangle$ defined in the obvious way, is the cartesian product of the X_i's. Thanks to $*$-autonomy, the coproduct of the X_i's is given by $\bigoplus_{i\in I} X_i = \left(\&_{i\in I} {X_i}^{\perp}\right)^{\perp}$ and $\mathbf{k}\langle \bigoplus_{i\in I} X_i\rangle \subseteq \prod_{i\in I} \mathbf{k}\langle X_i\rangle$ is the space of all families $(x_i)_{i\in I}$ of vectors such that $x_i = 0$ for almost all $i \in I$. Of course, the canonical linear topology on $\mathbf{k}\langle \&_{i\in I} X_i\rangle$ is the product topology, but the canonical topology on $\mathbf{k}\langle \bigoplus_{i\in I} X_i\rangle$ is much finer: it is generated by all products $\prod_{i\in I} V_i$ where V_i is a linear neighborhood of 0 in $\mathbf{k}\langle X_i\rangle$. For finite families of objects, products and coproducts coincide.

Let X be a finiteness space. We define $!X$ by $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and $\mathsf{F}(!X) = \{A \subseteq |!X| \mid \bigcup_{m\in A} |m| \in \mathsf{F}(X)\}$ and it can be proved that indeed $\mathsf{F}(!X) = \mathsf{F}(!X)^{\perp\perp}$ (again, see [TV10] for more general results of this kind). Given $x \in \mathbf{k}\langle X\rangle$ and $m \in |!X|$, we set $x^m = \prod_{a\in|X|} x_a^{m(a)} \in \mathbf{k}$ (this is a finite product since $|m|$ is a finite set), so that $x^! = (x^m)_{m\in|!X|} \in \mathbf{k}\langle !X\rangle$ by definition of $\mathsf{F}(!X)$. Let $M \in \mathbf{k}\langle !X \multimap Y\rangle$, it is not hard to see that one defines a map $\mathsf{Fun}(M) : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ by setting $\mathsf{Fun}(M)(x) = \left(\sum_{m\in|!X|} M_{m,b} x^m\right)_{b\in|Y|}$; all these sums are indeed finite, see [Ehr05] for more details. When the field $\mathbf{k}$ is infinite, the map $M \mapsto \mathsf{Fun}(M)$ is injective.

In [Ehr05], it is also proven that $!\_$ is a functor. Given $M \in \mathbf{k}\langle X \multimap Y\rangle$ one defines $!M \in \mathbf{k}\langle !X \multimap !Y\rangle$ by setting, for $m \in |!X|$ and $p \in |!Y|$, $(!M)_{m,p} = \sum_{r\in L(m,p)} \left[\begin{smallmatrix} p \\ r \end{smallmatrix}\right] M^r$ where $L(m,p) = \{r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|) \mid \sum_{b\in|Y|} r(a,b) = m(a)$ and $\sum_{a\in|X|} r(a,b) = p(b)\}$ (so that $r \in L(m,p) \Rightarrow \#m = \#r = \#p$) and $\left[\begin{smallmatrix} p \\ r \end{smallmatrix}\right] = \prod_{b\in|Y|} \frac{p(b)!}{\prod_{a\in|X|} r(a,b)!} \in \mathbb{N}^+$. This operation is functorial: $!\mathrm{Id} = \mathrm{Id}$ and $!M\,!N = !(M\,N)$, and we also have $\mathsf{Fun}(!M)(x) = !M\,x^! = (\,Mx)^!$. When $\mathbf{k}$ is infinite, this latter equation completely characterizes $!M$, by injectivity of the operation $\mathsf{Fun}(\_)$ in that case. This functor has a comonad structure, of which we recall here only the counit $\mathsf{d}_X \in \mathbf{k}\langle !X \multimap X\rangle$ given by $(\mathsf{d}_X)_{m,a} = \delta_{m,[a]}$ ($\delta_{i,j}$ is the Kronecker's symbol which takes value 1 if $i = j$ and 0 otherwise).

The bijection $|!(X \,\&\, Y)| \to |!X \otimes !Y|$ which maps the element $q \in |!(X \,\&\, Y)|$ to the pair $(m,p) \in |!X \otimes !Y|$ defined by $m(a) = q(1,a)$ and $p(b) = q(2,b)$ is a strong isomorphism of finiteness spaces. We also have a strong isomorphism from $!\top$ to 1. These strong ismorphisms induce natural isomorphisms $\mathsf{m}_{X,Y} \in \mathbf{Fin}(\mathbf{k})(!X \otimes !Y, !(X \,\&\, Y))$ and $\mathsf{m}_1 \in \mathbf{Fin}(\mathbf{k})(1, !\top)$ which induce the functor $!\_$ with a monoidality structure from $(\mathbf{Fin}(\mathbf{k}), \&, \top)$ to $(\mathbf{Fin}(\mathbf{k}), \otimes, 1)$, satisfying moreover the coherence diagram (5): to summarize, equipped with the structure described above, $\mathbf{Fin}(\mathbf{k})$ is a Seely category, that is, a categorical model of classical linear logic.

Applying the general recipe of Section 4, we get the contraction natural transformation $\mathsf{c}_X : !X \to !X \otimes !X$ and the weakening morphism $\mathsf{w}_X : !X \to 1$. We check that $(\mathsf{w}_X)_{m,*} = \delta_{m,[]}$ and that $(\mathsf{c}_X)_{m,(p,q)} = \delta_{m,p+q}$. We also get the co-contraction natural transformation $\overline{\mathsf{c}}_X : !X \otimes !X \to !X$ and the co-weakening morphism $\overline{\mathsf{w}}_X : 1 \to !X$. And we check that $(\overline{\mathsf{w}}_X)_{*,m} = \delta_{m,[]}$, and that $(\overline{\mathsf{c}}_X)_{(p,q),m} = \binom{p+q}{p} \delta_{m,p+q}$ where $\binom{m}{p} = \prod_{a \in |X|} \frac{m(a)!}{p(a)!(m(a)-p(a))!} \in \mathbb{N}^+$ is a generalized binomial coefficient.

We also have a co-dereliction natural transformation $\overline{\mathsf{d}}_X : X \to !X$ given by $(\overline{\mathsf{d}}_X)_{a,m} = \delta_{m,[a]}$, which is easily seen to satisfy the conditions (8), (9) and (10) of Section 4, so that $\mathbf{Fin}(\mathbf{k})$ is a model of full differential linear logic.

6.2.4 An intrinsic presentation of function spaces. We have seen that a morphism from X to Y of the linear category $\mathbf{Fin}(\mathbf{k})$ can be seen both as an element of $\mathbf{k}\langle X \multimap Y \rangle$ and as a continuous linear function from $\mathbf{k}\langle X \rangle$ to $\mathbf{k}\langle Y \rangle$. A morphism from X to Y in the Kleisli category $\mathbf{Fin}_!(\mathbf{k})$ is an element of $\mathbf{k}\langle !X \multimap Y \rangle$. Given $M \in \mathbf{k}\langle !X \multimap Y \rangle$, we have seen that we can define a function $\mathsf{Fun}(M) : \mathbf{k}\langle X \rangle \to \mathbf{k}\langle Y \rangle$ by $\mathsf{Fun}(M)(x) = M\ x^! = (\sum_{m \in |!X|} M_{m,b} x^m)_{b \in |Y|}$. Moreover, the correspondence $M \to \mathsf{Fun}(M)$ is functorial. We provide here an intrinsic characterization of these functions.

Let E and F be ltvs's. Let us say that a function $f : E \to F$ is *polynomial*[7] if there is $n \in \mathbb{N}$ and *hypocontinuous* i-linear maps $f_i : E^i \to F$ (for $i = 0, \dots, n$) such that

$$f(x) = f_0 + f_1(x) + \cdots + f_n(x, \dots, x)\,.$$

A polynomial map f of the form $f(x) = f_n(x, \dots, x)$, where f_n is an n-linear hypocontinuous function, is said to be *homogeneous of degree* n (this condition implies of course $\forall t \in \mathbf{k}\, f(tx) = t^n f(x)$, and when $\mathbf{k}$ is infinite, a polynomial function is homogeneous iff it satisfies this latter condition).

Let $\mathbf{Pol}_{\mathbf{k}}(E,F)$ be the $\mathbf{k}$-vector space of polynomial functions from E to F. This space can be endowed with the linear topology of uniform convergence on all linearly bounded subspaces, which admits the following generating filter base of open neighborhoods of 0: the basic opens are the linear subspaces $\mathsf{ann}(B,V) = \{f \in \mathbf{Pol}_{\mathbf{k}}(E,F) \mid f(B) \subseteq V\}$, where B is a linearly bounded subspace of E and V is a linear open subspace of F. Let $\widetilde{\mathbf{Pol}_{\mathbf{k}}}(E,F)$ be the completion[8] of that ltvs.

Theorem 25 *Assume that* $\mathbf{k}$ *is infinite. For any finiteness spaces* X *and* Y*,*

[7] When restricted to the exponential structure $\mathbf{Fin}(\mathbf{k})$, this notion coincides with the one of Section 2.2.5.

[8] The completion of an ltvs E is a pair $(\tilde{E}, h)$ where $\tilde{E}$ is a complete ltvs and $h : E \to \tilde{E}$ is a linear and continuous map uniquely defined by the following universal property: for any complete ltvs F and any linear continuous map $f : E \to F$ there is an unique linear and continuous map $\tilde{f} : \tilde{E} \to F$ such that $\tilde{f} \circ h = f$. Using standard techniques, one can prove that any ltvs admits a completion, which is unique up to unique isomorphism.

the ltvs $\mathbf{k}\langle !X \multimap Y\rangle$ *is linearly homeomorphic to* $\widetilde{\mathbf{Pol}_{\mathbf{k}}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$.

Proof. Let $h : \mathbf{k}\langle X\rangle^n \to \mathbf{k}\langle Y\rangle$ be an hypocontinuous n-linear function of matrix $M \in \mathbf{k}\langle X \otimes \cdots \otimes X \multimap Y\rangle$, so that $f(x_1, \dots, x_n) = M(x_1 \otimes \cdots \otimes x_n)$. Remember that, using contraction and dereliction, we have defined in 2.2.2 the morphism $\mathsf{d}_X^n \in \mathbf{k}\langle !X \multimap X \otimes \cdots \otimes X\rangle$. Then we have $N = M\ \mathsf{d}_X^n \in \mathbf{k}\langle !X \multimap Y\rangle$, and it is easy to see that $\mathsf{Fun}(N)(x) = h(x, \dots, x)$. In that way, we see that any polynomial map from $\mathbf{k}\langle X\rangle$ to $\mathbf{k}\langle Y\rangle$ is an element of $\mathbf{k}\langle !X \multimap Y\rangle$; we have an inclusion $\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle) \subseteq \mathbf{k}\langle !X \multimap Y\rangle$. Conversely, let $(m, b) \in |!X \multimap Y|$ with $m = [a_1, \dots, a_n]$. The map $f : \mathbf{k}\langle X\rangle^n \to \mathbf{k}$ defined by $f(x(1), \dots, x(n)) = x(1)_{a_1} \dots x(n)_{a_n}$ is multilinear and hypocontinuous. Hence the same holds for the map $x \mapsto f(x)e_b$ from $\mathbf{k}\langle X\rangle^n$ to $\mathbf{k}\langle Y\rangle$. Therefore we have $\mathbf{k}^{(|!X \multimap Y|)} \subseteq \mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$. Hence $\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$ is a dense subspace of $\mathbf{k}\langle !X \multimap Y\rangle$. To show that $\mathbf{k}\langle !X \multimap Y\rangle$ is the completion of $\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$ it suffices to show that the above defined linear topology on that space (uniform convergence on all linearly bounded subspaces) is the restriction of the topology of $\mathbf{k}\langle !X \multimap Y\rangle$.

Let $B \subseteq \mathbf{k}\langle X\rangle$ be a linearly bounded subspace and let $V \subseteq \mathbf{k}\langle Y\rangle$ be linear open. Let $v' \in \mathsf{F}(Y)^{\perp}$ be such that $\mathsf{V}(v') \subseteq V$. By Proposition 20, $|B| = \bigcup\{|x| \mid x \in B\} \in \mathsf{F}(X)$, so $\mathcal{M}_{\mathrm{fin}}(|B|) \in \mathsf{F}(!X)$. Let $M \in \mathsf{V}(\mathcal{M}_{\mathrm{fin}}(|B|) \times v') \subseteq \mathbf{k}\langle !X \multimap Y\rangle$, then $\widehat{M}(x)_b = 0$ for each $x \in B$ and $b \in v'$. So we have $\mathsf{V}(\mathcal{M}_{\mathrm{fin}}(|B|) \times v') \cap \mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle) \subseteq \mathsf{ann}(B, V)$.

Conversely let $U \in \mathsf{F}(!X)$ and $v' \in \mathsf{F}(Y)^{\perp}$, then we have $u = \bigcup_{m \in U} |m| \in \mathsf{F}(X)$ and hence the subspace $B \subseteq \mathbf{k}\langle X\rangle$ of all vectors which vanish outside u is linearly bounded. Let $M \in \mathbf{k}\langle !X \multimap Y\rangle$ be such that the map $\widehat{M} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ is polynomial and belongs to $\mathsf{ann}(B, \mathsf{V}(v'))$. Then for any $m = [a_1, \dots, a_n] \in \mathcal{M}_{\mathrm{fin}}(u)$ and $b \in v'$ we have $M_{m,b} = 0$ because this scalar is the coefficient of the monomial $\xi_1^{m(a_1)} \dots \xi_n^{m(a_n)}$ in the polynomial $P \in \mathbf{k}[\xi_1, \dots, \xi_n]$ such that $P(z_1, \dots, z_n) = \widehat{M}(x)_b$ where $x \in \mathbf{k}\langle X\rangle$ is such that $x_a = z_i$ if $a = a_i$ and $x_a = 0$ if $a \notin |m|$, and $P = 0$ because $\widehat{M}(B) \subseteq \mathsf{V}(v')$ by assumption (we also use the fact that $\mathbf{k}$ is infinite). Hence $M \in \mathsf{V}(U \times v')$ and we have shown that $\mathsf{ann}(B, \mathsf{V}(v')) \subseteq \mathsf{V}(U \times v') \cap \mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$, proving that this latter set is a neighborhood of 0 in the space of polynomials. □

The Taylor formula proved in [Ehr05] for the morphisms of this Kleisli category shows that actually any morphism is the sum of a converging series whose n-th term is an homogeneous polynomial of degree n.

As an example, take $E = \mathbf{k}[\xi] \simeq \mathbf{k}\langle !1 \multimap 1\rangle$. The corresponding topology on E is the discrete topology. A typical example of generalized polynomial map is the function $\varphi : E \to \mathbf{k}$ which maps a polynomial P to $P(P(0))$, in other words, $\varphi(x_0 + x_1\xi + \cdots + x_n\xi^n) = x_0 + x_1x_0 + \cdots + x_nx_0^n$. Considered as a generalized polynomial of infinitely many variables $x_0, x_1, \dots$, we see that φ is not of bounded degree, and so it is not polynomial. Nevertheless, it corresponds to a very simple and finite computation on polynomials.

6.2.5 Anti-derivatives. Just as **Rel**, the **Fin**(**k**) exponential structure is bi-commutative and Taylor in the sense of Section 2.2.4. Moreover, if **k** is of characteristic 0 (meaning that $\forall n \in \mathbb{N}\; n\,1 = 0 \Rightarrow n = 0$) it has anti-derivatives in the sense of 2.3, because the morphism $J_X = \mathrm{Id}_X + (\overline{\partial}_X\ \partial_X) : {!X} \to {!X}$ satisfies $(J_X)_{p,q} = (\#p + 1)\delta_{p,q}$ for all $p, q \in |{!X}|$ and hence is an isomorphism whose inverse I_X is given by $(I_X)_{p,q} = \frac{1}{\#p+1}\delta_{p,q}$.

7 Interpreting DiLL proofs

There are basically two approaches for interpreting nets or pre-nets in such categorical models. Let $\mathcal{C}$ be a model of DiLL in the sense of Section 4. The first thing to do is to interpret the formulas of LL as objects of $\mathcal{C}$, which is an obvious step, once an object $[\alpha]$ of $\mathcal{C}$ has been chosen for each atomic formula α, and the object $[\alpha]^\perp$ has been assigned to the negated atom $\overline{\alpha}$: in that way, an object $[A]$ of $\mathcal{C}$ is associated with each formula A and we have $[A^\perp] = [A]^\perp$.

7.1 General approach

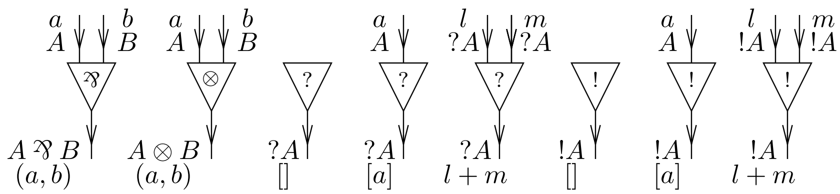
The abstract and most general approach does not allow to interpret all (typed) pre-nets, but only those which are correct in the sense of Section 1.5. To be more precise, we can interpret in $\mathcal{C}$ correctness deduction trees, because, with each of the rules of Section 1.5, one can easily associate categorical constructions of Section 4. Using the categorical axioms of that section, one can prove then that this interpretation does not depend on the correctness deduction tree, but only on the underlying pre-net. One proves moreover that the interpretation is invariant under the reduction rules of Section 1.6. To a net r with typed interface $I = (p_1 : A_1, \dots, p_n : A_n)$, one associates in that way an element $[r]^I$ of $\mathcal{C}(1, [A_1] \parr \cdots \parr [A_n])$.

7.2 Experiences

The other approach applies to **Rel**, and to the models based on that semantics (such as the model of finiteness spaces, in the case where $\mathbf{k} = \mathbb{B}$, which has not been dealt with in the present paper since $\mathbb{B}$ is not a field, but which is quite easy, see [Ehr05]).

This approach dates back to [Gir87] where an assignment of semantic tokens (elements of webs of coherence spaces) was called an *experience*. It is easy to adapt it to the current DiLL setting. Observe first that $[A \otimes B] = [A \parr B] = [A] \times [B]$ and that $[!A] = [?A] = \mathcal{M}_{\mathrm{fin}}([A])$ so that $[A^\perp] = [A]$.

Let r be a simple pre-net given together with a correct type assignment τ (see Section 1.4). An *experience* for (r, τ) is a mapping e which, with each oriented wire w of r, associates an element of $[\tau(w)]$ in such a way that $e(w^\perp) = e(w)$, and the following local constraints are satisfied:

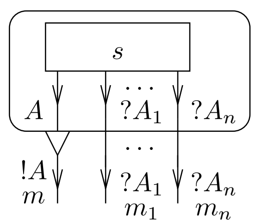


where each oriented wire is decorated with its type (assigned by τ) and by its semantic token (assigned by e).

Given a typed simple pre-net r and a type assignment τ which is correct for r and induces the typed interface $I = (p_1 : A_1, \ldots, p_n : A_n)$, we define $[(r, \tau)]^I_{\mathsf{rel}}$ as the set of all tuples $(a_1, \ldots, a_n) \in [A_1] \times \cdots \times [A_n]$ such that there is an experience e for (r, τ) with $e(w_i) = a_i$ where w_i is the oriented wire of r whose endpoint is p_i.

Given a typed pre-net $s = \{(r_1, \tau_1), \ldots, (r_h, \tau_h)\}$ with typed interface $I = (p_1 : A_1, \ldots, p_n : A_n)$ (remember that we have taken $\mathbf{k} = \mathbb{B}$ so a pre-net is a finite set of simple pre-nets), we set $[s]^I_{\mathsf{rel}} = [(r_1, \tau_1)]^I_{\mathsf{rel}} \cup \cdots \cup [(r_h, \tau_h)]^I_{\mathsf{rel}}$.

Concerning boxes such as



where s is a typed pre-net with typed interface $I = (p : A, p_1 : ?A_1, \ldots, p_n : ?A_n)$ the assignment of semantic tokens given by the figure above is correct if there is a $k \in \mathbb{N}$ and k tuples $(a^j, m^j_1, \ldots, m^j_n) \in [s]^I_{\mathsf{rel}}$ for $j = 1, \ldots, k$ such that $m = [a^1, \ldots, a^k]$ and $m_i = m^1_i + \cdots + m^k_i$ for $i = 1, \ldots, n$.

A simple inspection of cases shows that this semantics with typed pre-nets is invariant under the reduction rules of Section 1.6.

7.3 Comments

Of course, for typed pre-nets which are correct in the sense of Section 1.5, the general approach and the experience method give the same results, but the interesting point is that the experience method applies in **Rel** to all typed pre-nets, not only to the correct ones.

It is then quite interesting to understand how syntactic properties of pre-nets are related to semantic properties of their relational interpretation based on experiences. For instance, Girard showed directly in [Gir87] that, when an LL pre-net satisfies the acyclicity criterion (which implies that it is logically correct), then its relational interpretation[9] is a clique of the coherence space

[9] This is not a completely precise statement because the coherence space semantics and the relational semantics disagree on the interpretation of the exponentials: some tokens of the relational semantics are discarded by the coherence space semantics.

model.

More interestingly, one can try to characterize syntactically semantic properties of the interpretation of pre-nets in **Rel**. The first result of this kind is due to Rétoré [Ret97]: he proved that a typed pre-net whose relational interpretation is a clique is logically correct (this result was obtained in multiplicative LL). More results have been obtained by Pagani: extending this results to the exponentials [Pag06], he has introduced *visible acyclicity* a weaker notion of correctness for pre-nets. He also obtained similar results for DiLL, with respect this time to the finiteness space semantics.

Conclusion

This already too long article was not intended to provide an encyclopedic introduction to differential linear logic: many aspects have not been dealt with, or have simply been alluded to. It was only intended to highlight some aspects, insisting more on semantics than on syntax. The reason for this choice of presentation is that we do not know yet which aspects of the theory will show useful, or relevant to other topics. Our main motivation for developing this theory is not its usefulness, but the fact that it is extremely natural and that, once discovered the differential extension of LL, it was almost impossible not to "play with it". We also believe that it provides a radically new viewpoint on the exponentials of LL and we are convinced that applications will show up naturally.

References

[BCEM11] Antonio Bucciarelli, Alberto Carraro, Thomas Ehrhard, and Giulio Manzonetto. Full Abstraction for Resource Calculus with Tests. In *CSL*, Lecture Notes in Computer Science. Springer-Verlag, 2011. To appear.

[BCL99] Gérard Boudol, Pierre-Louis Curien, and Carolina Lavatelli. A semantics for lambda calculi with resource. *Mathematical Structures in Computer Science*, 9(4):437–482, 1999.

[BCS06] Richard Blute, Robin Cockett, and Robert Seely. Differential categories. *Mathematical Structures in Computer Science*, 16(6):1049–1083, 2006.

[Bie95] Gavin Bierman. What is a categorical model of intuitionistic linear logic? In Mariangiola Dezani-Ciancaglini and Gordon D. Plotkin, editors, *Proceedings of the second Typed Lambda-Calculi and Applications conference*, volume 902 of *Lecture Notes in Computer Science*, pages 73–93. Springer-Verlag, 1995.

[Bou93] Gérard Boudol. The lambda-calculus with multiplicities. Technical Report 2025, INRIA Sophia Antipolis, 1993.

[Cur09] Pierre-Louis Curien, editor. *Typed Lambda Calculi and Applications, 9th International Conference, TLCA 2009, Brasilia, Brazil, July 1-3, 2009. Proceedings*, volume 5608 of *Lecture Notes in Computer Science*. Springer, 2009.

[DB87] N.G. De Bruijn. Generalizing Automath by means of a lambda-typed lambda calculus. In D.W. Kueker, E.G.K. Lopez-Escobar, and C.H. Smith, editors, *Mathematical Logic and Theoretical Computer Science*, Lecture Notes in Pure and Applied Mathematics, pages 71–92. Marcel Dekker, 1987. Reprinted in: Selected papers on Automath, Studies in Logic, volume 133, pages 313-337, North-Holland, 1994.

[dC07] Daniel de Carvalho. *Sémantique de la Logique Linéaire et temps de calcul*. Thèse de doctorat, Université de la Méditerranée, 2007.

[dF09] Marc de Falco. An explicit framework for interaction nets. In Ralf Treinen, editor, *Rewriting Techniques and Applications (RTA'09)*, volume 5595 of *Lecture Notes in Computer Science*, pages 209–223. Springer-Verlag, 6 2009.

[DR99] Vincent Danos and Laurent Regnier. Reversible, irreversible and optimal lambda-machines. *Theoretical Computer Science*, 227(1-2):273–291, 1999.

[Ehr02] Thomas Ehrhard. On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science*, 12:579–623, 2002.

[Ehr05] Thomas Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4):615–646, 2005.

[Ehr10] Thomas Ehrhard. A finiteness structure on resource terms. In *LICS*, pages 402–410. IEEE Computer Society, 2010.

[EL10a] Thomas Ehrhard and Olivier Laurent. Acyclic solos and differential interaction nets. *Logical Methods in Computer Science*, 6(3), 2010.

[EL10b] Thomas Ehrhard and Olivier Laurent. Interpreting a finitary pi-calculus in differential interaction nets. *Information and Computation*, 208(6), 2010. Special Issue: 18th International Conference on Concurrency Theory (CONCUR 2007).

[ER03] Thomas Ehrhard and Laurent Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309(1-3):1–41, 2003.

[ER06] Thomas Ehrhard and Laurent Regnier. Böhm trees, Krivine machine and the Taylor expansion of ordinary lambda-terms. Unpublished long version, 2006.

[ER08] Thomas Ehrhard and Laurent Regnier. Uniformity and the Taylor expansion of ordinary lambda-terms. *Theoretical Computer Science*, 403(2-3):347–372, 2008.

[Fio07] Marcelo P. Fiore. Differential structure in models of multiplicative biadditive intuitionistic linear logic. In Simona Ronchi Della Rocca, editor, *TLCA*, volume 4583 of *Lecture Notes in Computer Science*, pages 163–177. Springer, 2007.

[Gim11] Stéphane Gimenez. Realizability proof for normalization of full differential linear logic. In C.-H. Luke Ong, editor, *TLCA*, volume 6690 of *Lecture Notes in Computer Science*, pages 107–122. Springer-Verlag, 2011.

[Gir86] Jean-Yves Girard. The system F of variable types, fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir88] Jean-Yves Girard. Normal functors, power series and the λ-calculus. *Annals of Pure and Applied Logic*, 37:129–177, 1988.

[Hut93] Michael Huth. Linear Domains and Linear Maps. In Stephen D. Brookes, Michael G. Main, Austin Melton, Michael W. Mislove, and David A. Schmidt, editors, *MFPS*, volume 802 of *Lecture Notes in Computer Science*, pages 438–453. Springer-Verlag, 1993.

[Köt69] Gottfried Köthe. *Topological Vector Spaces, vol. 1.* Number 159 in Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1969.

[Kri85] Jean-Louis Krivine. Un interprteur du lambda-calcul. Unpublished note, 1985.

[Kri07] Jean-Louis Krivine. A call-by-name lambda-calculus machine. *Higher-Order and Symbolic Computation*, 20(3):199–207, 2007.

[Laf90] Yves Lafont. Interaction nets. In *Seventeenth Annual Symposium on Principles of Programming Languages*, pages 95–108, San Francisco, California, 1990. ACM Press.

[Lef42] Solomon Lefschetz. *Algebraic topology.* Number 27 in American mathematical society colloquium publications. American Mathematical Society, 1942.

[Mac71] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, 1971.

[Mel09] Paul-André Melliès. Categorical semantics of linear logic. *Panoramas et Synthèses*, 27, 2009.

[Pag06] Michele Pagani. Acyclicity and coherence in multiplicative exponential linear logic. In Zoltán Ésik, editor, *CSL*, volume 4207 of *Lecture Notes in Computer Science*, pages 531–545. Springer-Verlag, 2006.

[Pag09] Michele Pagani. The cut-elimination theorem for differential nets with promotion. In Curien [Cur09], pages 219–233.

[PT09] Michele Pagani and Paolo Tranquilli. Parallel Reduction in Resource Lambda-Calculus. In Zhenjiang Hu, editor, *APLAS*, volume 5904 of *Lecture Notes in Computer Science*, pages 226–242. Springer, 2009.

[PT11] Michele Pagani and Paolo Tranquilli. The Conservation Theorem for Differential Nets. *Mathematical Structures in Computer Science*, 2011. To appear.

[Ret97] Christian Retoré. A semantic characterisation of the correctness of a proof net. *Mathematical Structures in Computer Science*, 7(5):445–452, 1997.

[Reu93] Christophe Reutenauer. *Free Lie Algebras*. Oxford University Press, 1993.

[Tas09a] Christine Tasson. Algebraic totality, towards completeness. In Curien [Cur09], pages 325–340.

[Tas09b] Christine Tasson. *Sémantiques et syntaxes vectorielles de la logique linéaire*. Thèse de doctorat, Université Paris Diderot – Paris 7, 2009.

[Tra09] Paolo Tranquilli. Confluence of pure differential nets with promotion. In Erich Grädel and Reinhard Kahle, editors, *CSL*, volume 5771 of *Lecture Notes in Computer Science*, pages 500–514. Springer-Verlag, 2009.

[Tra11] Paolo Tranquilli. Intuitionistic differential nets and lambda-calculus. *Theoretical Computer Science*, 412(20):1979–1997, 2011.

[TV10] Christine Tasson and Lionel Vaux. Transport of finiteness structures and applications. *Mathematical Structures in Computer Science*, 2010. To appear.

[Vau05] Lionel Vaux. The differential lambda-mu calculus. *Theoretical Computer Science*, 379(1-2):166–209, 2005.

[Vau07] Lionel Vaux. On linear combinations of lambda-terms. In *Term Rewriting and Applications*, volume 4533 of *Lecture Notes in Computer Science*, pages 374–388. Springer-Verlag, 2007.

[Win04] Glynn Winskel. Linearity and non linearity in distributed computation. In Thomas Ehrhard, Jean-Yves Girard, Paul Ruet, and Philip Scott, editors, *Linear Logic in Computer Science*, volume 316 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2004.