A 2-dimensional Adjunction between Triposes and Toposes

Jonas Frey

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Outline

1. Triposes and toposes
2. Towards a universal characterization of the tripos-to-topos construction
3. dc-categories and semi-lax adjunctions

Toposes

A topos is a category with finite limits, exponentials and a subobject classifier.
Toposes can be viewed as *mathematical universes*. More precisely, in a topos, we can interpret *intuitionistic higher order logic*.

2-categories of toposes

What should be the one-cells?

Possible choices:

- Logical functors : Too restrictive
- Geometric morphisms : Good, but the tentative unit of the biadjunction we want to present is not a geometric morphism
- Cartesian (finite limit preserving) functors : Right choice
- Regular functors : Have special status among cartesian functors

Geometric morphisms can be recovered later as adjunctions of cartesian functors.

Triposes

- Larger class of models for intuitionistic higher order logic
- Introduced 1980 by Hyland, Johnstone and Pitts to construct the effective topos

Definition of Tripos

Let $\mathcal{C}$ be a cartesian category. A *tripos over* $\mathcal{C}$ is a fibration

$$\mathcal{P} : \mathcal{X} \to \mathcal{C}$$

such that

1. All fibres of $\mathcal{P}$ are pre-Heyting algebras
2. Reindexing along morphisms in $\mathcal{C}$ preserves all structure of pre-Heyting algebras
3. $\mathcal{P}$ is internally complete and cocomplete. This means that for all $f : A \to B$ in $\mathcal{C}$, the reindexing map $f^* : \mathcal{P}_B \to \mathcal{P}_A$ has left and right adjoints
$$\exists_f \dashv f^* \dashv \forall_f.$$
4. $\mathcal{P}$ has *power objects*, i.e., for all $A \in \mathrm{Obj}(\mathcal{C})$ we are given an object $\mathfrak{P}A$ and a predicate $(\in_A) \ \in \mathcal{P}_{A\times\mathfrak{P}A}$ such that for all predicates $\varphi \in \mathcal{P}_{C\times A}$ there exists a map $\{\varphi\} : C \to \mathfrak{P}A$ such that
$$c, a \mid\vdash \varphi(c, a) \leftrightarrow a \in_A \{\phi\}(c)$$
holds in $\mathcal{P}$.

Tripos morphisms

Given triposes $\mathcal{P} : \mathcal{X} \to \mathcal{C}$ and $\mathcal{Q} : \mathcal{Y} \to \mathcal{D}$, a morphism between them is a pair

$$(F : \mathcal{C} \to \mathcal{D}, \quad \Phi : \mathcal{X} \to \mathcal{Y})$$

of functors such that

1. The diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

   commutes (on the nose).
2. Φ maps cartesian arrows to cartesian arrows.
3. F preserves finite limits and Φ preserves finite meets.

If Φ furthermore commutes with existential quantification, then we call the tripos morphism *regular*.

2-cells of triposes

A 2-cell

$$\eta : (F, \Phi) \to (G, \Gamma) : \mathcal{P} \to \mathcal{Q}$$

is a natural transformation

$$\eta : F \to G$$

such that for all $A \in \mathrm{Obj}(\mathcal{C})$ and all $\psi \in \mathrm{Obj}(\mathcal{P}_A)$, we have

$$x \mid (\Phi\psi)(x) \vdash (\Gamma\psi)(\eta_A(x))$$

in the logic of $\mathcal{Q}$.

Embedding toposes into triposes

For a given category $\mathcal{C}$, we denote by $\boldsymbol{M}(\mathcal{C})$ the full subcategory of $\mathcal{C}{\downarrow}\mathcal{C}$ on the monomorphisms.

For each topos $\mathcal{E}$, its *subobject fibration*

$$\partial_1{}^1 : \boldsymbol{M}(\mathcal{E}) \to \mathcal{E}$$

is a tripos, which we denote by $\boldsymbol{S}\mathcal{E}$.

It is straightforward to check that this gives rise to a 2-functor $\boldsymbol{S}$ from toposes to triposes.

[1] ∂_1 is the codomain projection

The tripos-to-topos construction

The topos $\boldsymbol{T}\mathcal{P}$

For a tripos $\mathcal{P}$ on $\mathcal{C}$, we can construct a topos $\boldsymbol{T}\mathcal{P}$ as follows:

The **objects** of $\boldsymbol{T}\mathcal{P}$ are pairs $A = (|A|, \sim_A)$, where $|A| \in \mathrm{Obj}(\mathcal{C})$, $(\sim_A) \ \in \mathcal{P}(|A| \times |A|)$, and the judgements

$$x \sim_A y \vdash y \sim_A x$$
$$x \sim_A y, y \sim_A z \vdash x \sim_A z$$

hold in the logic of $\mathcal{P}$

Intuition: "$\sim_A$ is a partial equivalence relation on $|A|$ in the logic of $\mathcal{P}$"

The tripos-to-topos construction

The topos $\boldsymbol{T}\mathcal{P}$ (continued)

Morphisms of $\boldsymbol{T}\mathcal{P}$ are given by functional relations with respect to $\mathcal{P}$.

More precisely, a morphism from A to B is a $(\dashv\vdash)$-equivalence class of predicates on $|A| \times |B|$ such that for some (or equivalently any) representative ϕ the following judgements hold in $\mathcal{P}$.

$$\begin{aligned} \phi(x,y) &\vdash x \sim_A x \wedge y \sim_B y \\ \phi(x,y), x \sim_A x', y \sim_B y' &\vdash \phi(x',y') \\ \phi(x,y), \phi(x,y') &\vdash y \sim_B y' \\ x \sim_A x &\vdash \exists y \,.\, \phi(x,y) \end{aligned}$$

The tripos-to-topos construction

The topos $\boldsymbol{T}\mathcal{P}$ (continued)

Given morphisms

$$A \xrightarrow{[\phi]} B \xrightarrow{[\gamma]} C\,,$$

their **composition** is given by $[\gamma \circ \phi]$, where $\gamma \circ \phi \in \mathcal{P}_{|A| \times |C|}$ is the predicate

$$x, z \mid \exists y \,.\, \phi(x, y) \wedge \gamma(y, z).$$

The **identity** morphism on A is $[\sim_A]$.

The tripos-to-topos construction

Mapping tripos morphisms to functors between toposes

- Easy for *regular* tripos morphisms:
  Given a regular tripos morphism

$$(F, \Phi) : \mathcal{P} \to \mathcal{Q},$$

the functor

$$\boldsymbol{T}(F, \Phi) : \boldsymbol{T}\mathcal{P} \to \boldsymbol{T}\mathcal{Q}$$

is given by

$$\begin{array}{rcc} (|A|, \sim_A) & \mapsto & (F(|A|), \Phi(\sim_A)) \\ ([\phi] : (|A|, \sim_A) \to (|B|, \sim_B)) & \mapsto & [\Phi\phi] \end{array}$$

The tripos-to-topos construction

Mapping tripos morphisms to functors between toposes

- This method does not work if (F, Φ) is not regular, because then, $\Phi\phi$ is not total in general
- Interestingly, this can be circumvented by using a completion process for objects in $\boldsymbol{T}\mathcal{P}$.
- Construction becomes more clumsy
- Find an elegant *characterization*!

Motivating example

- Every complete Heyting algebra A give rise to a tripos $\tilde{A}$ over **Set**:
  - Fibre over I is A^I
  - Reindexing is given by precomposition
- Meet preserving maps between complete Heyting algebras give rise to tripos morphisms

Consider the succession of tripos morphisms

$$\tilde{\mathbb{B}} \xrightarrow{\tilde{\delta}} \widetilde{\mathbb{B} \times \mathbb{B}} \xrightarrow{\tilde{\wedge}} \tilde{\mathbb{B}},$$

where $\mathbb{B} = \{\text{true}, \text{false}\}$ with false $\leq$ true.
What do we get when applying the tripos-to-topos construction?

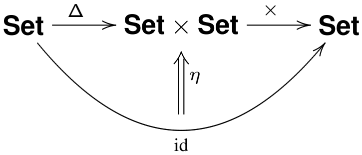
Motivating example

Answer:

$$\mathbf{Set} \xrightarrow{\Delta} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set}\,,$$

Motivating example

However, the composition gets mapped to the identity functor!



The tripos-to-topos construction seems to be an *oplax* functor!

Towards a universal characterization of the tripos-to-topos construction

- We want to characterize the tripos-to-topos construction as being left adjoint to $\boldsymbol{S}$ (the forgetful functor from toposes to triposes)
- This can not be an ordinary biadjunction, as the tripos-to-topos construction seems to be oplax, and ordinary biadjunctions live in the framework of bicategories and *pseudofunctors*.
- However, we still have something that looks like a unit and gives rise to a 'universal lifting property' (explained below).

Towards a universal characterization of the tripos-to-topos construction

The ‘unit’ of the ‘adjunction’

For each tripos $\mathcal{P} : \mathcal{X} \to \mathcal{C}$, there is a tripos transformation

$$(D, \Xi) : \mathcal{P} \to \boldsymbol{ST}\mathcal{P}$$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \Xi\ } & \boldsymbol{M}(\boldsymbol{T}\mathcal{P}) \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \partial_1} \\ \mathcal{C} & \xrightarrow[\ D\ ]{} & \boldsymbol{T}\mathcal{P} \end{array}$$

D is the so-called ‘constant objects functor’, it is defined as

$$\begin{aligned} A &\mapsto (A, =_A) \\ f &\mapsto [x, y \mid f(x) = y] \end{aligned}$$

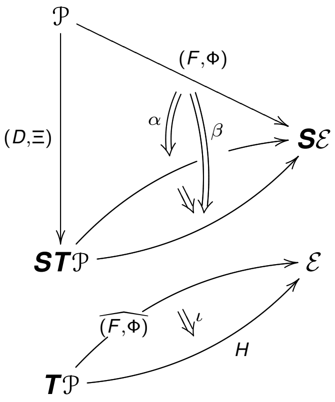
Exercise: For the definition of Ξ, make yourself clear how one can associate subobjects of DA to predicates on A in $\mathcal{P}$.

The universal lifting property

It turns out that that we have a lifting property for (D, Ξ) that has a slight resemblance to the condition for left adjointability of functors in one dimension.

For each tripos morphism (F, Φ), there is a cartesian functor $\widehat{(F, \Phi)}$ and a tripos transformation α such that for all H and β, there is a unique mediating ι.

In other words, the category $(\mathcal{P} \swarrow \boldsymbol{S})((D, \Xi), (F, \Phi))$ has an initial object $(\widehat{(F, \Phi)}, \alpha)$.



The universal lifting property

The universal lifting property suffices to construct an oplax functor, however it does *not* determine the tentative unit (D, Ξ) up to equivalence.

We will now define a three-dimensional category in which the 2-category of triposes and the 2-category of triposes are objects, and the tripos-to-topos construction is an ordinary biadjunction.

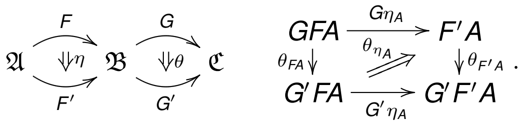
In this structure, the above ‘universal lifting property’ will be part of a characterization of *left adjointablility*.

In comparison to the tripos-to-topos construction, we will from now on revert all 2-cells, such that everything is *lax* instead of *oplax*

dc-categories

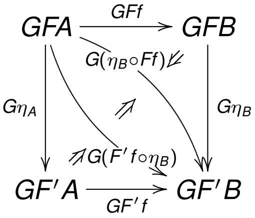
- The canonical tricategory is given by bicategories, pseudofunctors, pseudo-natural transformations and modifications.
- When we try to define a tricategory out of lax functors and lax transformations, we run into two problems:

First problem: Given pseudofunctors F, F', G, G' and lax transformations η, θ as in the left diagram below, there are two generally non-isomorphic ways to define $(\theta \circ \eta)_A : GFA \to G'F'A$:



dc-categories

Second problem: If the functor G is also lax, then the composition $G \circ \eta$ is not even definable! If we try to compose constraint cells of G and η to construct the constraint cell $(G\eta)_f$, we run into a problem:



dc-categories avoid these problems while still having lax features!

A **dc-category** is just a 2-category $\mathfrak{A}$ together with a designated subclass $\mathfrak{A}_r$ of the class of all 1-cells such that

- $\mathfrak{A}_r$ contains all equivalences,
- $\mathfrak{A}_r$ is closed under composition, and
- $\mathfrak{A}_r$ is closed under vertical isomorphisms; i.e if $f \in \mathfrak{A}_r$ and $f \cong g$, then $g \in \mathfrak{A}_r$.

We call the arrows in $\mathfrak{A}_r$ *regular* arrows, and denote them by '$\longmapsto$' in diagrams.

A **semi-lax functor** between dc-categories $\mathfrak{A}$ and $\mathfrak{B}$ is a lax functor $(F, \phi) : \mathfrak{A} \to \mathfrak{B}$ such that

- F maps regular arrows in $\mathfrak{A}$ to regular arrows in $\mathfrak{B}$,
- all $\phi_A : \mathrm{id}_{FA} \to F(\mathrm{id}_A)$ are invertible,
- $\phi_{(f,g)} : Fg \circ Ff \to F(g \circ f)$ is invertible whenever g is regular

A **semi-lax transformation** between semi-lax functors F, G is a lax natural transformation $\eta : F \to G$ such that

- For each object A, η_A is regular, and
- η_f is invertible whenever f is regular.

dc-categories

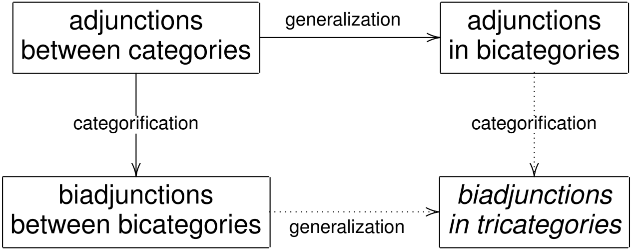
Exercise: Verify that semi-lax functors and transformations can be composed just like pseudofunctors and pseudo-natural transformations. In particular, check that the disturbing 2-cells mentioned 3 slides earlier become invertible.

Conjecture: dc-categories, semi-lax functors, semi-lax transformations and modifications form a *tricategory*.

This seems reasonable, because semi-lax functors and transformations are very similar to pseudofunctors and pseudo-natural transformations in their behaviour. (However, I did not even manage to comprehend the proof that the pseudofunctors and pseudo-natural transformations form a tricategory)

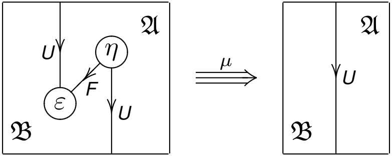
Abstract biadjunctions

- Adjunctions between categories can be generalized to adjunctions in bicategories, and they can be categorified to adjunctions between bicategories.
- If we combine these processes, we get *biadjunctions in tricategories*.

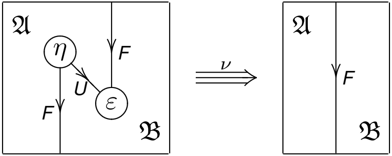


Abstract biadjunctions

To categorify the definition of adjunctions via triangle-equalities, we replace the triangle equalities by isomorphic *3-cells*



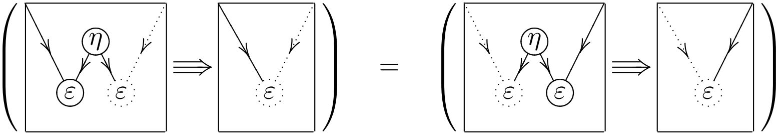
and



The most interesting question is ‘What are the new axioms?’

Abstract biadjunctions

In semi string diagram style, the axioms for abstract biadjunctions are



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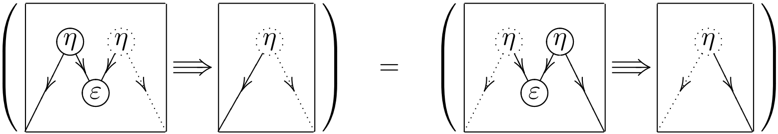


This elegant and comprehensible representation is due to John Baez [HDA4], if we write the equations out as pasting diagrams or even purely symbolic, things get badly readable because of the constraint cells.

Semi-lax adjunctions

A *semi-lax adjunction* is what we get if we interpret abstract definition of biadjunction in the three-dimensional structure of dc-categories.

We now state the central theorems.

Theorem 1: If $(F, U, \eta, \varepsilon, \mu, \nu)$ is a semi-lax adjunction, then U is a pseudofunctor.

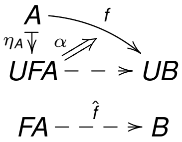
This is remarkable, as it reveals an asymmetry in the concept of semi-lax adjunction.

Semi-lax adjunctions

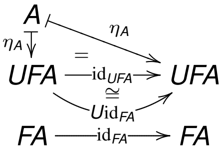
Theorem 2: Let $\mathfrak{A}, \mathfrak{B}$ be dc-categories and let $(U, \phi) : \mathfrak{B} \to \mathfrak{A}$ be a pseudo functor that maps regular arrows to regular arrows. Then U has a left semi-lax adjoint *iff*

1. For each $A \in \mathrm{Obj}(\mathfrak{A})$ there is an $FA \in \mathrm{Obj}(\mathfrak{B})$ and a regular arrow $\eta_A : A \mapsto UFA$ such that for all $B \in \mathrm{Obj}(\mathfrak{B})$ and $f : A \to UB$, the category $(A \nearrow U)(\eta_A, f)$ has a terminal object $(\hat{f}, \alpha_f)$.
2. If $f : A \to UB$ is regular then $\hat{f}$ is also regular and α_f is invertible.
3. $(\mathrm{id}_{FA}, \phi_{FA}^{-1} \circ \eta_A)$ is terminal in $(A \nearrow U)(\eta_A, \eta_A)$.
4. For all $f : A \to UB$ and all *regular* $g : B \mapsto C$, $(g\hat{f}, (\phi_{(\hat{f},g)}^{-1} \circ \eta_A)(Ug \circ \alpha_f))$ is terminal in $(A \nearrow U)(\eta_A, Ugf)$.

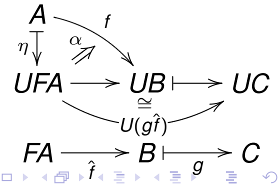
Ad (1):



Ad (3):



Ad (4):



Semi-lax adjunctions

Theorem 3: The forgetful functor ***S*** from toposes to triposes has a semi-lax left adjoint.

Conclusion:

What have we achieved?

- We found a universal characterization of the tripos-to-topos construction.
- We found an interesting tricategory(?) with lax features.