CCS for Trees

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Abstract. CCS can be considered as a most natural extension of finite state automata in which *interaction* is made possible thanks to parallel composition. We propose here a similar extension for top-down tree automata. We introduce a parallel composition which is parameterized by a graph at the vertices of which subprocesses are located. Communication is allowed only between subprocesses related by an edge in this graph. We define an observational equivalence based on barbs as well as weak bisimilarity equivalence and prove an adequacy theorem relating these two notions.

Introduction

There is no need to insist on the importance of tree automata [CDG+07] in modern theoretical and applied computer science: they are pervasive in logic, verification, rewriting, structured documents handling, constraint solving etc. Tree automata are similar to usual finite word automata with the difference that they recognize trees instead of words (sequences of letters). Let Σ be a ranked signature (Σ_n is the set of function symbols of arity n). A Σ-tree is just a term written with the signature Σ. A *top-down tree automaton* has a finite number of *states* and transitions labeled by elements of Σ: a transition labeled by $f \in \Sigma_n$ has a *source* and a sequence of n *targets* which all are states of the automaton. A word automaton can be seen as a tree automaton over a signature Σ such that Σ_n is empty for all $n > 1$ and Σ_0 has an unique distinguished element $*$.

The definition of tree recognition by a top-down tree automaton A is quite simple: a tree $f(t_1, \dots, t_n)$ is recognized by A at state X means that A has an f-labeled transition whose source is X and target is $(X_1, \dots, X_n)$ and t_i is recognized by A at state X_i for each $i = 1, \dots, n$. There is also a notion of bottom-up tree automata, that we do not consider in this work; these two notions are equivalent in term of the recognized languages, as long as one considers *non-deterministic* automata.

Automata feature a *dualist* vision of computation with an essential dichotomy between programs (automata) and data (words, trees), very much in the spirit of Turing machines (based on the machine/tape dichotomy). The process algebra CCS, introduced in the early 1980's by Milner [Mil80], encompasses this restriction, extending finite automata with interactive capabilities. In this framework, finite automata (labeled with letters $a, b, \ldots$) can typically interact with other automata (labeled with dual letters $\overline{a}, \overline{b}, \ldots$), as soon as they are combined through a new binary operation: *parallel composition*. But much more general interaction scenarii are of course possible in CCS. This fundamental invention led to very fruitful new lines of research in the theory of concurrent processes and to the introduction of new process algebra, among which the π-calculus [MPW92] is not the less remarkable, with many spectacular applications to cryptography, bioinformatics etc.

In this paper, we propose a similar "interactive closure" of tree automata, a new version of CCS which extends tree automata just as ordinary CCS extends word automata. The natural idea is of course to add a parallel operation, but this requires some care. Indeed when a prefixed process $f \cdot (P_1, \ldots, P_n)$ (after a prefix $f \in \Sigma_n$, it is natural to have n subprocesses, and not only one, as explained in [CQJ08]) interacts with a dually prefixed one $\overline{f} \cdot (Q_1, \ldots, Q_n)$, we should remove the prefixes (just as in CCS) and then authorize interaction between the subprocess P_i with all processes which could communicate with its father $f \cdot (P_1, \ldots, P_n)$ as well as with Q_i, *but not with the Q_j's for $j \neq i$*; neither should the P_i's be allowed to communicate with each other in the resulting process. The same should hold of course for the Q_i's. This led us to the idea that a parallel composition should be a *graph*, at the vertices of which subprocesses (which are guarded sums) should be located; the edges of this graph specify which interactions are allowed. In Section 1, we introduce the syntax of this extension CCTS, restricting ourselves to a fragment where all sums are guarded; indeed, the corresponding fragment of CCS is known to be sensible and well behaved.

In Section 2, we introduce an operational semantics for CCTS by defining a single rewriting rule. This rule generalizes the $a/\overline{a}$ reduction of CCS to the case where a can be an n-ary function symbol. This rule implements the idea of restricted communication capabilities explained above. In order to define an operational equivalence on processes, we adapt the concept of *weak barbed congruence* [MS92,SW01] which is a natural way of saying that two processes behave in the same way, in all possible contexts. As usual, this notion is quite difficult to handle and we introduce therefore a notion of weak bisimilarity in Section 3 and prove that two weakly bisimilar processes are weakly barbed congruent in Section 4. For this, we define a labeled transition system on processes, and the definition of its transitions involves crucially the locations (graph vertices). The notion of bisimulation itself has to take these locations carefully into account.

In Section 2, we also argue that our version of CCS is a conservative extension of both tree automata and ordinary CCS: by this we mean that it admits restrictions which coincide with these two formalisms. Moreover, we show that tree recognition can be expressed simply in terms of interaction, using only the rewriting semantics. Though quite simple, this result uses in an essential way the restricted communication capabilities of CCTS.

These results suggest that CCTS is a sound and interesting extension of CCS. The most novel feature is that subprocesses are located at the vertices of a graph whose edges indicate which communications are possible, and the topology of this graph evolves during reduction. When no edge relates two processes, they can evolve independently, in a truly concurrent way, whereas the presence of an edge means that the corresponding processes will possibly synchronize in the future. Another interesting property of this approach is the importance of *locations* which suggests connections with the work of Castellani [Cas01], though locations are used in a different way: in this latter work, communication is possible when the involved processes are located at the same place.

This paper extends an earlier work of the second author and al. [CQJ08], where parallel composition however was not dealt with.

1 Syntax of processes

We use letters $\boldsymbol{P}, \boldsymbol{Q} \ldots$ to denote vectors $(P_1, \ldots, P_n)$, $(Q_1, \ldots, Q_n)$ etc. Let Loc be a countable set whose elements are called *locations* denoted with letters $p, q \ldots$ with or without subscripts or superscripts.

1.1 Graphs

Let E and F be disjoint sets and let $p \in E$. We set $E\,[F/p] = (E \setminus \{p\}) \cup F$. In other words, $E\,[F/p]$ is the set obtained from E by substituting the element p with the set F.

By a graph we mean a pair $G = (|G|, \frown_G)$, where $|G|$ is a finite subset of Loc and $\frown_G$ is a symmetric and antireflexive relation on $|G|$. Let G and H be graphs with $|G| \cap |H| = \emptyset$ and let $p \in |G|$. We define a graph $G\,[H/p]$ as follows:

- $|G\,[H/p]| = |G|\,[|H|/p]$
- and, given $q, r \in |G\,[H/p]|$, we say that $q \frown_{G[H/p]} r$ if $q \frown_G r$ or $q \frown_H r$ or $q \frown_G p$ and $r \in |H|$ or $r \frown_G p$ and $q \in |H|$.

1.2 Processes

We assume to be given a countable set of *processes variables* $\mathcal{V}$, denoted with letters $X, Y \ldots$ with or without subscripts or superscripts.

Let $\Sigma = (\Sigma_n)_{n \in \mathbf{N}}$ be a signature. With any symbol $f \in \Sigma_n$, we associate a co-symbol $\bar{f}$ distinct from all the elements of Σ_n and we set $\bar{\Sigma}_n = \Sigma_n \cup \{\bar{f} \mid f \in \Sigma_n\}$. In that way, we define an extended signature $\bar{\Sigma} = (\bar{\Sigma}_n)_{n \in \mathbf{N}}$. For $f \in \Sigma_n$, we set $\bar{\bar{f}} = f$.

We define the set of CCTS processes by induction.

- If $X \in \mathcal{V}$ then X is a process.
- If $X \in \mathcal{V}$ and P is a process, then $\mu X \cdot P$ is a process in which X is bound.

- If $f \in \bar{\Sigma}_n$ and $P_1, \dots, P_n$ are processes, then $f \cdot (P_1, \dots, P_n)$ is a process.
- If G is a Loc-graph (that is $|G| \subseteq \mathsf{Loc}$) and Φ is a finite function from $|G|$ to processes, then $G\langle\Phi\rangle$ is a process, to be understood as the parallel composition of the processes $\Phi(p)$ for $p \in |G|$. The processes $\Phi(p)$ are called the *components* of $G\langle\Phi\rangle$.
- 0 is a process and if P and Q are processes, then $P + Q$ is a process.
- If P is a process and I is a finite subset of Σ, then $P \setminus I$ is a process.

We define now the notion of *canonical* process: it is a process where all sums are guarded.

- If $X \in \mathcal{V}$ then X is a canonical process.
- A *canonical guarded sum* is either 0 or a process of the shape $f \cdot (P_1, \dots, P_n) + S$ where $f \in \bar{\Sigma}_n$, S is a canonical guarded sum and $P_1, \dots, P_n$ are canonical processes.
- A *canonical recursive guarded sum* is either a canonical guarded sum or a process of the shape $\mu X \cdot S$ where S is a canonical recursive guarded sum.
- If G is a Loc-graph and Φ is a finite function from $|G|$ to canonical recursive guarded sums, then $G\langle\Phi\rangle$ is a canonical process.
- If P is a canonical process and I is a finite subset of Σ, then $P \setminus I$ is a canonical process.

With any canonical recursive guarded sum S, we associate a canonical guarded sum $\mathsf{cs}(S)$ as follows:

$$\mathsf{cs}(S) = \begin{cases} S & \text{if } S \text{ is a canonical guarded sum} \\ \mathsf{cs}(T\,[S/X]) & \text{if } S = \mu X \cdot T \end{cases}$$

where the substitution $T\,[S/X]$ is defined in the obvious way. The notion of free and bound variable does not deserve further comments, μ being of course a binder.

All the processes we consider in this paper are canonical.

α-conversions of locations. Two processes P and P' such that there exists a bijection $\varphi : |P| \to |P'|$ which is a graph isomorphism (that is $p \frown_P q \Leftrightarrow \varphi(p) \frown_{P'} \varphi(q)$) and $P'(\varphi(p)) = P(p)$ for all $p \in |P|$ are said to be *α-equivalent*. Processes are always considered up to α-equivalence, and very often we need disjointness assumptions on the webs of processes involved in some reasonings: such assumptions can easily be enforced by means of α-conversion.

1.3 More notations

We denote with Proc the set of all canonical processes. If $P = G\langle\Phi\rangle$ is a canonical process, we use $|P| = |G|$. Also, for $p \in |P|$, we often write $P(p)$ instead of $\Phi(p)$, and we denote as $\frown_P$ the graph relation of G.

The empty process (the only one such that $|P| = \emptyset$) is denoted as ε.

Given two graphs G and H with disjoint webs, and a subset D of $|G| \times |H|$ we define a graph $K = G \oplus_D H$ by $|K| = |G| \cup |H|$ and, given $p, q \in |K|$, we stipulate that $p \frown_K q$ if $p \frown_G q$ or $p \frown_H q$ or $(p, q) \in D$ or $(q, p) \in D$: if $D = \emptyset$ then we set $G \oplus H = G \oplus_D H$.

Given processes $S = K\langle\Theta\rangle$ and $P = G\langle\Phi\rangle$ and a relation $D \subseteq |S| \times |P|$, one defines the process $S \oplus_D P$ as $(K \oplus_D H)\langle\Theta \cup \Phi\rangle$. When D is empty we simply denote this sum as $S \oplus P$, and more generally, we denote as $\oplus\boldsymbol{P}$ the sum $P_1 \oplus \cdots \oplus P_n$ of the processes $P_1, \dots, P_n$ where $\boldsymbol{P} = (P_1, \dots, P_n)$. In particular, when $D = |S| \times |P|$, the process $S \oplus_D P$ will be denoted as $S \mid P$ and called the *full parallel composition* of S and P. It corresponds to the standard parallel composition of process algebras, where all processes can freely interact with each other.

With the same notations as above, if $s \in |K|$, we denote as $S\,[P/s]$ the process $K\,[G/s]\,\langle\Theta'\rangle$ where $\Theta'(s') = \Theta(s')$ if $s' \neq s$ and $\Theta'(s) = P$.

2 Operational semantics

2.1 Internal reduction

Let P and P' be processes. We say that P *reduces* to P' if there are $p, q \in |P|$ such that $p \frown_P q$, $\mathsf{cs}(P(p)) = f \cdot (P_1, \dots, P_n) + S$, $\mathsf{cs}(P(q)) = \overline{f} \cdot (Q_1, \dots, Q_n) + T$ and P' is defined as follows: $|P'| = (|P| \setminus \{p, q\}) \cup \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $\frown_{P'}$ is the least symmetric relation on $|P'|$ such that, for any, $p', q' \in |P'|$, one has $p' \frown_{P'} q'$ in one of the following cases:

1. $p' \frown_{P_i} q'$ or $p' \frown_{Q_i} q'$ for some $i = 1, \dots, n$
2. $p' \in |P_i|$ and $q' \in |Q_i|$ for some $i = 1, \dots, n$ (*the same i for both*)
3. $p' \in \bigcup_{i=1}^{n} |P_i|$ and $q' \notin \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $p \frown_P q'$, or $q' \in \bigcup_{i=1}^{n} |Q_i|$ and $p' \notin \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $p' \frown_P q$
4. $p' \notin \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $q' \notin \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $p' \frown_P q'$.

We finish the definition of P' by saying that $P'(p') = P_i(p')$ if $p' \in |P_i|$, $P'(p') = Q_i(p')$ if $p' \in |Q_i|$ (for $i = 1, \dots, n$) and $P'(p') = P(p')$ if $p' \notin \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$.

This crucial definition deserves some comments. The process P has two subprocesses located at p and q, with dual prefixes: $f \cdot \boldsymbol{P}$ and $\overline{f} \cdot \boldsymbol{Q}$. The fact that p and q are connected in P ($p \frown_P q$) means that these processes can interact. This interaction consists in suppressing both prefixes and in replacing the vertice p of the graph G of P by the graph $G_1 \oplus \cdots \oplus G_n$ (where G_i is the graph of P_i) and the graph q by the graph $H_1 \oplus \cdots \oplus H_n$ (where H_i is the graph of Q_i) within the graph G of P. The connection between p and q in P is inherited by the vertices of G_i and H_i in P', but a process located on G_i (one of the components of P_i) cannot communicate with a process located on H_j with $j \neq i$. The connections between p and other vertices of P, distinct from q, are also inherited by the vertices of all G_i's and similarly for the H_i's.

We denote with $\to$ the internal reduction relation and with $\to^*$ its reflexive and transitive closure.

Example 1. Let $a \in \Sigma_0$ and $f \in \Sigma_2$. Consider the process $P = \overline{a} \mid a \mid f \cdot (a, \overline{a}) \mid \overline{f} \cdot (a, \overline{a})$ (we write simply "a" instead of $a \cdot ()$). In other words, the graph of P is a complete graph with 4 vertices, say $1, 2, 3, 4$, and we have $P(1) = a$, $P(2) = \overline{a}$, $P(3) = f \cdot (a, \overline{a})$ and $P(4) = \overline{f} \cdot (a, \overline{a})$. Since 3 and 4 are connected in that graph and the corresponding prefixes f and $\overline{f}$ are dual, we can reduce P to a process P' such that $|P'| = \{1, 2, 5, 6, 7, 8\}$ (remember that we work up to α-equivalence, so the names of locations are irrelevant) with $P'(1) = a$, $P'(2) = \overline{a}$, $P'(5) = a$, $P'(6) = \overline{a}$, $P'(7) = a$, and $P'(8) = \overline{a}$, and the edges of P' are all $\{i, j\}$ with $i \in \{1, 2\}$ and $j \neq i$, $\{5, 7\}$ and $\{6, 8\}$. So, in P', the interaction of a located at 5 with $\overline{a}$ located at 8 is not possible, but of course a located at 5 can interact with $\overline{a}$ located at 2. Performing that reduction, we get P'' with $|P''| = \{1, 6, 7, 8\}$ and the edges of P'' are all $\{1, j\}$ with $j \neq 1$ and $\{6, 8\}$, with $P''(1) = a$, $P''(6) = \overline{a}$, $P''(7) = a$ and $P''(8) = \overline{a}$. In P'', the only possible reductions are between a located at 1 and $\overline{a}$ located at 6 or 8. Both lead to the process $a \oplus \overline{a}$ where no reduction is possible.

2.2 Connection with tree automata

A top-down tree automaton is a pair $A = (\mathcal{Q}, \mathcal{T})$ where $\mathcal{Q}$ is a finite subset of $\mathcal{V}$, whose elements are called *states*, and $\mathcal{T}$ is a set of triples $(X, f, (X_1, \ldots, X_n))$ where $f \in \Sigma_n$ and $X_1, \ldots, X_n \in \mathcal{Q}$ and whose elements are called *transitions*. The *language recognized by A at state $X \in \mathcal{Q}$*, denoted as $\mathsf{L}(A, X)$, is the least set of Σ-trees such that $f(t_1, \ldots, t_n) \in \mathsf{L}(A, X)$ as soon as there are $X_1, \ldots, X_n \in \mathcal{Q}$ such that $(X, f, (X_1, \ldots, X_n)) \in \mathcal{T}$ and $t_i \in \mathsf{L}(A, X_i)$ for $i = 1, \ldots, n$.

We associate a process $\langle A \rangle_X$ with any pair (A, X) where $A = (\mathcal{Q}, \mathcal{T})$ is a tree automaton and $X \in \mathcal{Q}$. More precisely we define $\langle A \rangle_X^{\mathcal{X}}$ where $\mathcal{X} \subseteq \mathcal{V}$ (intuitively, $\mathcal{X}$ is the set of already defined processes), and then we set $\langle A \rangle_X = \langle A \rangle_X^{\emptyset}$. If $X \notin \mathcal{X}$, then $\langle A \rangle_X^{\mathcal{X}} = \mu X \cdot S$ where S is the sum of all prefixed processes $f \cdot (\langle A \rangle_{X_1}^{\mathcal{X} \cup \{X\}}, \ldots, \langle A \rangle_{X_n}^{\mathcal{X} \cup \{X\}})$ where $(X, f, (X_1, \ldots, X_n)) \in \mathcal{T}$, and if $X \in \mathcal{X}$, then $\langle A \rangle_X^{\mathcal{X}} = X$. This inductive definition is well founded because the parameter $\mathcal{X}$ increases strictly at each inductive step, and remains included in $\mathcal{Q}$. Moreover, the invariant that all the free variables of $\langle A \rangle_X^{\mathcal{X}}$ is a subset of $\mathcal{X}$ is preserved by the inductive step, and hence $\langle A \rangle_X$ is closed.

Lemma 1. *With the notations above, $\mathsf{cs}(\langle A \rangle_X)$ is the sum of all prefixed processes $f \cdot (\langle A \rangle_{X_1}, \ldots, \langle A \rangle_{X_n})$ where $(X, f, (X_1, \ldots, X_n)) \in \mathcal{T}$.*

Proof. More generally, $\mathsf{cs}(\langle A \rangle_X^{\{X_1, \ldots, X_p\}} [\langle A \rangle_{X_1}/X_1, \ldots, \langle A \rangle_{X_p}/X_p])$ is equal to the sum above, for any subset $\{X_1, \ldots, X_p\}$ of $\mathcal{Q}$ (with the X_i's pairwise distinct). The proof is a simple induction on $q - p$, where q is the cardinality of $\mathcal{Q}$. □

We represent dually any Σ-tree $t = f(t_1, \ldots, t_n)$ as a process $\overline{t}$ by setting $\overline{t} = \overline{f} \cdot (\overline{t_1}, \ldots, \overline{t_n})$. The following results expresses that our process algebra, together with its internal reduction, is a conservative extension of tree automata by showing that tree recognition boils down to a (very) particular case of interaction between processes.

Theorem 1. *Let $A = (\mathcal{Q}, \mathcal{T})$ be a tree automaton, let $X \in \mathcal{X}$ and let t be a Σ-tree. Then $t \in \mathsf{L}(A, X)$ iff $\langle A \rangle_X \mid \bar{t} \rightarrow^* \varepsilon$.*

Proof. This is straightforward, once observed that, if $t = f(t_1, \dots, t_n)$ and if $(X, f, (X_1, \dots, X_n)) \in \mathcal{T}$, one has $\langle A \rangle_X \mid \bar{t} \rightarrow (\langle A \rangle_{X_1} \mid \overline{t_1}) \oplus \cdots \oplus (\langle A \rangle_{X_n} \mid \overline{t_n})$, thanks to Lemma 1. Observe then that $(\langle A \rangle_{X_1} \mid \overline{t_1}) \oplus \cdots \oplus (\langle A \rangle_{X_n} \mid \overline{t_n})$ reduces to ε iff each process $\langle A \rangle_{X_i} \mid \overline{t_i}$ reduces to ε since these processes cannot interact with each other. If $\mathcal{T}$ has no element of the shape $(X, f, (X_1, \dots, X_n))$, then the process $\langle A \rangle_X \mid \bar{t}$ does not reduce. $\square$

2.3 Connection with CCS for words

We assume here that $\Sigma_n = \emptyset$ for all $n > 1$ and that $\Sigma_0 = \{*\}$. Then a Σ-tree is the same thing as a Σ_1-word, written $a_1 \dots a_p *$. We restrict our attention to processes in which all the graphs parameterizing parallel compositions are complete, so that any process is of the shape $S_1 \mid \cdots \mid S_p$ where each S_i is a recursive canonical guarded sum $\mu \boldsymbol{X} \cdot (a_1 \cdot P_1 + \cdots + a_m \cdot P_m)$: this restriction of our process algebra coincides with guarded CCS. Observe also that, if P is a process in this restricted setting (arities ≤ 1 and all parallel compositions are complete graphs), and if $P \rightarrow P'$, then P' belongs to the same restriction and the reduction $P \rightarrow P'$ is a standard τ-reduction of CCS. In that way we see that our process algebra is also a conservative extension of ordinary guarded CCS.

2.4 Weak barbed bisimilarity

Let $f \in \bar{\Sigma}$ and let P be a process. We say that f is a *barb* of P, and write $P \downarrow_f$, if there exists $p \in |P|$ such that $\mathsf{cs}(P(p))$ is of shape $f \cdot (P_1, \dots, P_n) + S$.

A relation $\mathcal{B} \subseteq \mathsf{Proc}^2$ is a *weak barbed bisimulation* if it is symmetric and satisfies the following conditions. For any $P, Q \in \mathsf{Proc}$ such that $P \mathrel{\mathcal{B}} Q$,

- for any $P' \in \mathsf{Proc}$, if $P \rightarrow^* P'$, then there exists $Q' \in \mathsf{Proc}$ such that $Q \rightarrow^* Q'$ and $P' \mathrel{\mathcal{B}} Q'$ (one says that $\mathcal{B}$ is a *weak reduction bisimulation*);
- for any $P' \in \mathsf{Proc}$ and any $f \in \bar{\Sigma}$, if $P \rightarrow^* P'$ and $P' \downarrow_f$, then there exists $Q' \in \mathsf{Proc}$ such that $Q \rightarrow^* Q'$ and $Q' \downarrow_f$ (one says that $\mathcal{B}$ is *weak barb preserving*).

The diagonal relation $\{(P, P) \mid P \in \mathsf{Proc}\}$ is a weak barbed bisimulation, and if $\mathcal{B}$ and $\mathcal{B}'$ are weak barbed bisimulations, then so is $\mathcal{B} \cup \mathcal{B}'$. We say that $P, Q \in \mathsf{Proc}$ are *weakly barbed bisimilar* if there exists a weak barbed bisimulation $\mathcal{B}$ such that $P \mathrel{\mathcal{B}} Q$. Notation: $P \stackrel{\cdot}{\approx} Q$.

Lemma 2. *Weak barbed bisimilarity is an equivalence relation.*

Proof. Straightforward, using the above properties of $\stackrel{\cdot}{\approx}$. $\square$

2.5 Weak barbed congruence

Let Y be a variable; a Y*-context* is a process R which contains exactly one free occurrence of Y, which does not occur in a subprocess of R of the shape $\mu \boldsymbol{X} \cdot R'$ (in other words, Y must really occur only once in R). If R and S are Y-contexts, so is $R\,[S/Y]$.

A relation $\mathcal{R} \subseteq \mathsf{Proc}^2$ is a *congruence* if it is reflexive and such that, for any Y-context R, one has $P \mathrel{\mathcal{R}} Q \Rightarrow R\,[P/Y] \mathrel{\mathcal{R}} R\,[Q/Y]$.

Proposition 1. *For any reflexive relation* $\mathcal{R} \subseteq \mathsf{Proc}^2$*, there exists a largest congruence* $\overline{\mathcal{R}}$ *contained in* $\mathcal{R}$*. This relation is characterized by:* $P \mathrel{\overline{\mathcal{R}}} Q$ *iff for any* Y*-context* R *one has* $R\,[P/Y] \mathrel{\mathcal{R}} R\,[Q/Y]$*. If* $\mathcal{R}$ *is an equivalence relation, so is* $\overline{\mathcal{R}}$*.*

Proof. The first statement results from the fact that congruences are closed under arbitrary unions and that $\mathcal{R}$ contains the identity relation which is a congruence. As to the second statement, let $\mathcal{E}$ be the relation defined by $P \mathrel{\mathcal{E}} Q$ iff for any Y-context R one has $R\,[P/Y] \mathrel{\mathcal{R}} R\,[Q/Y]$. Then $\mathcal{E}$ is a congruence which is contained in $\mathcal{R}$ (since we can take $R = Y$) and hence $\mathcal{E} \subseteq \overline{\mathcal{R}}$. Conversely, assume that $P \mathrel{\overline{\mathcal{R}}} Q$ and let R be a Y-context. Since $\overline{\mathcal{R}}$ is a congruence, we have $R\,[P/Y] \mathrel{\overline{\mathcal{R}}} R\,[Q/Y]$ and hence $R\,[P/Y] \mathrel{\mathcal{R}} R\,[Q/Y]$ since $\overline{\mathcal{R}} \subseteq \mathcal{R}$ by definition of $\overline{\mathcal{R}}$ and hence $P \mathrel{\mathcal{E}} Q$. The last statement results from the second one since $\mathcal{E}$ is an equivalence relation when $\mathcal{R}$ is an equivalence relation. □

The largest congruence contained in $\dot{\approx}$ is denoted as $\cong$ and is called *weak barbed congruence*: it is our main notion of operational equivalence on processes. It is an equivalence relation by the proposition above and by Lemma 2. Moreover, we have $P \cong Q$ iff for any Y-context R, we have $R\,[P/Y] \mathrel{\dot{\approx}} R\,[Q/Y]$.

3 Localized tree-transition systems of processes

Let P and P' be processes. Let $f \in \bar{\Sigma}_n$ and let $L_1, \dots, L_n$ be pairwise disjoint finite subsets of Loc. Let $p \in \mathsf{Loc}$. We write $P \xrightarrow{p:f\cdot(\boldsymbol{L})} P'$ if $p \in |P|$, $\mathsf{cs}(P(p)) = f \cdot (P_1, \dots, P_n) + S$ with $L_i = |P_i|$ for $i = 1, \dots, n$ and $P' = P\,[\oplus \boldsymbol{P}/p]$.

We write $P \xrightarrow[\rho]{\tau} P'$ if $P \to P'$ in the sense of 2.1 and, with the notations of that section, $\rho : |P'| \to |P|$ is the *residual function* defined $\rho'(p') = p$ if $p' \in \bigcup_i |P_i|$, $\rho'(p') = q$ if $p' \in \bigcup_i |Q_i|$, and $\rho(p') = p'$ otherwise.

We define the reflexive-transitive closure $\xrightarrow[\rho]{\tau *}$ by: $P \xrightarrow[\rho]{\tau *} P'$ if there exist $P_1, \dots, P_n$ and $\rho_1, \dots, \rho_{n-1}$ such that $P = P_1$, $P_n = P'$ and $P_i \xrightarrow[\rho_i]{\tau} P_{i+1}$ for $i = 1, \dots, n-1$, and $\rho = \rho_1 \circ \cdots \circ \rho_{n-1}$.

We write $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\rho,\rho'}{\Longrightarrow}} P'$ if $P \xrightarrow[\rho]{\tau *} P_1 \xrightarrow{p:f\cdot(\boldsymbol{L})} P_1' \xrightarrow[\rho']{\tau *} P'$.

3.1 Localized weak bisimilarity

We propose now a notion of bisimilarity as a tool for proving weak barbed congruence of processes. The definition is coalgebraic and is based on a concept of bisimulation which, due to the importance of the graph structure in the operational semantics of CCTS, strongly uses locations. This definition should be considered only as a first attempt towards a theory of weak bisimulation for CCTS, and it is possible that the bisimilarity that we propose here is too restrictive for interesting uses.

A *localized relation* (on processes) is a set $\mathcal{R} \subseteq \mathsf{Proc} \times \mathcal{P}(\mathsf{Loc}^2) \times \mathsf{Proc}$ such that, if $(P, E, Q) \in \mathcal{R}$ then $E \subseteq |P| \times |Q|$. Such a relation is *symmetric* if $(P, E, Q) \in \mathcal{R} \Rightarrow (Q, {}^{t}E, P) \in \mathcal{R}$ where ${}^{t}E = \{(q, p) \mid (p, q) \in E\}$.

A *(localized) weak bisimulation* is a symmetric localized relation such that

- if $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda]{\tau} P'$ then $Q \xrightarrow[\rho]{\tau*} Q'$ with $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that, if $(p', q') \in E'$ then $(\lambda(p'), \rho(q')) \in E$ (this latter condition will be called *condition on residuals*)
- if $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow{p:f\cdot(\boldsymbol{L})} P'$ then $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(p', \rho'(q')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$, or else $p' \notin \bigcup_{i=1}^{n} L_i$, $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$ and $(p', \rho\rho'(q')) \in E$ (this latter condition will be called *condition on residuals*).

Lemma 3. *Let $\mathcal{R}$ be a weak bisimulation. If $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda]{\tau*} P'$, then $Q \xrightarrow[\rho]{\tau*} Q'$ with $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(\lambda'(p'), \rho'(q')) \in E$.*

Proof. Simple induction on the length of the sequence of reductions $P \xrightarrow[\lambda]{\tau*} P'$. □

Lemma 4. *If $P \xrightarrow[\lambda]{\tau*} P_1$, $P_1 \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda_1,\lambda_1'}{\Longrightarrow}} P_1'$ and $P_1' \xrightarrow[\lambda']{\tau*} P'$ then $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda\lambda_1,\lambda_1'\lambda'}{\Longrightarrow}} P'$.*

Proof. Results immediately from the definitions. □

Lemma 5. *A symmetric localized relation $\mathcal{R} \subseteq \mathsf{Proc} \times \mathcal{P}(\mathsf{Loc}^2) \times \mathsf{Proc}$ is a weak bisimulation iff the following properties hold.*

- *If $(P, E, Q) \in \mathcal{R}$ and $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda'}{\Longrightarrow}} P'$, then $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho'}{\Longrightarrow}} Q'$ with $(\lambda(p), \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(\lambda'(p'), \rho'(q')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$, or else $\lambda'(p') \notin \bigcup_{i=1}^{n} L_i$, $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$ and $(\lambda\lambda'(p'), \rho\rho'(q')) \in E$.*
- *If $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda]{\tau*}$, then $Q \xrightarrow[\rho]{\tau*} Q'$ with $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(\lambda'(p'), \rho'(q')) \in E$.*

Proof. See the Appendix section. □

Lemma 6. *Let $\mathcal{I}$ be the localized relation defined by: $(P, E, Q) \in \mathcal{I}$ if $P = Q$ and $E = \mathrm{Id}_{|P|}$. Then $\mathcal{I}$ is a weak bisimulation.*

Proof. Straightforward. □

If $\mathcal{R}$ and $\mathcal{R}'$ are weak bisimulations, so is $\mathcal{R} \cup \mathcal{R}'$: this results immediately from the definition. We say that P and Q are weakly bisimilar (notation $P \approx Q$) if there exists a weak bisimulation $\mathcal{R}$ and a set $E \subseteq |P| \times |Q|$ such that $(P, E, Q) \in \mathcal{R}$.

Let $\mathcal{R}$ and $\mathcal{S}$ be localized relations. We define a localized relation $\mathcal{S} \circ \mathcal{R}$ as follows: $(P, H, R) \in \mathcal{S} \circ \mathcal{R}$ if $H \subseteq |P| \times |R|$ and there exist Q, E and F such that $(P, E, Q) \in \mathcal{R}$, $(Q, F, R) \in \mathcal{S}$ and $F \circ E \subseteq H$.

Lemma 7. *If $\mathcal{R}$ and $\mathcal{S}$ are weak bisimulations, then so is $\mathcal{S} \circ \mathcal{R}$.*

Proof. See the Appendix section. □

We say that two processes P and Q are weakly bisimilar, and write $P \approx Q$, if there exists a weak bisimulation $\mathcal{R}$ and a relation $E \subseteq |P| \times |Q|$ such that $(P, E, Q) \in \mathcal{R}$.

Proposition 2. *The relation $\approx$ is an equivalence relation on processes.*

Proof. Reflexivity results from Lemma 6, and symmetry from the symmetry hypothesis on weak bisimulations. Transitivity is a straightforward consequence of Lemma 7. □

Proposition 3. *If $P \approx Q$ then $P \dot{\approx} Q$.*

Proof. Let $\mathcal{R}$ be a weak bisimulation. Let $\mathcal{B}$ be the binary relation on processes defined by: $(P, Q) \in \mathcal{B}$ if there exists $E \subseteq |P| \times |Q|$ such that $(P, E, Q) \in \mathcal{R}$. We contend that $\mathcal{B}$ is a weak barbed bisimulation, and this will prove the proposition. First observe that $\mathcal{B}$ is symmetric because $\mathcal{R}$ is a symmetric localized relation.

Let $(P, Q) \in \mathcal{B}$ and assume first that $P \to^* P'$, that is $P \xrightarrow[\lambda]{\tau*} P'$ for some residual function λ. Let $E \subseteq |P| \times |Q|$ be such that $(P, E, Q) \in \mathcal{R}$. By Lemma 5, one has $Q \xrightarrow[\rho]{\tau*} Q'$ for some residual function ρ, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$ and therefore $(P', Q') \in \mathcal{B}$ as required; this shows that $\mathcal{B}$ is a weak reduction bisimulation.

Assume now that $(P, Q) \in \mathcal{B}$ and that $P \to^* P'$ with $P' \downarrow_f$ (with $f \in \bar{\Sigma}$ of arity n), meaning that $P' \xrightarrow{p' : f \cdot (L)} P''$. Let $E \subseteq |P| \times |Q|$ be such that $(P, E, Q) \in \mathcal{R}$. By Lemma 5, one has $Q \xrightarrow[\rho]{\tau*} Q'$ for some residual function ρ, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$. Since $(P', E', Q') \in \mathcal{R}$ and since $\mathcal{R}$ is a weak bisimulation, we have $Q' \overset{q' : f \cdot (M)}{\underset{\rho', \rho''}{\Longrightarrow}} Q''$ and therefore $Q' \to^* Q'_1$ with $Q'_1 \downarrow_f$. This shows that $\mathcal{B}$ is weak barb preserving since $Q \to^* Q'_1$. □

4 Weak bisimilarity is a congruence

More notations. We say that a triple of relations (D, D', E) with $D \subseteq A \times B$, $D' \subseteq A \times B'$ and $E \subseteq B \times B'$ is *adapted*, if, for any $(a, b, b') \in A \times B \times B'$, with $(b, b') \in E$, one has $(a, b) \in D$ iff $(a, b') \in D'$.

Let $\mathcal{R}$ be a localized relation on processes. One defines a new localized relation $\mathcal{R}'$ by stipulating that $(U, F, V) \in \mathcal{R}'$ if there is a process S, and a triple $(P, E, Q) \in \mathcal{R}$ as well are two relations $C \subseteq |S| \times |P|$ and $D \subseteq |S| \times |Q|$ such that $U = S \oplus_C P$, $V = S \oplus_D Q$ (these notations are introduced in Section 1.3), the triple of relations (C, D, E) is adapted and F is the relation $\mathrm{Id}_{|S|} \cup E \subseteq |U| \times |V|$. This localized relation will be called *the parallel extension* of $\mathcal{R}$.

Lemma 8. *If $\mathcal{R}$ is symmetric, then so is its parallel extension $\mathcal{R}'$.*

Proof. Observe that (C, D, E) is adapted iff $(D, C, {}^{\mathrm{t}}E)$ is adapted. □

The next proposition is an essential tool for proving that weak bisimulation is a congruence.

Proposition 4. *If $\mathcal{R}$ is a weak bisimulation, so is its parallel extension $\mathcal{R}'$.*

Proof. See the Appendix section. □

Theorem 2. *The weak bisimilarity relation $\approx$ is a congruence.*

Proof. See the Appendix section. □

We can prove now the main technical result of the paper.

Theorem 3. *Let P and Q be processes. If $P \approx Q$ (P and Q are weakly bisimilar) then $P \cong Q$ (P and Q are weakly barb congruent).*

Proof. Assume that $P \approx Q$ and let R be a Y-context. We have $R[P/Y] \approx R[Q/Y]$ by Theorem 2 and hence $R[P/Y] \stackrel{\cdot}{\approx} R[Q/Y]$ by Proposition 3. □

Conclusion

We have presented an extension of CCS which deals with trees instead of words, and various concepts and tools associated with this new process algebra. The notion of barbed bisimilarity, as it is defined here, is a straightforward generalization of the corresponding notion for CCS and therefore is hardly questionable, but we cannot say the same of weak bisimilarity. It will be crucial to understand if weak bisimilarity is equivalent to weak barbed congruence here and, if not, to look for a more liberal notion of weak bisimilarity in order to get such a full abstraction property. Another more conceptual task will be to extend this approach to more expressive settings such as for instance the π-calculus, and of course to understand if CCTS can be encoded in such settings.

This work also originated from the encodings of the π-calculus and of the solos calculus in differential interaction nets by the first author and Laurent [EL10]. In these nets, which are graphical objects, parallel compositions appear as complete graphs, and it is clear that more general graphs (actually, arbitrary graphs) could be encoded as well in the very same formalism. A graphical approach to CCTS, in the spirit of interaction nets, will be presented in a forthcoming paper.

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Appendix: more proofs

Proof of Lemma 5

The stated property are obviously sufficient, we prove that the first one is necessary (necessity of the second one is Lemma 3). Assume that $(P, E, Q) \in \mathcal{R}$

and $P \overset{p:f\cdot(L)}{\underset{\lambda,\lambda'}{\Longrightarrow}} P'$, that is $P \xrightarrow[\lambda]{\tau*} P_1 \xrightarrow{p:f\cdot(L)} P'_1 \xrightarrow[\lambda']{\tau*} P'$. By Lemma 3 one has $Q \xrightarrow[\rho]{\tau*} Q_1$ with $(P_1, E_1, Q_1) \in \mathcal{R}$ where E_1 is such that $(p_1, q_1) \in E_1 \Rightarrow (\lambda(p_1), \rho(q_1)) \in E$. Since $P_1 \xrightarrow{p:f\cdot(L)} P'_1$ and $(P_1, E_1, Q_1) \in \mathcal{R}$, one has $Q_1 \overset{q:f\cdot(M)}{\underset{\rho_1,\rho'_1}{\Longrightarrow}} Q'_1$ with $(p, \rho_1(q)) \in E_1$ and $(P'_1, E'_1, Q'_1) \in \mathcal{R}$ where E'_1 is such that if $(p'_1, q'_1) \in E'_1$ then $(p'_1, \rho'_1(q'_1)) \in L_i \times M_i$ for some i or $p'_1 \notin \bigcup_i L_i$, $\rho'_1(q'_1) \notin \bigcup_i M_i$ and $(p'_1, \rho_1\rho'_1(q'_1)) \in E_1$. Since $P'_1 \xrightarrow[\lambda']{\tau*} P'$ and $(P'_1, E'_1, Q'_1) \in \mathcal{R}$, we can apply Lemma 3 again which shows that $Q'_1 \xrightarrow[\rho']{\tau*} Q'$ with $(P', E', Q') \in \mathcal{R}$ where E' is such that $(p', q') \in E' \Rightarrow (\lambda'(p'), \rho'(q')) \in E'_1$. By Lemma 4, we have $Q \overset{q:f\cdot(M)}{\underset{\rho\rho_1,\rho'_1\rho'}{\Longrightarrow}} Q'$ and remember that $(P', E', Q') \in \mathcal{R}$. We have $(p, \rho_1(q)) \in E_1$ and hence $(\lambda(p), \rho\rho_1(q)) \in E$ by definition of E_1. Last, the condition on residuals obviously holds.

Proof of Lemma 7

First, observe that $\mathcal{S} \circ \mathcal{R}$ is symmetric. We apply Lemma 5. Let $(P, H, R) \in \mathcal{S} \circ \mathcal{R}$ and assume that $P \overset{p:f\cdot(L)}{\underset{\lambda,\lambda'}{\Longrightarrow}} P'$. Let Q, E and F be such that $(P, E, Q) \in \mathcal{R}$, $(Q, F, R) \in \mathcal{S}$ and $F \circ E \subseteq H$. Then we have $Q \overset{q:f\cdot(M)}{\underset{\rho,\rho'}{\Longrightarrow}} Q'$ with $(\lambda(p), \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ with E' such that if $(p', q') \in E'$ then $(\lambda'(p'), \rho'(q')) \in \bigcup_i L_i \times M_i$ or $\lambda'(p') \notin \bigcup_i L_i$, $\rho'(q') \notin \bigcup_i M_i$ and $(\lambda\lambda'(p'), \rho\rho'(q')) \in E$. Therefore we have $R \overset{r:f\cdot(N)}{\underset{\sigma,\sigma'}{\Longrightarrow}} R'$ with $(\rho(q), \sigma(r)) \in F$ and $(Q', F', R') \in \mathcal{S}$ with F' such that if $(q', r') \in F'$ then $(\rho'(q'), \sigma'(r')) \in \bigcup_i M_i \times N_i$ or $\rho'(q') \notin \bigcup_i M_i$, $\sigma'(r') \notin \bigcup_i N_i$ and $(\rho\rho'(q'), \sigma\sigma'(r')) \in F\}$. So we have $(\lambda(p), \sigma(r)) \in F \circ E \subseteq H$. Let $H' = \{(p', r') \in |P'| \times |R'| \mid (\lambda'(p'), \sigma'(r')) \in \bigcup_i L_i \times N_i \text{ or } (\lambda\lambda'(p'), \sigma\sigma'(r')) \in H\}$. We conclude by showing that $F' \circ E' \subseteq H'$, since this shows that $(P', H', R') \in \mathcal{S} \circ \mathcal{R}$. Let $(p', r') \in F' \circ E'$, there exists q' such that $(p', q') \in E'$ and $(q', r') \in F'$. There are two possibilities:

- if $(\lambda'(p'), \rho'(q')) \in L_i \times M_i$ for a (necessarily unique) i, then we must have $(\rho'(q'), \sigma'(r')) \in M_i \times N_i$ (for the same i) and hence $(\lambda'(p'), \sigma'(r')) \in L_i \times N_i$;
- if $\lambda'(p') \notin \bigcup_i L_i$ and $\rho'(q') \notin \bigcup M_i$, we cannot have $(\rho'(q'), \sigma'(r')) \in M_j \times N_j$ for any j and hence we must have $\sigma'(r') \notin \bigcup_i N_i$, so we have $(\lambda\lambda'(p'), \rho\rho'(q')) \in E$ and $(\rho\rho'(q'), \sigma\sigma'(r')) \in F$. Therefore $(\lambda\lambda'(p'), \sigma\sigma'(r')) \in F \circ E \subseteq H$.

Proof of Proposition 4

Symmetry of $\mathcal{R}'$ results from the symmetry of $\mathcal{R}$ and from Lemma 8. Let $(U, F, V) \in \mathcal{R}'$ with $U = S \oplus_C P$, $V = S \oplus_D Q$, $(P, E, Q) \in \mathcal{R}$, (C, D, E) adapted and $F = \mathrm{Id}_{|S|} \cup E$.

Case of a τ-transition. Assume that $U \xrightarrow[\lambda]{\tau} U'$. We must show that $V \xrightarrow[\rho]{\tau*} V'$ with $(U', F', V') \in \mathcal{R}'$ and $(\lambda(u'), \rho(v')) \in F$ for each $(u', v') \in F'$ (condition on residuals). There are three cases as to the locations of the two guarded sums involved in that reduction.

Assume first that they are located in S, in other words there are $s, t \in |S|$ with $s \frown_S t$, $S(s) = f\cdot\boldsymbol{S}+\tilde{S}$ ($\tilde{S}$ is a guarded sum) and $S(t) = \overline{f}\cdot\boldsymbol{T}+\tilde{T}$ ($\tilde{T}$ is a guarded sum), and we have $S \xrightarrow[\mu]{\tau} S'$ with $S' = (|S| \setminus \{s,t\}) \cup \bigcup_i |S_i| \cup \bigcup |T_i|$ and $\frown_{S'}$ is the least symmetric relation on $|S'|$ such that $s' \frown_{S'} t'$ if $s', t' \notin \bigcup_i |S_i| \cup \bigcup_i |T_i|$ and $s' \frown_S t'$, or $(s',t') \in |S_i| \times |T_i|$ for some i, or $s' \in \bigcup_i |S_i|$, $t' \notin \bigcup_i |S_i| \cup \bigcup_i |T_i|$ and $s \frown_S t'$, or $s' \notin \bigcup_i |S_i| \cup \bigcup_i |T_i|$, $t' \in \bigcup_i |T_i|$ and $s' \frown_S t$, or $s' \frown_{S_i} t'$ or $s' \frown_{T_i} t'$ for some i. The residual function μ is given by $\mu(s') = s$ if $s' \in \bigcup_i |S_i|$, $\mu(s') = t$ if $s' \in \bigcup_i |T_i|$ and $\mu(s') = s'$ otherwise. We have $U' = S' \oplus_{C'} P$ where C' is defined as follows: $(s', p) \in C'$ if $(\mu(s'), p) \in C$, and moreover $\lambda = \mu \cup \mathrm{Id}_{|P|}$. Then we have similarly $V = S \oplus_D Q \xrightarrow[\rho]{\tau} V' = S' \oplus_{D'} Q$ with $\rho = \mu \cup \mathrm{Id}_{|Q|}$, and D' defined by: $(s', q) \in D'$ iff $(\mu(s'), q) \in D$. The triple (C', D', E) is obviously adapted since (C, D, E) is adapted and therefore we have $(U', F', V') \in \mathcal{R}'$ where $F' = \mathrm{Id}_{|S'|} \cup E$. Moreover, the condition on residuals is satisfied, since, given $(u', v') \in F'$, we have either $u' = v' \in |S'|$ and then $\lambda(u') = \mu(v') \in |S|$ or $(u', v') \in E$ and $(\lambda(u'), \mu(v')) = (u', v') \in E$. In both cases $(\lambda(u'), \mu(v')) \in F$.

Assume next that they are located in P, in other words there are $p, r \in |P|$ with $P(p) = f\cdot\boldsymbol{P}+\tilde{P}$ (where $\tilde{P}$ is a guarded sum) and $P(r) = \overline{f}\cdot\boldsymbol{R}+\tilde{R}$ (where $\tilde{R}$ is a guarded sum), and we have $P \xrightarrow[\mu]{\tau} P'$ with $P' = (|P| \setminus \{p,r\}) \cup \bigcup_i |P_i| \cup \bigcup_i |R_i|$ and $\frown_{P'}$ is the least symmetric relation on $|P'|$ such that $p' \frown_{P'} r'$ if $p', r' \notin \bigcup_i |P_i| \cup \bigcup_i |R_i|$ and $p' \frown_P r'$ or $(p', r') \in |P_i| \times |R_i|$ for some i or $p' \in \bigcup_i |P_i|$, $r' \notin \bigcup_i |P_i| \cup \bigcup_i |R_i|$ and $p \frown_P r'$, or $p' \notin \bigcup_i |P_i| \cup \bigcup_i |R_i|$, $r' \in \bigcup_i |R_i|$ and $p' \frown_P r$, or $p' \frown_{P_i} r'$ or $p' \frown_{R_i} r'$ for some i. The residual function μ is given by $\mu(p') = p$ if $p' \in \bigcup_i |P_i|$, $\mu(p') = r$ if $p' \in \bigcup_i |R_i|$ and $\mu(p') = p'$ otherwise. With these notations, the process U' is $U' = S \oplus_{C'} P'$ where $C' \subseteq |S| \times |P'|$ is defined by $(s, p') \in C'$ if $(s, \mu(p')) \in C$ and the residual function λ is defined as $\lambda = \mathrm{Id}_{|S|} \cup \mu$. Since $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\mu]{\tau *} P'$, one has $Q \xrightarrow[\nu]{\tau *} Q'$ with $(P', E', Q') \in \mathcal{R}$ where $E' \subseteq |P'| \times |Q'|$ satisfies the condition on residuals $(p', q') \in E' \Rightarrow (\mu(p'), \nu(q')) \in E$. Let $C' \subseteq |S| \times |P'|$ be defined by $(s, p') \in C'$ if $(s, \mu(p')) \in C$ and we define similarly $D' \subseteq |S| \times |Q'|$: $(s, q') \in D'$ if $(s, \nu(q')) \in D$. Setting $V' = S \oplus_{D'} Q'$, we have $V \xrightarrow[\rho]{\tau *} V'$ where $\rho = \mathrm{Id}_{|S|} \cup \nu$. The triple (C', D', E') is adapted: let $(p', q') \in E'$ and let $s \in |S|$. If $(s, p') \in C'$, we have $(s, \mu(p')) \in C$. Since $(\mu(p'), \nu(q')) \in E$ (by definition of E'), we have $(s, \nu(q')) \in E$ because (C, D, E) is adapted. That is $(s, q') \in D'$. The converse implication is proved similarly. Let $F' = \mathrm{Id}_{|S|} \cup E' \subseteq |U'| \times |V'|$, we have therefore $(U', F', V') \in \mathcal{R}'$ (by definition of $\mathcal{R}'$). Last we check the condition on residuals. Let $(u', v') \in F'$, then either $u' = v' \in |S|$ and then $\lambda(u') = u' = v' = \rho(v')$ or $u' \in |P'|$, $v' \in |Q'|$ and $(u', v') \in E'$ and then $(\lambda(u'), \rho(v')) = (\mu(u'), \nu(v')) \in E$ by the condition on residuals satisfied by E.

Assume last that one of the involved guarded sums is located in S and that the other one is located in P, this is of course the most interesting case in this first part of the proof. By definition of internal reduction (see Section 2.1) we have $s \in |S|$ and $p \in |P|$ with $(s,p) \in C$ and with $S(s) = \overline{f} \cdot \boldsymbol{S} + \tilde{S}$ and $P(p) = f \cdot \boldsymbol{P} + \tilde{P}$ with the usual notational conventions, and $U' = S' \oplus_{C'} P'$ where $S' = S\,[\oplus\boldsymbol{S}/s]$, $P' = P\,[\oplus\boldsymbol{P}/p]$, and $C' \subseteq |S'| \times |P'|$ is defined as follows: $(s',p') \in C'$ if $(s',p') \in |S_i| \times |P_i|$ for some i, or $s' \notin \bigcup_i |S_i|$ and $p' \in \bigcup_i |P_i|$ and $(s',p) \in C$, or $s' \in \bigcup_i |S_i|$ and $p' \notin \bigcup_i |P_i|$ and $(s,p') \in C$, or $s' \notin \bigcup_i |S_i|$ and $p' \notin \bigcup_i |P_i|$ and $(s',p') \in C$. The residual map $\lambda : |U'| = |S'| \cup |P'| \to |U| = |S| \cup |P|$ is defined by $\lambda(u') = u'$ if $u' \in (|S'| \setminus \bigcup_i |S_i|) \cup (|P'| \setminus \bigcup_i |P_i|)$, $\lambda(s') = s$ if $s' \in \bigcup_i |S_i|$ and $\lambda(p') = p$ if $p' \in \bigcup_i |P_i|$.

We have $P \xrightarrow{p:f\cdot(\boldsymbol{L})} P'$ (where $L_i = |P_i|$ for each $i = 1,\dots,n$) and hence, since we have assumed that $(P,E,Q) \in \mathcal{R}$, we have $Q \underset{\rho,\rho'}{\overset{q:f\cdot(\boldsymbol{M})}{\Longrightarrow}} Q'$ with $(p,\rho(q)) \in E$ and $(P',E',Q') \in \mathcal{R}$ where E' is such that if $(p',q') \in E'$ then $(p',\rho'(q')) \in L_i \times M_i$ for some i, or $p' \notin \bigcup_i L_i$, $\rho'(q') \notin \bigcup M_i$ and $(p',\rho\rho'(q')) \in E$. So

$$Q \underset{\rho}{\xrightarrow{\tau*}} Q_1 \xrightarrow{q:f\cdot(\boldsymbol{M})} Q_1' \underset{\rho'}{\xrightarrow{\tau*}} Q'\,.$$

Then we have $V \underset{\mu}{\xrightarrow{\tau*}} V_1$ with $V_1 = S \oplus_{D_1} Q_1$ where $D_1 \subseteq |S| \times |Q_1|$ is defined by $(s,q_1) \in D_1$ if $(s,\rho(q_1)) \in D$, and $\mu = \mathrm{Id}_{|S|} \cup \rho$. We have $q \in |Q_1|$ with $Q_1(q) = f \cdot \boldsymbol{R} + \tilde{R}$ and $|R_i| = M_i$ for $i = 1,\dots,n$. Moreover, $(p,\rho(q)) \in E$ and $(s,p) \in C$, and since (C,D,E) is adapted, we have $(s,\rho(q)) \in D$, that is $(s,q) \in D_1$. Therefore, since $S(s) = \overline{f} \cdot \boldsymbol{S} + \tilde{S}$, we have $V_1 \underset{\theta}{\xrightarrow{\tau}} V_1' = S' \oplus_{D_1'} Q_1'$ where $D_1' \subseteq |S'| \times |Q_1'|$ is defined as follows. Given $(s',q_1') \in |S'| \times |Q_1'|$, we have $(s',q_1') \in D_1'$ if $s' \in |S_i|$ and $q_1' \in |R_i|$ for some $i = 1,\dots,n$ or $s' \notin \bigcup_i |S_i|$, $q_1' \in \bigcup_i |R_i|$ and $(s',q) \in D_1$ (that is $(s',\rho(q)) \in D$), or $s' \in \bigcup_i |S_i|$, $q_1' \notin \bigcup_i |R_i|$ and $(s,q_1') \in D_1$ (that is $(s,\rho(q_1')) \in D$) or $s' \notin \bigcup_i |S_i|$, $q_1' \notin \bigcup_i |R_i|$ and $(s',q_1') \in D_1$ (that is $(s',\rho(q_1')) \in D$). The function θ is defined by $\theta(v_1') = v_1'$ if $v_1' \in (|S| \setminus \bigcup_i |S_i|) \cup (|Q_1| \setminus \bigcup_i |R_i|)$, $\theta(s') = s$ if $s' \in \bigcup_i |S_i|$ and $\theta(q_1') = q$ if $q_1' \in \bigcup_i |R_i|$. Last, since $Q_1' \underset{\rho'}{\xrightarrow{\tau*}} Q'$, we have $V_1' = S' \oplus_{D_1'} Q_1' \underset{\mu'}{\xrightarrow{\tau*}} V' = S' \oplus_{D'} Q'$ where $\mu' = \mathrm{Id}_{|S|} \cup \rho'$ and D' is the set of all $(s',q') \in |S'| \times |Q'|$ such that $(s',\rho'(q')) \in D_1'$. So we have $V \underset{\mu\theta\mu'}{\xrightarrow{\tau*}} V'$. Let $F' \subseteq |U'| \times |V'|$ be defined as the set of all (u',v') such that $u' = v' \in |S'|$ or $(u',v') \in |P'| \times |Q'|$ and $(u',v') \in E'$. It is clear then that $(u',v') \in F' \Rightarrow (\lambda(u'),\mu\theta\mu'(v')) \in F$.

To finish, we must prove that $(U',F',V') \in \mathcal{R}'$ and to this end it suffices to show that the triple of relations (C',D',E') is adapted. Let $s' \in |S'|$, $p' \in |P'|$ and $q' \in |Q'|$ with $(p',q') \in E'$.

Assume first that $(s',p') \in C'$ and let us show that $(s',q') \in D'$, that is $(s',\rho'(q')) \in D_1'$. Coming back to the definition of C', we see that there are four cases to consider. Assume first that $(s',p') \in |S_i| \times |P_i|$ for some i. Since $p' \in |P_i| = L_i$, we must have $\rho'(q') \in M_i = |Q_i|$ because $(p',q') \in E'$. But then $(s',\rho'(q')) \in D_1'$ as required. Assume now that $s' \notin \bigcup_i |S_i|$ and $p' \in \bigcup_i |P_i|$, and that we have $(s',p) \in C$. Let i be such that $p' \in |P_i| = L_i$; by definition of E' we know that $\rho'(q') \in M_i$. Since $(p,\rho(q)) \in E$ and (C,D,E) is adapted, we have $(s',\rho(q)) \in D$. Combining these two facts and coming back to the definition of D_1', we see that $(s',\rho'(q')) \in D_1'$ as required. Next assume that $s' \in \bigcup_i |S_i|$ and

$p' \notin \bigcup_i |P_i|$ so that we have $(s, p') \in C$ (by definition of C'). By definition of E' again, we see that $\rho'(q') \notin \bigcup_i M_i$ and that $(p', \rho\rho'(q')) \in E$ because $(p', q') \in E'$. Therefore we have $(s, \rho\rho'(q')) \in D$ and, coming back to the definition of D'_1, this means that $(s', \rho'(q'_1))$ as required. Assume last that $s' \notin \bigcup_i |S_i|$ and that $p' \notin \bigcup_i |P_i|$. By definition of C', we have $(s', p') \in C$. Since $(p', q') \in E'$ and $p' \notin \bigcup_i |P_i|$, we know that $\rho'(q') \notin \bigcup_i M_i$ and that $(p', \rho\rho'(q')) \in E$. Since (C, D, E) is adapted we get $(s', \rho\rho'(q')) \in D$ and hence $(s', \rho'(q')) \in D'_1$ by definition of D'_1 and this ends the proof of this implication.

Let us prove now the converse implication, assuming that $(s', \rho'(q')) \in D'_1$; we contend that $(s', p') \in C'$. Again there are four cases to deal with: the four possibilities in the definition of D'_1. Assume first that $s' \in |S_i|$ and that $\rho'(q') \in M_i$ for some i. Since $(p', q') \in E'$, we must have $p' \in L_i = |P_i|$ and hence $(s', p') \in C'$ as required. Assume now that $s' \in \bigcup_i |S_i|$ and $\rho'(q') \notin \bigcup_i M_i$, and in that case we know that $(s, \rho\rho'(q')) \in D$. We know that $(p', q') \in E'$ and hence, since $\rho'(q') \notin \bigcup_i M_i$ we must have $p' \notin \bigcup_i L_i$ and $(p', \rho\rho'(q')) \in E$. Since (C, D, E) is adapted, we have $(s, p') \in C$ and hence (s', p') by definition of C' (since we have seen that $p' \notin \bigcup_i L_i$). Assume next that $s' \notin \bigcup_i |S_i|$ and that $\rho'(q') \in \bigcup_i M_i$; in that case we know that $(s', \rho(q)) \in D$. Let i be such that $\rho'(q') \in M_i$. Since $(p', q') \in E'$, we know that $p' \in L_i$. Since $(p, \rho(q)) \in E$, we have $(s', p) \in C$ because (C, D, E) is adapted. Therefore $(s', p') \in C'$ because $p' \in \bigcup_i L_i$. Assume last that $s' \notin \bigcup_i |S_i|$ and $\rho'(q') \notin \bigcup_i M_i$. Then we know that $(s', \rho\rho'(q')) \in D$, still by definition of D'_1. Moreover, since $(p', q') \in E'$, we must have $p' \notin \bigcup_i L_i$ and $(p', \rho\rho'(q')) \in E$. It follows that $(s', p') \in C$ because (C, D, E) is adapted, and therefore we have $(s', p') \in C'$, by definition of C' and because $p' \notin \bigcup_i L_i$.

This ends the first part of the proof.

Case of a labeled transition. We assume now that $U \overset{r:f\cdot(\boldsymbol{L})}{\longrightarrow} U'$. Since $U = S \oplus_C P$, we consider two cases as to the location of r.

If $r \in |S|$ then we have $S(r) = f \cdot \boldsymbol{S} + \tilde{S}$ and $S \overset{r:f\cdot(\boldsymbol{L})}{\longrightarrow} S'$ where $S' = S\,[\oplus \boldsymbol{S}/r]$ (so that $L_i = |S_i|$ for each i), and $U' = S' \oplus_{C'} P$ where $C' \subseteq |S'| \times |P|$ is defined as follows: we have $(s', p) \in C'$ if $(s', p) \in C$ and $s' \notin \bigcup_i |S_i|$, or $s' \in \bigcup_i |S_i|$ and $(s, p) \in C$. Let $D' \subseteq |S'| \times |Q|$ be defined similarly: $(s', q) \in D'$ if $(s', q) \in D$ and $s' \notin \bigcup_i |S_i|$, or $s' \in \bigcup_i |S_i|$ and $(s, q) \in D$. We check that (C', D', E) is adapted: let $(p, q) \in E$, $s' \in |S'|$ and assume that $(s', p) \in C'$. If $s' \notin \bigcup_i |S_i|$ then we know that $(s', p) \in C$ and hence $(s', q) \in D$ since (C, D, E) is adapted. The converse implication is proved in the same way. Let $V' = S' \oplus_{D'} Q$, we have therefore $(U', F', V') \in \mathcal{R}'$ where $F' = \mathrm{Id}_{|S'|} \cup E$. We have $(r, r) \in F$, $V \overset{r:f\cdot(\boldsymbol{L})}{\longrightarrow} V'$ and, given $(u', v') \in F'$, we have either $(u', v') \in \bigcup_i (L_i \times L_i)$ (and actually $u' = v'$) or $u' \notin \bigcup_i L_i$, $v' \notin \bigcup_i L_i$ and $(u', v') \in F$ as easily checked. Therefore the condition on residuals is satisfied.

The last case to consider is when $r = p \in |P|$ and then we have $P(p) = f \cdot \boldsymbol{P} + \tilde{P}$ and $P \overset{p:f\cdot(\boldsymbol{L})}{\longrightarrow} P'$. Then we have $U' = S \oplus_{C'} P'$ where $C' \subseteq |S| \times |P'|$ is defined as follows: $(s, p') \in C'$ if $p' \in \bigcup_i |P_i|$ and $(s, p) \in C$, or $p' \notin \bigcup_i |P_i|$ and $(s, p') \in C$. Since $(P, E, Q) \in \mathcal{R}$ we have $Q \underset{\rho,\rho'}{\overset{q:f\cdot(\boldsymbol{M})}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$ and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$ and, for any $(p', q') \in E'$, one has $(p', \rho'(q')) \in L_i \times M_i$ for some i, or else $p' \notin \bigcup_i L_i$, $\rho'(q') \notin \bigcup_i M_i$ and $(p', \rho\rho'(q')) \in E$. Then we have $V \overset{q:f\cdot(M)}{\underset{\mu,\mu'}{\Longrightarrow}} V'$ where $V' = S \oplus_{D'} Q'$ with $D' \subseteq |S| \times |Q'|$ is defined by: $(s, q') \in D'$ if $\rho'(q') \in \bigcup_i M_i$ and $(s, \rho(q)) \in D$ or $q' \notin \bigcup_i M_i$ and $(s, \rho\rho'(q')) \in D$. Moreover $\mu = \mathrm{Id}_{|S|} \cup \rho$ and $\mu' = \mathrm{Id}_{|S|} \cup \rho'$. Let $F' \subseteq |U'| \times |V'|$ be defined by $F' = \mathrm{Id}_{|S|} \cup E'$. Let $(u', v') \in F'$, if $u' \in |S|$ or $v' \in |S|$, we must have $u' = v'$. If $u' \notin |S|$ and $v' \notin |S|$ then we have $(u', v') \in E'$ and hence either there exists i such that $u' \in L_i$ and $\mu'(v') = \rho'(v') \in M_i$, or $u' \notin \bigcup_i L_i$, $\rho'(v') \notin \bigcup_i M_i$ and $(u', \rho\rho'(v')) \in E$, that is $(u', \mu\mu'(v')) \in F$. Moreover, the triple (C', D', E') is adapted: let $(p', q') \in E'$ and $s \in |S|$ and assume first that $(s, p') \in C'$. If $(p', \rho'(q')) \in L_i \times M_i$ for some i, then we have $(s, p) \in C$ by definition of C' and hence $(s, \rho(q)) \in D$ since (C, D, E) is adapted and $(p, \rho(q)) \in E$, therefore $(s, \rho'(q')) \in D'$ (by definition of D'). Otherwise we have $p' \notin \bigcup_i L_i$ and $\rho'(q') \notin \bigcup_i M_i$ and $(p', \rho\rho'(q')) \in E$. Since $(s, p') \in C'$ and $p' \notin \bigcup_i L_i$ we have $(s, p') \in C$ and hence $(s, \rho\rho'(q')) \in D$ because (C, D, E) is adapted. This means that $(s', q') \in D'$, by definition of D'. Assume now that $(s', q') \in D'$ and let us prove that $(s', p') \in C'$. If $\rho'(q') \notin \bigcup_i M_i$ then we have $(s, \rho\rho'(q')) \in D$ (by definition of D'). Since $(p', q') \in E'$ and $\rho'(q') \notin \bigcup_i M_i$, we know that $p' \notin \bigcup |P_i|$ and that $(p', \rho\rho'(q')) \in E$ and therefore, since (C, D, E) is adapted, we get $(s, p') \in C$, and hence $(s, p') \in C'$ by definition of C' (in the case where $p' \notin \bigcup |P_i|$). If $\rho'(q') \in M_i$, we have $p' \in L_i$ and $(p, \rho(q)) \in E$ (again because $(p', q') \in E'$). Moreover, by definition of D' we have, in that case, $(s, \rho(q)) \in D$ and therefore $(s, p) \in C$ because (C, D, E) is adapted. We conclude as required that $(s, p') \in C'$ by definition of C' and because $p' \in \bigcup_i L_i$.

Proof of Theorem 2

Let $\mathcal{R}$ be a weak bisimulation. Let R be a Y-context. We define a new localized relation $R[\mathcal{R}/Y]$ as follows: if $R = Y$ then $R[\mathcal{R}/Y] = \mathcal{R}$; if $R \neq Y$ then we stipulate that $(P', E', Q') \in R[\mathcal{R}/Y]$ if there exists $(P, E, Q) \in \mathcal{R}$ and $E' = \mathrm{Id}_{|R|}$, $P' = R[P/Y]$ and $Q' = R[Q/Y]$ (observe that $|P'| = |Q'| = |R|$ because $R \neq Y$). We define a localized relation $\mathcal{R}^+$ as the union of $\mathcal{I}$ (the set of all triples (U, E, U) where U is any process and $E = \mathrm{Id}_{|U|}$), of the parallel extension $\mathcal{R}'$ (see Proposition 4) of $\mathcal{R}$ and of all the relations of the shape $R[\mathcal{R}/Y]$ for all Y-contexts R. We contend that $\mathcal{R}^+$ is a weak bisimulation.

Let $(U, F, V) \in \mathcal{R}^+$ and assume that we are in one of the two following situations $U \xrightarrow[\mu]{\tau} U'$ (called case (**1**) in the sequel) or $U \xrightarrow{p:f\cdot(L)} U'$ (called (**2**) in the sequel). We must show that $V \xrightarrow[\nu]{\tau*} V'$ with $(U', F', V') \in \mathcal{R}^+$ for some $F' \subseteq |U'| \times |V'|$ such that for any $(u', v') \in F'$, one has $(\mu(u'), \nu(v')) \in F$ in case (**1**) and that $V \overset{q:f\cdot(M)}{\underset{\nu,\nu'}{\Longrightarrow}} V'$ with $(U', F', V') \in \mathcal{R}^+$ in case (**2**), for some $F' \subseteq |U'| \times |V'|$ such that, for any $(u', v') \in F'$, one has $(u', \nu'(v')) \in L_i \times M_i$ for some i, or $u' \notin \bigcup_i L_i$, $\rho'(v') \notin \bigcup_i M_i$ and $(u', \rho\rho'(v')) \in F$.

The case where $(U, F, V) \in \mathcal{I}$ is trivial.

If $(U, F, V) \in \mathcal{R}'$ we apply directly Proposition 4 in both cases.

Assume now that $(U, F, V) \in R[\mathcal{R}/Y]$ for some Y-context R, so that $U = R[P/Y]$, $V = R[Q/Y]$ with $(P, E, Q) \in \mathcal{R}$ and $F = E$ if $R = Y$ and $F = \mathrm{Id}_{|R|}$ otherwise. If $R = Y$ we use directly the fact that $\mathcal{R}$ is a weak bisimulation to exhibit V' and F' satisfying the required conditions. So we assume from now on that $R \neq Y$ so that $F = \mathrm{Id}_{|R|}$. By definition of a Y-context, there is exactly one $r \in |R|$ such that Y occurs free in $R(r)$. Then $R(r)$ can be written uniquely as $R(r) = g \cdot \boldsymbol{R} + \tilde{R}$ where Y does not occur in $\tilde{R}$ and occurs in exactly one of the processes $\boldsymbol{R} = (R_1, \dots, R_n)$; without loss of generality we can assume that R_1 is a Y-context and that Y does not occur free in $R_2, \dots, R_n$.

Assume first that $R_1 \neq Y$. In both cases (**1**) and (**2**), we have $U' = R'[P/Y]$ with $R \xrightarrow[\mu]{\tau} R'$ (case (**1**)) or $R \overset{p:\underline{f}\cdot(\boldsymbol{L})}{\longrightarrow} R'$ (case (**2**)). Let $V' = R'[Q/Y]$. In case (**1**), we have $V \xrightarrow[\mu]{\tau} V'$ and in case (**2**) we have $V \overset{q:\underline{f}\cdot(\boldsymbol{L})}{\longrightarrow} V'$, and since $R' \neq Y$ (by our hypothesis on R_1), we have $(U', \mathrm{Id}_{|R'|}, V') \in \mathcal{R}^+$ because $(P, E, Q) \in \mathcal{R}$. The condition on residuals is obviously satisfied in both cases.

Assume now that $R_1 = Y$. Suppose first that we are in case (**1**). There are two cases to consider as to the locations $s, t \in |U|$ of the sub-processes involved in the transition $U \xrightarrow[\mu]{\tau} U'$. The case where $s \neq r$ and $t \neq r$ is similar to the case above where $R_1 \neq Y$. By symmetry we are left with the case where $s = r$ (and hence $t \neq r$). So $U(t) = R(t) = \overline{f} \cdot \boldsymbol{T} + \tilde{T}$ and the guarded sum $R(r)$ has an unique summand which is involved in the transition $U \xrightarrow[\mu]{\tau} U'$ (called *active summand* in the sequel), and this summand is of the shape $f \cdot \boldsymbol{S}$. If the active summand is $g \cdot \boldsymbol{R}$ then $U(r) = f \cdot (P, R_2, \dots, R_n) + \tilde{S}$ and U' can be written $U' = R' \oplus_C P$ for some process R' which can be defined using only R, and $C \subseteq |R'| \times |P|$. Explicitly, R' is defined as follows: $|R'| = (|R| \setminus \{r, t\}) \cup \bigcup_{i=2}^{n} |R_i| \cup \bigcup_{i=1}^{n} |T_i|$ and $\frown_{R'}$ is the least symmetric relation on $|R'|$ such that $r' \frown_{R'} t'$ if $r' \frown_{R_i} t'$ for some $i = 2, \dots, n$ or $r' \frown_{T_i} t'$ for some $i = 1, \dots, n$, or $r' \in |R_i|$ and $t' \in |T_i|$ for some $i \in \{2, \dots, n\}$, or $r' \notin \bigcup_{i=2}^{n} |R_i|$, $t' \in \bigcup_{i=1}^{n} |T_i|$ and $r' \frown_R t$, or $r' \in \bigcup_{i=2}^{n} |R_i|$, $t' \notin \bigcup_{i=1}^{n} |T_i|$ and $r \frown_R t'$, or $r', t' \notin \bigcup_{i=2}^{n} |R_i| \cup \bigcup_{i=1}^{n} |T_i|$ and $r' \frown_R t'$. The relation C is defined as follows: given $(r', p) \in |R'| \times |P|$, one has $(r', p) \in C$ if $r' \in |T_1|$, or $r' \notin \bigcup_{i=2}^{n} |R_i| \cup \bigcup_{i=1}^{n} |T_i|$ and $r' \frown_R r$. The residual function $\mu : |U'| \to |U|$ is given by $\mu(r') = r$ if $r' \in |P| \cup \bigcup_{i=2} \in |R_i|$, $\mu(r') = t$ if $r' \in \bigcup_{i=1}^{n} |T_i|$ and $\mu(r') = r'$ when r' belongs to none of these two sets. Let $V' = R' \oplus_D Q$, where $D \subseteq |R'| \times |Q|$ is defined exactly like C (just replace P by Q in the definition). Then (C, D, E) is adapted (because the property for $(r', p) \in |R'| \times |P|$ of belonging or not to C depends only on r', and does not depend on p, and similarly for D). We can mimic that reduction on V, so that $V \xrightarrow[\nu]{\tau} V'$ for the residual function ν which is defined like μ (replacing Q by P). We have $(U', F', V') \in \mathcal{R}' \subseteq \mathcal{R}^+$ where $F' = \mathrm{Id}_{|R'|} \cup E$. Given $(u', v') \in F'$, we have $\mu(u') = \nu(v')$, that is $(\mu(u'), \nu(v')) \in F$ so that the condition on residuals holds. Assume now that the active summand is not $g \cdot \boldsymbol{R}$. In that case we also have $V \xrightarrow[\mu]{\tau} U'$ (both P and Q vanish in the corresponding reductions), and we are done because $(U', \mathrm{Id}_{|U'|}, U') \in \mathcal{I} \subseteq \mathcal{R}^+$.

We suppose now that we are in case (**2**). Assume first that $p \neq r$. In that case we have $R \xrightarrow{p:f\cdot(L)} R'$ and $U' = R'[P/Y]$ and we also have $V \xrightarrow{p:f\cdot(L)} V' = R'[Q/Y]$ so $(U', \mathrm{Id}_{|R'|}, V') \in R'[\mathcal{R}/Y] \subseteq \mathcal{R}^+$, and the condition on residuals is obvious. Assume now that $p = r$. Then exactly one of the summands of the guarded sum $R(r)$ is the prefixed process performing the action f in the considered transition on U (the active summand). The case where the active summand is not $g \cdot (P, R_2, \ldots, R_n)$ is completely similar to the previous one (P vanishes in the transition). Assume that the active summand is $g \cdot (P, R_2, \ldots, R_n)$, then $U' = R' \oplus_C P$ where R' is defined by $|R'| = (|R| \setminus \{r\}) \cup \bigcup_{i=2}^n |R_i|$ and $\frown_{R'}$ is the least symmetric relation on $|R'|$ such that $r' \frown_{R'} t'$ if $r' \frown_{R_i} t'$ for some $i = 2, \ldots, n$ or $r' \in \bigcup_{i=2} |R_i|$, $t' \notin \bigcup_{i=2}^n |R_i|$ and $r \frown_R t'$ or $r', t' \notin \bigcup_{i=2}^n |R_i|$ and $r' \frown_R t'$. The relation $C \subseteq |R'| \times |P|$ is defined by $(r', p) \in C$ if $r' \notin \bigcup_{i=2}^n |R_i|$ and $r' \frown_R r$. Then we have $V \xrightarrow{p:f\cdot(M)} V'$ (with $M_1 = |Q|$ and $M_i = L_i = |R_i|$ for $i = 2, \ldots, n$) with $V' = R' \oplus_D Q$ where D is defined like C (replacing P by Q in the definition). Then we have $(U', F', V') \in \mathcal{R}' \subseteq \mathcal{R}^+$ where $F' = \mathrm{Id}_{|R'|} \cup E$ since (C, D, E) is obviously adapted (as above). Moreover the condition on residuals is obviously satisfied. This ends the proof of the fact that $\mathcal{R}^+$ is a weak bisimulation.

We can now prove that $\approx$ is a congruence. Assume that $P \approx Q$ and let R be a Y-context. Let $E \subseteq |P| \times |Q|$ and let $\mathcal{R}$ be a weak bisimulation such that $(P, E, Q) \in \mathcal{R}$. Then we have $(R[P/Y], \mathrm{Id}_{|R|}, R[Q/Y]) \in R[\mathcal{R}/Y] \subseteq \mathcal{R}^+$ and hence $R[P/Y] \approx R[Q/Y]$ since $\mathcal{R}^+$ is a weak bisimulation. $\square$