

A proof of strong normalization by reducibility modulo for $\lambda_{\sigma w}$

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This note presents a proof of strong normalization for $\lambda_{\sigma w}$ with names using the reducibility modulo technique [Rit94].

1 A proof of strong normalization of $\lambda_{\sigma w}$

We admit all along this proof the following properties:

- *Confluence and termination* of the substitution part of $\lambda_{\sigma w}$ written \longrightarrow_{es} [ACCL90].
- *Subject reduction* of $\lambda_{\sigma w}$ [ACCL90]

The scheme of the proof is the following one:

1. We (re-)define $\lambda_{\sigma w}$ and its typing system and remark that this version of $\lambda_{\sigma w}$ has the subject reduction property.
2. We define a new calculus ($\lambda_{\sigma w/\equiv}$) by replacing a strongly normalizing subset of the reduction rules of $\lambda_{\sigma w}$ by the corresponding equations on $\lambda_{\sigma w}$ -terms.
3. We show the strong normalization for $\lambda_{\sigma w/\equiv}$ well typed terms.
4. By a technical Lemma we deduce the strong normalization of well typed $\lambda_{\sigma w}$ terms.

1.1 The calculus $\lambda_{\sigma w}$

1.1.1 Syntax

Types

$$\begin{array}{l} A ::= \iota \quad \text{Base types} \\ \quad | A \rightarrow A \quad \text{Functional types} \end{array}$$

Substitutions

$$\begin{array}{lcl}
s ::= id & \text{Identity} \\
| (x/M).s & \text{Cons} \\
| s \circ t & \text{Concatenation}
\end{array}$$

Terms

$$\begin{array}{lcl}
M ::= x & \text{Variable} \\
| (M M) & \text{Application} \\
| \lambda x:A.M & \text{Abstraction} \\
| M[s] & \text{Substitution}
\end{array}$$

We may omit the parenthesis in an application term if they are clear from the context.

Expressions ($\lambda_{\sigma w}$ -terms)

$$\begin{array}{lcl}
e ::= M & \text{Terms} \\
| s & \text{Substitutions}
\end{array}$$

1.1.2 Reduction rules

$$\begin{array}{lclcl}
(\lambda x.M)[s] N & \longrightarrow & M[(x/N).s] & & (Abs_var_1) \\
\lambda x.M N & \longrightarrow & (\lambda x.M)[id] N & & (Abs_var_2) \\
(M N)[s] & \longrightarrow & M[s]N[s] & & (Sub_app) \\
(s \circ t) \circ u & \longrightarrow & s \circ (t \circ u) & & (Sub_ass_env) \\
x[id] & \longrightarrow & x & & (Sub_var_1) \\
x[(x/N).s] & \longrightarrow & N & & (Sub_var_2) \\
y[(x/N).s] & \longrightarrow & y[s] \text{ if } y \neq x & & (subvarthree) \\
M[s][t] & \longrightarrow & M[s \circ t] & & (Sub_clos) \\
((x/M).s) \circ t & \longrightarrow & (x/M[t]).(s \circ t) & & (Sub_concat) \\
id \circ s & \longrightarrow & s & & (Sub_id)
\end{array}$$

Notation 1.1 For the rest of this section we will note this reduction relation $\longrightarrow_{\lambda_{\sigma w}}$ when a simple \longrightarrow may be ambiguous.

1.1.3 Typing rules

Definition 1.1 (Environment) An *environment* is a set of pairs of the form $x : A$ where x is a variable and A is a type.

$$\begin{array}{c}
\frac{\text{the } x_i \text{ are distinct variables}}{x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i} \quad (\text{Proj}) \\
\\
\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (M N) : B} \quad (\text{App}) \\
\\
\frac{x : A, \Gamma \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \rightarrow B} \quad (\rightarrow \text{Right})
\end{array}$$

$$\frac{\Gamma \vdash s \triangleright \Gamma' \quad \Gamma' \vdash M : A}{\Gamma \vdash M[s] : A} \quad (\text{Sub})$$

$$\frac{}{\Gamma \vdash id \triangleright \Gamma} \quad (\text{Sub_axiom})$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash s \triangleright \Gamma'}{\Gamma \vdash (x/M).s \triangleright x : A, \Gamma'} \quad (\text{Sub_cons})$$

$$\frac{\Gamma \vdash t \triangleright \Gamma' \quad \Gamma' \vdash s \triangleright \Gamma''}{\Gamma \vdash s \circ t \triangleright \Gamma''} \quad (\text{Sub_concat})$$

A term M (resp. a substitution s) is said to be *well-typed* if there exist a type A (resp. an environment Δ) and an environment Γ such that $\Gamma \vdash M : A$ (resp. $\Gamma \vdash s \triangleright \Delta$) can be derived from the previous set of typing rules. A term M is said to be *of type* A if there is an environment Γ such that $\Gamma \vdash M : A$ can be derived from the previous set of typing rules.

Lemma 1.2

1. If $\Gamma \vdash M : A$, then $\Gamma, x : B \vdash M : A$.
2. If $\Gamma \vdash s \triangleright \Delta$ and $\Gamma \vdash s \triangleright \Delta'$, then $\Delta = \Delta'$.

Definition 1.2 (Void substitutions) The set of *void substitutions* is defined to be the least set of substitutions stable by concatenation and containing *id*

Lemma 1.3 If $\Delta \vdash s \triangleright \emptyset$, then $\Delta = \emptyset$ and s is a void substitution.

Proof. By induction on the height of the typing derivation of $\Delta \vdash s \triangleright \emptyset$. ■

Definition 1.3 For any strongly normalizing expression e we note $\nu(e)$ the length of the longest reduction sequence starting at e .

1.2 Definition of $\lambda_{\sigma w/\equiv}$

Definition 1.4 The congruence \equiv on $\lambda_{\sigma w}$ -terms is defined to be the least reflexive, symmetric and transitive relation closed under contexts and substitutions and containing the axioms

$$(M \ N)[s] \quad =_{\text{Sub_app}} \quad M[s] \ N[s] \quad (1)$$

$$(s \circ t) \circ u \quad =_{\text{Sub_ass_env}} \quad s \circ (t \circ u) \quad (2)$$

$$M[s][t] \quad =_{\text{Sub_clos}} \quad M[s \circ t] \quad (3)$$

We will consider in this section the reduction system $\lambda_{\sigma w/\equiv}$, where $a \longrightarrow \lambda_{\sigma w/\equiv} b$ if and only if there exist a', b' such that

$$a \equiv a' \longrightarrow_{\mathcal{R}} b' \equiv b, \text{ where } \mathcal{R} = \lambda_{\sigma w} \setminus \{\text{Sub_app}, \text{Sub_ass_env}, \text{Sub_clos}\}$$

This definition can also be interpreted as a reduction relation on **equivalence classes**, i.e., $[a] \longrightarrow_{\lambda_{\sigma w/\equiv}} [b]$ if and only if $a \equiv a' \longrightarrow_{\mathcal{R}} b' \equiv b$. We may use both interpretations according to the context.

Remark 1.4 By subject reduction property, there is no problem to define well-typed $\lambda_{\sigma w/\equiv}$ -terms.

1.3 Strong normalization for $\lambda_{\sigma w/\equiv}$

For the rest of this section we will denote by \longrightarrow the $\lambda_{\sigma w/\equiv}$ reduction relation.

Lemma 1.5 If s is a void substitution, then s is \longrightarrow -strongly normalizing.

Proof.

By equation (2), there is only two possibilities:

- $s = id$, then s is in \longrightarrow -normal form so it is \longrightarrow -strongly normalizing.
- $s = id \circ s'$ where s' is a void substitution. Then it is easy to show that s is strongly normalizing by induction on the number of concat in s .

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Definition 1.5 (Neutral terms and substitutions)

- A term M is *neutral* if and only if it is neither of the form $(\lambda x.N)[s]$ nor $\lambda x.N$.
- A substitution s is *neutral* if and only if it is not of the form $(x/M).t$.

Definition 1.6 (Reducible terms and substitutions) The set of *reducible terms for a type in an environment* Γ is defined by induction on types as follows:

$$\llbracket \iota \rrbracket_{\Gamma} =_{def} \{M \mid \Gamma \vdash M : \iota \text{ and } M \text{ is strongly normalizing}\}.$$

$$\llbracket A \rightarrow B \rrbracket_{\Gamma} =_{def} \{M \mid \Gamma \vdash M : A \rightarrow B \text{ and } \forall \Delta \text{ such that } \forall N \in \llbracket A \rrbracket_{\Gamma \Delta}, (M N) \in \llbracket B \rrbracket_{\Gamma \Delta}\}$$

The set of *reducible substitutions for an environment* Γ in an environment Δ is defined as follows:

$$\llbracket \Gamma \rrbracket_{\Delta} =_{def} \{s \mid \Delta \vdash s \triangleright \Gamma \text{ and } \forall (x:A) \in \Gamma, x[s] \in \llbracket B \rrbracket_{\Delta}\}$$

Remark 1.6 We remark that, by definition, any substitution s such that $\Delta \vdash s \triangleright \emptyset$ is in $\llbracket \emptyset \rrbracket_{\Delta}$.

Corollary 1.7 Void substitutions are in $\llbracket \emptyset \rrbracket_{\emptyset}$.

Notation 1.8

- For any term M we say that M is reducible if there is a type A and an environment Γ such that $M \in \llbracket A \rrbracket_{\Gamma}$.
- For any substitution s we say that s is reducible if there are two environments Γ and Δ such that $s \in \llbracket \Gamma \rrbracket_{\Delta}$.

Remark 1.9 In the rest of this section we may omit environments if they are clear from the context or if they are not necessary in the statements.

Lemma 1.10

1. If $M \in \llbracket C \rrbracket_{\Gamma}$, then M is strongly normalizing.
2. If $\Gamma \vdash (xM_1 \dots M_n) : C$ and $M_1 \dots M_n$ are strongly normalizing, then $(xM_1 \dots M_n) \in \llbracket C \rrbracket_{\Gamma}$.
3. If $M \in \llbracket C \rrbracket_{\Gamma}$ and $M \longrightarrow M'$ then $M' \in \llbracket C \rrbracket_{\Gamma}$.
4. If M is a neutral of type C and all its one-step reducts are reducible expressions, then M is reducible.

Proof. We first show the four properties for terms by induction on the type C .

Base case ι :

1. By definition of $\llbracket \iota \rrbracket_{\Gamma}$.
2. Since $M_1 \dots M_n$ are strongly normalizing, then the only reduction sequences starting at $(xM_1 \dots M_n)$ proceed independently in the terms $M'_i s$, and all these reduction sequences terminate. As a consequence, the term $(xM_1 \dots M_n)$ is strongly normalizing and thus by definition $(xM_1 \dots M_n) \in \llbracket \iota \rrbracket_{\Gamma}$.
3. Let M be in $\llbracket \iota \rrbracket_{\Gamma}$. By definition $\Gamma \vdash M : \iota$ and M is strongly normalizing. Then, any M' such that $M \longrightarrow M'$ is also strongly normalizing. By subject reduction property, $\Gamma \vdash M' : \iota$ and thus by definition M' is in $\llbracket \iota \rrbracket_{\Gamma}$.
4. Let M be a well-typed (*i.e.* $\Gamma \vdash M : \iota$) and neutral term such that all its one-step reducts are in $\llbracket \iota \rrbracket_{\Gamma}$. By definition all these one-steps reducts are strongly normalizing so that M is strongly normalizing and thus by definition M is in $\llbracket \iota \rrbracket_{\Gamma}$.

Inductive case

1. Let M be in $\llbracket A \rightarrow B \rrbracket_{\Gamma}$. By induction hypothesis (property 2 with $n = 0$) there is a fresh variable x of type A in $\llbracket A \rrbracket_{\Gamma, xA}$ so that by definition of $\llbracket A \rightarrow B \rrbracket_{\Gamma}$, the term $(M x)$ is in $\llbracket B \rrbracket_{\Gamma, xA}$. By induction hypothesis (property 1) on B , $(M x)$ is strongly normalizing and then so is M .
2. Let $(x M_1 \dots M_n)$ be a term such that $\Gamma \vdash (x M_1 \dots M_n) : A \rightarrow B$ and M_1, \dots, M_n are strongly normalizing. Let N be *any* term in $\llbracket A \rrbracket_{\Gamma\Delta}$. By induction hypothesis (property 1) we know that N is strongly normalizing, and by subject reduction property, $\Gamma\Delta \vdash (x M_1 \dots M_n N) : B$. As a consequence, by induction hypothesis (property 2) we have that $(x M_1 \dots M_n N)$ is in $\llbracket B \rrbracket_{\Gamma\Delta}$, and thus by definition $(x M_1 \dots M_n)$ is in $\llbracket A \rightarrow B \rrbracket_{\Gamma}$.
3. Let M be in $\llbracket A \rightarrow B \rrbracket_{\Gamma}$ and let consider the reduction step $M \longrightarrow M'$. By subject reduction property, $\Gamma \vdash M' : A \rightarrow B$. Take any term $N \in \llbracket A \rrbracket_{\Gamma\Delta}$. By definition of $\llbracket A \rightarrow B \rrbracket_{\Gamma}$, $(MN) \in \llbracket B \rrbracket_{\Gamma\Delta}$. Since $(MN) \longrightarrow (M'N)$, then by induction hypothesis (property 3) on B , we have that $(M'N) \in \llbracket B \rrbracket_{\Gamma\Delta}$. And then $M' \in \llbracket A \rightarrow B \rrbracket_{\Gamma}$.
4. Let M be a neutral term such that $\Gamma \vdash M : A \rightarrow B$ and all its one-step reducts are in $\llbracket A \rightarrow B \rrbracket_{\Gamma}$. Let N be in $\llbracket A \rrbracket_{\Gamma\Delta}$. We have to show that $(MN) \in \llbracket B \rrbracket_{\Gamma\Delta}$. Since (MN) is neutral and $\Gamma\Delta \vdash (M N) : B$, then by induction hypothesis (property 4) it is sufficient to show that all its one-step reducts are in $\llbracket B \rrbracket_{\Gamma\Delta}$. By induction hypothesis (property 1) we know that N is strongly normalizing, so we can reason by induction on $\nu(N)$ as follows:

The one-steps reducts of $(M N)$ are:

- $(M N')$, with $N \longrightarrow N'$. Then $\nu(N') < \nu(N)$ and the property holds by induction hypothesis.
- $(M' N)$, with $M \longrightarrow M'$. Since M' is in $\llbracket A \rightarrow B \rrbracket_{\Gamma}$ by hypothesis, then $(M' N)$ is in $\llbracket B \rrbracket_{\Gamma\Delta}$ by definition.
- There is no other possible case since M is neutral.

We can then conclude that $M \in \llbracket A \rightarrow B \rrbracket_{\Gamma}$.

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Lemma 1.11

1. If $s \in \llbracket \Gamma \rrbracket_{\Delta}$, then s is strongly normalizing.
2. If $s \in \llbracket \Gamma \rrbracket_{\Delta}$ and $s \longrightarrow s'$ then $s' \in \llbracket \Gamma \rrbracket_{\Delta}$.
3. If s is a neutral substitution such that $\Delta \vdash s \triangleright \Gamma$ and all its one-step reducts are reducible expressions, then $s \in \llbracket \Gamma \rrbracket_{\Delta}$.

Proof. We prove the properties by cases on Γ .

- Let us suppose $\Gamma = \emptyset$. Then by Lemma 1.3, Δ is also empty and the only substitutions in $\llbracket \emptyset \rrbracket_\emptyset$ are void substitutions. By Lemma 1.5 void substitutions are strongly normalizing so that Property 1 holds.

Now, if $s \in \llbracket \emptyset \rrbracket_\emptyset$ and $s \longrightarrow s'$, by the subject reduction property, $\emptyset \vdash s' \triangleright \emptyset$ so that by Remark 1.6 $s' \in \llbracket \emptyset \rrbracket_\emptyset$ and thus Property 2 also holds.

To show the third property suppose that s is neutral and $\emptyset \vdash s \triangleright \emptyset$. Then by definition $s \in \llbracket \emptyset \rrbracket_\emptyset$ and thus Property 3 also holds.

- Let us now suppose $\Gamma \neq \emptyset$.
 1. Let s be in $\llbracket \Gamma \rrbracket_\Delta$. Take $(x:A) \in \Gamma$. Then, by definition of $\llbracket \Gamma \rrbracket_\Delta$, the term $x[s]$ is in $\llbracket A \rrbracket_\Delta$. Since the properties hold for terms, then $x[s]$ is strongly normalizing and then s is strongly normalizing.
 2. Let s be in $\llbracket \Gamma \rrbracket_\Delta$. Then $\Delta \vdash s \triangleright \Gamma$. Take $(x:A) \in \Gamma$ and let s' such that $s \longrightarrow s'$. By definition of $\llbracket \Gamma \rrbracket_\Delta$, $x[s] \in \llbracket A \rrbracket_\Delta$ and since the properties hold for terms, then $x[s'] \in \llbracket A \rrbracket_\Delta$. By subject reduction property, $\Delta \vdash s' \triangleright \Gamma$ so that by definition of $\llbracket \Gamma \rrbracket_\Delta$, $s' \in \llbracket \Gamma \rrbracket_\Delta$.
 3. Let s be a well-typed (*i.e.* $\Delta \vdash s \triangleright \Gamma$) and neutral substitution and let us suppose that all the one-step reducts of s are in $\llbracket \Gamma \rrbracket_\Delta$. Take $(x:A)$ in Γ . Since s is neutral, $x[s]$ may either reduce to $x[s']$ with $s \longrightarrow s'$, or to x (if $s = id$). In the first case we have that $\Delta \vdash s' \triangleright \Gamma$ holds by subject reduction property, and s' is reducible by hypothesis, so that $x[s'] \in \llbracket A \rrbracket_\Delta$ by definition and we are done. In the second case, we have to show that $x \in \llbracket A \rrbracket_\Delta$, but Δ must be equal to Γ since $\Delta \vdash s \triangleright \Gamma$ so that $x \in \llbracket A \rrbracket_\Gamma$ holds by Lemma 1.10 (property 2). As a consequence $s \in \llbracket \Gamma \rrbracket_\Delta$.

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Now we will state (and prove) some Lemmas which prove the reducibility of certain expressions given the reducibility of some of their reducts.

Lemma 1.12 Let Δ be an environment. Let M be in $\llbracket A \rrbracket_\Delta$ and s be in $\llbracket \Gamma \rrbracket_\Delta$. If x is a fresh variable, then $t = (x/M).s \in \llbracket x:A, \Gamma \rrbracket_\Delta$.

Proof. First of all we remark that $\Delta \vdash t \triangleright x:A, \Gamma$. To prove that t is in $\llbracket x:A, \Gamma \rrbracket_\Delta$ it is sufficient to prove that $\forall (y:B) \in (x:A, \Gamma)$ we have that $y[t]$ is in $\llbracket B \rrbracket_\Delta$.

Take $(y:B) \in (x:A, \Gamma)$. Then $y[t]$ is neutral. Now, if we show that all its one-step reducts are in $\llbracket B \rrbracket_\Delta$ we may conclude that $y[t]$ is in $\llbracket B \rrbracket_\Delta$ by property 4 of Lemma 1.10. Now, since M and s are respectively in $\llbracket A \rrbracket_\Delta$ and $\llbracket \Gamma \rrbracket_\Delta$, then they are strongly normalizing by Lemma 1.10 and 1.11 and thus we may proceed by induction on $\nu(M) + \nu(s)$.

Now, the one-step reducts of $y[t]$ are:

- $y[(x/M').s]$ with $M \longrightarrow M'$. Then we conclude by induction hypothesis.
- $y[(x/M).s']$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
- M if $x = y$. Then we conclude by hypothesis.
- $y[s]$ if $x \neq y$. Then we conclude by hypothesis.

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Lemma 1.13 If $s \circ t \in \llbracket \Gamma \rrbracket_\Delta$, $M[t] \in \llbracket A \rrbracket_\Delta$, and if x is a fresh variable, then $u = ((x/M).s) \circ t$ is in $\llbracket (x:A, \Gamma) \rrbracket_\Delta$.

Proof. First of all we have to prove that u is well typed in Δ . But $\Delta \vdash s \circ t \triangleright \Gamma$ implies that there exists Γ' such that $\Delta \vdash t \triangleright \Gamma'$, $\Gamma' \vdash s \triangleright \Gamma$, and $\Delta \vdash M[t]:A$. On the other hand, Lemma 1.2 implies $\Delta \vdash t \triangleright \Gamma'$ and $\Gamma' \vdash M:A$, so that

$$\frac{\Delta \vdash t \triangleright \Gamma' \quad \frac{\Gamma' \vdash M:A \quad \Gamma' \vdash s \triangleright \Gamma}{\Gamma' \vdash (x/M).s \triangleright x:A, \Gamma}}{\Delta \vdash ((x/M).s) \circ t \triangleright x:A, \Gamma}$$

It suffices to prove that for all $(y:B) \in (x:A, \Gamma)$, $y[u]$ is in $\llbracket B \rrbracket_\Delta$. Let $y:B$ be in $(x:A, \Gamma)$. Since the expressions $M[t]$ and $s \circ t$ are strongly normalizing by Lemma 1.10 and 1.11, we can prove the property by induction on $\nu(M[t]) + \nu(s \circ t)$.

Now, the term $y[u]$ is neutral so it is sufficient to prove that all the one-step reducts of $y[u]$ are in $\llbracket B \rrbracket_\Delta$. This reducts are:

- $y[((x/M').s) \circ t]$ with $M \longrightarrow M'$. Then we conclude by induction hypothesis.
- $y[((x/M).s') \circ t]$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
- $y[((x/M).s) \circ t']$ with $t \longrightarrow t'$. Then we conclude by induction hypothesis.
- $y[(x/M[t]).(s \circ t)]$. Since $M[t]$ and $s \circ t$ are respectively in $\llbracket A \rrbracket_\Delta$ and $\llbracket \Gamma \rrbracket_\Delta$ by hypothesis, then the substitution $(x/M[t]).(s \circ t)$ is in $\llbracket x:A, \Gamma \rrbracket_\Delta$ by Lemma 1.12, and thus $y[(x/M[t]).(s \circ t)]$ is in $\llbracket B \rrbracket_\Delta$.

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Lemma 1.14 Let Δ and Γ be environments. Let $R = (\lambda x:A.M)[s]$ be a term such that $\Delta \vdash R:A \rightarrow B$ and $s \in \llbracket \Gamma \rrbracket_\Delta$. If for all Δ' and for all $N \in \llbracket A \rrbracket_{\Delta\Delta'}$ we have $M[(x/N).s] \in \llbracket B \rrbracket_{\Delta\Delta'}$, then $R \in \llbracket A \rightarrow B \rrbracket_\Delta$

Proof. By definition of reducibility of R , it suffices to show that for all $N \in \llbracket A \rrbracket_{\Delta\Delta'}$, $(R N) \in \llbracket B \rrbracket_{\Delta\Delta'}$.

Let N be in $\llbracket A \rrbracket_{\Delta\Delta'}$. Since $M[(x/N).s] \in \llbracket B \rrbracket_{\Delta\Delta'}$ then it is strongly normalizing, so we can reason by induction on $\nu(M) + \nu(N) + \nu(s)$. Now, $(R N)$ is neutral, and thus it is sufficient to prove that all the one-step reducts of $(R N)$ are in $\llbracket B \rrbracket_{\Delta\Delta'}$.

Now, $(R N)$ can only reduce to:

- $(\lambda x.M')[s] N$ with $M \longrightarrow M'$. We conclude by induction hypothesis.
- $(\lambda x.M)[s'] N$ with $s \longrightarrow s'$. We conclude by induction hypothesis.
- $(\lambda x.M)[s] N'$ with $N \longrightarrow N'$. We conclude by induction hypothesis.
- $M[(x/N).s]$ which is in $\llbracket B \rrbracket_{\Delta\Delta'}$ by hypothesis.

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We are now almost ready to prove that all well-typed expressions are reducible.

Theorem 1.15 Let Δ and Γ be valid environments and s be a substitution in $\llbracket \Gamma \rrbracket_{\Delta}$.

- For every substitution t such that $\Gamma \vdash t \triangleright \Gamma'$ for some valid environment Γ' , $t \circ s$ is in $\llbracket \Gamma' \rrbracket_{\Delta}$.
- For all term M such that $\Gamma \vdash M : A$ for some type A , $M[s]$ is in $\llbracket A \rrbracket_{\Delta}$.

Proof. By induction on the structure of the term M or the substitution t .

- If $e = id$, then $id \circ s$ is neutral, so that we show by induction on $\nu(s)$ (since s is strongly normalizing by Lemma 1.11) that all the one-step reducts of $id \circ s$ are in $\llbracket \Gamma \rrbracket_{\Delta}$. These reducts are:
 - $id \circ s'$ with $s \longrightarrow s'$. The property holds by induction hypothesis.
 - s . The property holds by hypothesis.
- If $e = x$, then Γ is not empty since x must be typed in Γ . Moreover, $x : A$ is in Γ and thus $x[s]$ is in $\llbracket A \rrbracket_{\Delta}$ by hypothesis.
- If $e = (x/M).v$, then by induction hypothesis $M[s]$ and $v \circ s$ are respectively in $\llbracket A \rrbracket_{\Delta}$ and $\llbracket \Gamma' \rrbracket_{\Delta}$ for a type A and an environment Γ' . Thus, by Lemma 1.13 $((x/M).v) \circ s$ is in $\llbracket x : A, \Gamma' \rrbracket_{\Delta}$.
- If $e = v \circ u$, then there exist two environments Γ' and Γ'' such that $\Gamma \vdash u \triangleright \Gamma''$ and $\Gamma'' \vdash v \triangleright \Gamma'$. By application of induction hypothesis we obtain consequently $u \circ s$ in $\llbracket \Gamma'' \rrbracket_{\Delta}$ and $v \circ (u \circ s)$ in $\llbracket \Gamma' \rrbracket_{\Delta}$. Since $v \circ (u \circ s) =_{Sub_ass_env} (v \circ u) \circ s$ then the property holds.
- If $e = (MN)$, then by induction hypothesis $M[s]$ and $N[s]$ are respectively in $\llbracket B \rightarrow A \rrbracket_{\Delta}$ and $\llbracket B \rrbracket_{\Delta}$ for some type B . By definition of reducibility $(M[s]N[s])$ is in $\llbracket A \rrbracket_{\Delta}$. But $(M[s]N[s]) =_{Sub_app} (MN)[s]$, so $(MN)[s]$ is in $\llbracket B \rrbracket_{\Delta}$.

- If $e = \lambda x.M$ then by Lemma 1.12, $(x/N).s$ is in $\llbracket x : A, \Gamma \rrbracket_{\Delta\Delta'}$ for all Δ' and all N in $\llbracket A \rrbracket_{\Delta\Delta'}$. Without loss of generality we can assume that x does not appear in Δ' . Thus we have $\Delta\Delta' \vdash s \triangleright \Gamma\Delta'$. By induction hypothesis $M[(x/N).s]$ is $\llbracket B \rrbracket_{\Delta\Delta'}$ and then by Lemma 1.14, $(\lambda x.M)[s]$ is also in $\llbracket B \rrbracket_{\Delta}$.
- If $e = M[t]$, then by induction hypothesis $t \circ s$ is in $\llbracket \Gamma' \rrbracket_{\Delta}$ for some Γ' and thus by induction hypothesis again $M[t \circ s]$ is in $\llbracket A \rrbracket_{\Delta}$. Since $M[t \circ s] =_{Sub.clos} M[t][s]$ then we are done. ▪

We can now claim the next theorem.

Theorem 1.16 $\longrightarrow_{\lambda_{\sigma w}/\equiv}$ is strongly normalizing.

Proof. First of all we remark that for any environment Γ the substitution id is in $\llbracket \Gamma \rrbracket_{\Gamma}$: for that it is sufficient to prove that for every $x : A$ in Γ , $x[id]$ is in $\llbracket A \rrbracket_{\Gamma}$ for a type A . But $x[id]$ is a neutral term so it is sufficient to prove that all the one-steps reducts of $x[id]$ are in $\llbracket A \rrbracket_{\Gamma}$. But the only possible reduct of $x[id]$ is x and x is neutral, so to show that x is in $\llbracket A \rrbracket_{\Gamma}$ it is sufficient to show that all its one-step reducts are in $\llbracket A \rrbracket_{\Gamma}$. As x is in normal form then Property 4 of Lemma 1.10 gives us that x is reducible.

Now, since id is reducible, then by Theorem 1.15 the term $M[id]$ is reducible for all well-typed term M . Property 1 of Lemma 1.10 allows us to conclude that $M[id]$ is strongly normalizing and thus that M is strongly normalizing. ▪

1.4 Strong normalization for $\lambda_{\sigma w}$

We are now able to deduce the strong normalization of $\lambda_{\sigma w}$ from that of $\lambda_{\sigma w}/\equiv$. For that we will use the following abstract Lemma:

First of all we introduce an abstract Lemma:

Lemma 1.17 Let $A = \langle \mathcal{O}, R_1 \cup R_2 \rangle$ be an abstract reduction system such that:

- R_2 is strongly normalizing;
- there exists a reduction system $S = \langle \mathcal{O}', R' \rangle$ and translation \mathcal{T} from \mathcal{O} to \mathcal{O}' such that:

$$\begin{aligned} - a \longrightarrow_{R_1} b &\text{ implies } \mathcal{T}(a) \longrightarrow_{R'} \mathcal{T}(b), \\ - a \longrightarrow_{R_2} b &\text{ implies } \mathcal{T}(a) = \mathcal{T}(b). \end{aligned}$$

Then for all term $a \in \mathcal{O}$ such that $\mathcal{T}(a)$ is R' -strongly normalizing we have that a is $(R_1 \cup R_2)$ -strongly normalizing.

Proof. We proceed by contradiction. Let $a \in \mathcal{O}$ be a term such that a is not $(R_1 \cup R_2)$ -strongly normalizing. Since $\mathcal{T}(a)$ is R' -strongly normalizing we remark that any infinite reduction of a contains infinitely many steps of R_1 : if it only contains finitely many steps of R_1 , then it would have to contain infinitely many steps of R_2 contradicting the R_2 strong normalization. So, an infinite sequence of reductions starting with a may be written as:

$$a \longrightarrow^*_{R_2} a_1 \longrightarrow^+_{R_1} a_2 \longrightarrow^*_{R_2} a_3 \longrightarrow^+_{R_1} a_4 \dots$$

But such a sequence translates in S to the following one:

$$\mathcal{T}(a) = \mathcal{T}(a_1) \longrightarrow^+_{R'} \mathcal{T}(a_2) = \mathcal{T}(a_3) \longrightarrow^+_{R'} \mathcal{T}(a_4) \longrightarrow^+_{R'} \dots$$

contradicting the R' -strong normalization of $\mathcal{T}(a)$. ▪

We can now conclude the following:

Theorem 1.18 $\lambda_{\sigma w}$ is strongly normalizing on well typed terms.

Proof. We note here R_2 the reduction relation generated by the three rules (Sub_app), (Sub_ass_env) and (Sub_clos) of $\lambda_{\sigma w}$, R_1 the reduction relation generated by the other rules of $\lambda_{\sigma w}$ and \mathcal{O} the set of well typed terms of $\lambda_{\sigma w}$. We also note \mathcal{O}' the set of \equiv -equivalence classes of well-typed terms $\lambda_{\sigma w}$ and R' the reduction relation $\lambda_{\sigma w/\equiv}$. We define \mathcal{T} to be the canonical projection from \mathcal{O} to \mathcal{O}' (that is $\mathcal{T}(a) = [a]$). This translation \mathcal{T} clearly satisfies the second condition of Lemma 1.17. Now, we know by Theorem 1.16 that $\lambda_{\sigma w/\equiv}$ is strongly normalizing on $\lambda_{\sigma w}$ well-typed terms, that is, that $\lambda_{\sigma w/\equiv}$ is strongly normalizing on well-typed \equiv -equivalence classes. By application of Lemma 1.17 we get that well-typed terms of $\lambda_{\sigma w}$ are $\lambda_{\sigma w}$ -strongly normalizing. ▪

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