On the 2-categorical duals of (full and) faithful functors

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Abstract

We consider the classes of functors $F : \mathbb{A} \to \mathbb{B}$ such that for all $\mathbb{C}$, the *precomposition functors* $(-) \circ F : \mathbb{C}^{\mathbb{B}} \to \mathbb{C}^{\mathbb{A}}$ are faithful respectively full and faithful. Since F is (full and) faithful if and only if all *postcomposition functors* $F \circ (-) : \mathbb{A}^{\mathbb{C}} \to \mathbb{B}^{\mathbb{C}}$ are (full and) faithful, these classes of functors can be considered 2-categorical duals of the classes of (full and) faithful functors.

John Baez and Michael Shulman mention these classes of functors briefly in [2, page 48], and give sufficient conditions for functors to be contained. We give necessary and sufficient conditions and some examples.

Remark

It was pointed out by Mathieu Dupont that the results presented in this note were in fact already proved as soon as 2001 by by Admek, El Bashir, Sobral and Velebil in [1]. I'll leave the file available for the moment, because it is linked on the n-Category Café.

1 Introduction

Injectivity and surjectivity are fundamental properties of functions between sets. If we go one step up on the dimensional ladder, the question arises if there are fundamental properties of functors between categories with a comparable status. Natural candidates are faithfulness, fullness and essential surjectivity. It is remarkable that these concepts arise in a hierarchy and have to be accumulated; fullness without faithfulness, e.g., is not well behaved. For a discussion of these phenomena with a n-categorical perspective, see [2, Section 5.5].

In one dimension we have that injecivity and surjectivity of functions is equivalent to injectivity of *post-* and *precomposition* operations, and thus dual to each other. For faithfulness, fullness and essential surjecivity, however, there is no such duality. Thus, it is natural to ask whether the dual concepts are well behaved.

These duals are indeed of practical interest and arise naturally in the context of Kan extensions, because these can be defined as adjoints to *precomposition* mappings. The left Kan extension of set valued presheafs along a functor H between small categories, for example, is a reflection if and only if H belongs to the class of functors that is dual to 'full and faithful'.

In the present work, we present characterizations of the dual properties of *faithfullnes* and *full- and faithfulness.*

2 Definitions and conventions

Definition 2.1 Let $\mathfrak{A}$ be a 2-category and $f : A \to B$ a 1-cell in $\mathfrak{A}$.

1. We call f faithful on the left (right), if for all objects X of $\mathfrak{A}$, the postcomposition functor $\mathfrak{A}(X, f)$ (respectively the precomposition functor $\mathfrak{A}(f, X)$) is faithful.

2. We call f full on the left (right), if for all objects X of $\mathfrak{A}$, the postcomposition functor $\mathfrak{A}(X, f)$ (respectively the precomposition functor $\mathfrak{A}(f, X)$) is full.

3. We call f full and faithful on the left (right), if it is full on the left (right) and faithful on the left (right). $\diamondsuit$

We will often abbreviate and write e.g. *left faithful* instead of *faithful on the left.*

Observation 2.2 *Functors between categories are* left *(full and) faithful iff they are (full and) faithful.*

Note that the same is *not* true for functors which are only full (combinatorial counterexamples are easy to find).

The goal of this note is to give characterizations of *right* (full and) faithful functors.

3 Right faithful functors

In [2, page 48], John Baez and Michael Shulman remark that a sufficient criterion for a functor to be right faithful ('2-epic' in their terminology) is that it is essentially surjective[1]. They also state that this criterion is not necessary and give the following weaker sufficient criterion.

Lemma 3.1 *Let $F : \mathbb{A} \to \mathbb{B}$ be a functor. If each object B of $\mathbb{B}$ is a retract of some FA for $A \in \mathsf{Obj}(\mathbb{A})$, then F is right faithful.*

Proof. Straightforward. ∎

In fact, as we will show now, this criterion is also necessary.

Lemma 3.2 *Let $F : \mathbb{B} \to \mathbb{A}$ be a right faithful functor. Then every $B \in \mathsf{Obj}(\mathbb{B})$ is a retract of some FA with $A \in \mathsf{Obj}(\mathbb{A})$.*

[1] This is interesting since the essentially surjective functors also admit a 2-categorical characterization, by orthogonality, and in fact they seem to be the more interesting arrow class from the logical point of view. See also [5, 4].

Proof. Let $B \in \mathsf{Obj}(\mathbb{B})$. If F is right faithful, then, in particular, the precomposition mapping

$$\widehat{\mathbb{B}}(YB, P) \longrightarrow \widehat{\mathbb{A}}((YB)F, PF) \qquad (*)$$

is injective for all presheaves P on $\mathbb{B}$. This is equivalent to saying that the composite

$$\widehat{\mathbb{B}}(YB, P) \longrightarrow \widehat{\mathbb{A}}((YB)F, PF) \xrightarrow{\cong} \widehat{\mathbb{B}}(\mathrm{Lan}_F((YB)F), P)$$

with the adjoint transpose map for the left Kan extension along F is injective, because the latter map is a bijection. But this composite is the same as precomposition in $\widehat{\mathbb{B}}$ with the counit $\varepsilon_{YB} : \mathrm{Lan}_F((YB)F) \to YB$ of the left Kan extension. We conclude that the mapping $(*)$ is injective for all P if and only if ε_{YB} is an epimorphism in $\widehat{\mathbb{B}}$.

Now a natural transformation between presheafs of sets is an epimorphism iff it is pointwise surjective, and thus we have to consider $\varepsilon_{YB,B'}$ for $B' \in \mathsf{Obj}(\mathbb{A})$. Because left Kan extensions of presheafs can be computed by the formula $(\mathrm{Lan}_F\, Q)B = \int^A \mathbb{B}(B, FA) \times QA$ (see e.g. [3, X.4.1]), $\varepsilon_{YB,B'}$ can (up to isomorphism) be written as

$$\int^A \mathbb{B}(B', FA) \times \mathbb{B}(FA, B) \xrightarrow{(f,g)\mapsto gf} \mathbb{B}(B', B).$$

Now if $\varepsilon_{YB,B}$ is surjective, then id_B has a preimage which gives the required retract. ∎

Example. If an adjunction $F \vdash U : \mathbb{B} \to \mathbb{A}$ has a split mono unit η, then U is right faithful, because every A is a retract of UFA. An example for this situation is given by the adjunction $\Delta \vdash \times : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ for the cartesian product in a category where it exists. Its unit η with components $\eta_A = \delta_A : A \to A \times A$ is split by any of the projections. ♢

4 Right full and faithful functors

Analogously to the right faithful case, Baez and Shulman observed that a sufficient criterion for a functor to be ritht full and faithful ('1-epic' in their terminology) is that it is full and essentially surjective[2].

We give a necessary and sufficient criterion.

Theorem 4.1 *The following properties are equivalent for a functor $F : \mathbb{A} \to \mathbb{B}$.*

1. *F is right full and faithful.*
2. *The natural transformation ζ with components*

$$\zeta_{B',B} : \int^A \mathbb{B}(B', FA) \times \mathbb{B}(FA, B) \xrightarrow{(f,g)\mapsto gf} \mathbb{B}(B', B)$$

is an isomorphism.

[2]And again, the class of arrows which are full and essentially surjective seems to be the logically more interesting class of arrows and can be characterized by orthogonality.

3. *Every $B \in \mathsf{Obj}(\mathbb{B})$ is a retract of some FA_B by means of morphisms $m_B : B \rightarrowtail FA_B$ and $e_B : FA_B \twoheadrightarrow B$; and for every morphism $f : B' \to B$ in $\mathbb{B}$, $(m_{B'}, fe_{B'})$ and $(m_B f, e_B)$ are in the same equivalence class in $\int^A \mathbb{B}(B, FA) \times \mathbb{B}(FA, C)$.*

Proof. 1. $\Rightarrow$ 2. Rereading the proof of Lemma 3.2, we see that if the precomposition mapping $(*)$ is a bijection (which it is for a right full and faithful functor), then ε_{YB} is an isomorphism. Because ζ has the same components as ε_{YB}, this implies that ζ is also an isomorphism.

2. $\Rightarrow$ 3. As in Lemma 3.2, (m_B, fe_B) can be chosen among the representatives of $\zeta^{-1}_{B,B}(\mathrm{id}_B)$. The pairs $(m_{B'}, fe_{B'})$ and $(m_B f, e_B)$ are equivalent, because they are both mapped to f by $\zeta_{B',B}$.

3. $\Rightarrow$ 1. Let $G, H : \mathbb{B} \to \mathbb{C}$ and let $\phi : GF \to HF$. We have

$$(B' \xrightarrow{f} FA, FA \xrightarrow{g} B) \sim (B' \xrightarrow{h} FA', FA' \xrightarrow{k} B) \\ \Rightarrow \quad Hg \circ \phi_A \circ Gf = Hk \circ \phi_{A'} \circ Gh, \quad (\dagger)$$

where $\sim$ denotes the equivalence relation in the coend, because this holds for the generators of the equivalence relation.

To prove the proposition, we have to construct a $\psi : G \to H$ such that $\psi F = \phi$.

Define $\psi_B := He_B \circ \phi_{A_B} \circ Gm_B$.

The naturality of ψ at $f : B' \to B$ follows then directly by applying $(\dagger)$ to the pairs $(m_{B'}, fe_{B'})$ and $(m_B f, e_B)$, which are equivalent by assumption. ■

Examples of right full and faithful functors can be found very easily, because, for every reflection $F \dashv U$, F is right full and faithful. This is straightforward by condition 2. of the above theorem.

It is however a bit disappointing that the elementary characterization 3. is not as easy as for righy faithful functors, because it involves the somewhat awkward equivalence relation. There is an easier *sufficient* condition for right full-and-faithfulness, which says that in addition to the fact that the objects of $\mathbb{B}$ should be retracts of FA's, the morphisms of $\mathbb{B}$ should also be retracts of images of morphisms in $\mathbb{A}$ (in the most straightforward sense). For left adjoints of reflections, this always holds. It remains to be clarified if maybe this criterion is also necessary.

References

[1] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. On functors which are lax epimorphisms. *Theory Appl. Categ.*, 8:509–521 (electronic), 2001.

[2] John C. Baez and Michael Shulman. Lectures on n-categories and cohomology, October 2007.

[3] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[4] Ross Street. Characterizations of bicategories of stacks. In *Category theory (Gummersbach, 1981)*, volume 962 of *Lecture Notes in Math.*, pages 282–291. Springer, Berlin, 1982.

[5] Ross Street. Two-dimensional sheaf theory. *J. Pure Appl. Algebra*, 23(3):251–270, 1982.