Dc-categories and the tripos-to-topos construction

Jonas Frey

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Part 1

The tripos-to-topos construction

The tripos to topos construction

- The tripos-to-topos construction was defined in 1980 by Hyland, Johnstone and Pitts [Hyland et al., 1980] as a tool to construct the effective topos.
- It allows to construct interesting toposes that are not Grothendieck toposes.
- It relates two classes of models of intuitionistic higher order logic.

Definition of Tripos

$\mathcal{A}$ Let $\mathbb{C}$ be a category with finite limits. A tripos over $\mathbb{C}$ is a functor

$$\mathcal{P} : \mathbb{C}^{\text{op}} \to \textbf{Poset},$$

such that

1. For each $A \in \mathbb{C}$ $\mathcal{P}(A)$ is a Heyting algebra[1].
2. For all $f : A \to B$ in $\mathbb{C}$ the maps $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ preserve all structure of Heyting algebras.
3. For all $f : A \to B$ in $\mathbb{C}$, the maps $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ have left and right adjoints
$$\exists_f \dashv \mathcal{P}(f) \dashv \forall_f$$
subject to the Beck-Chevalley condition.
4. For each $A \in \mathbb{C}$ there exists $\mathfrak{P}A \in \mathbb{C}$ and $(\ni_A) \in \mathcal{P}(\mathfrak{P}A \times A)$ such that for all $\psi \in \mathcal{P}(C \times A)$ there exists $\chi_\psi : C \to \mathfrak{P}A$ such that
$$\mathcal{P}(\chi_\psi \times A)(\ni_A) = \psi.$$

[1] A Heyting algebra is a poset which is bicartesian closed as a category.

The topos $\boldsymbol{T}\mathcal{P}$

For a tripos $\mathcal{P}$ on $\mathbb{C}$, we can construct a topos $\boldsymbol{T}\mathcal{P}$ as follows:

- The objects of $\boldsymbol{T}\mathcal{P}$ are pairs $A = (|A|, \sim_A)$, where $|A| \in \mathrm{obj}(\mathbb{C})$, $(\sim_A) \ \in \mathcal{P}(|A| \times |A|)$, and the judgements

$$x \sim_A y \vdash y \sim_A x$$
$$x \sim_A y, y \sim_A z \vdash x \sim_A z$$

  hold in the logic of $\mathcal{P}$.

  Intuition: "$\sim_A$ is a partial equivalence relation on $|A|$ in the logic of $\mathcal{P}$"

The topos $\boldsymbol{T}\mathcal{P}$

- A morphism from A to B is a predicate $\phi \in \mathcal{P}(|A| \times |B|)$ the following judgements hold in $\mathcal{P}$.

(strict)	$\phi(x,y) \vdash x \sim_A x \wedge y \sim_B y$
(cong)	$\phi(x,y), x \sim_A x', y \sim_B y' \vdash \phi(x',y')$
(singval)	$\phi(x,y), \phi(x,y') \vdash y \sim_B y'$
(tot)	$x \sim_A x \vdash \exists y \,.\, \phi(x,y)$

The topos $T\mathcal{P}$

- The composition of two morphisms

$$A \xrightarrow{\phi} B \xrightarrow{\gamma} C,$$

is given by

$$(\gamma \circ \phi)(a, c) \quad \equiv \quad \exists b \,.\, \phi(a, b) \wedge \gamma(b, c).$$

- The identity morphism on A is $\sim_A$.

Is this construction functorial?

We need a notion of morphism between triposes!

Tripos morphisms

A tripos morphism between triposes $\mathcal{P} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Poset}$ and $\mathcal{Q} : \mathbb{D}^{\mathrm{op}} \to \mathbf{Poset}$ is a pair (F, Φ) of a functor

$$F \;:\; \mathbb{C} \to \mathbb{D}$$

and a natural transformation

$$\Phi \;:\; \mathcal{P} \to \mathcal{Q} \circ F$$

such that

1. F preserves finite products
2. For every $C \in \mathbb{C}$, Φ_C preserves finite meets.

If Φ commutes with existential quantification, i.e.

$$\Phi_D(\exists_f \psi) = \exists_{Ff} \Phi_C(\psi)$$

for all $f : C \to D$ in $\mathbb{C}$ and $\psi \in \mathcal{P}(C)$, then we call the tripos morphism regular.

2-cells of triposes

A 2-cell

$$\eta : (F, \Phi) \to (G, \Gamma) : \mathcal{P} \to \mathcal{Q}$$

is a natural transformation

$$\eta : F \to G$$

such that for all $C \in \mathbb{C}$ and all $\psi \in \mathcal{P}(C)$, we have

$$\Phi_C(\psi) \;\leq\; \mathcal{Q}(\eta_C)(\Gamma_C(\psi)).$$

Mapping tripos morphisms to functors between toposes

Given a regular tripos morphism

$$(F, \Phi) : \mathcal{P} \to \mathcal{Q},$$

we can define a functor

$$\boldsymbol{T}(F, \Phi) : \boldsymbol{T}\mathcal{P} \to \boldsymbol{T}\mathcal{Q}$$

by

$$\begin{array}{rcl} (|A|, \sim_A) & \mapsto & (F(|A|), \Phi(\sim_A)) \\ (\gamma : (|A|, \sim_A) \to (|B|, \sim_B)) & \mapsto & \Phi\gamma \end{array}$$

This works because the definition of partial equivalence relations, functional relations and composition only uses $\wedge$ and $\exists$, which are preserved by regular tripos morphisms.

Mapping tripos morphisms to functors between toposes

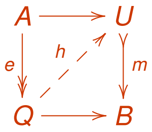
- This method does not work if (F, Φ) is not regular.
- However, there *is* a way to construct functors from plain tripos morphisms.
- For this, we have to decompose the construction in two steps:

  Triposes $\to$ Q-Toposes $\to$ Toposes
- 'Q-topos' is a notion that is related to, but weaker than quasitopos, definition follows.

Q-Toposes

Definition

- A monomorphism $e : U \to B$ in a category $\mathcal{C}$ is called strong, if for every commutative square

  where e is an epimorphism, there exists a (unique) h.
- A q-topos is a category $\mathcal{C}$ with finite limits, an exponentiable classifier of strong monomorphisms, and pullback stable quotients of strong equivalence relations.

Given a tripos $\mathcal{P} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Poset}$, we construct a q-topos $\boldsymbol{F}\mathcal{P}$

- Same objects as $\boldsymbol{T}\mathcal{P}$ (i.e., partial equivalence relations in $\mathcal{P}$)
- Morphisms of type $(C, \rho) \to (D, \sigma)$ are morphisms $f : C \to D$ in $\mathbb{C}$ such that

$$\rho(x, y) \vdash \sigma(fx, fy),$$

  quotiented by an equivalence relation:
  f, g are identified, iff

$$\rho(x, x) \vdash \sigma(fx, gx).$$

- Composition and identities are inherited from $\mathbb{C}$.

Now given an arbitrary tripos morphism

$$(F, \Phi) : \mathcal{P} \to \mathcal{Q},$$

we can define a functor

$$\boldsymbol{F}(F, \Phi) : \boldsymbol{F}\mathcal{P} \to \boldsymbol{F}\mathcal{Q}$$

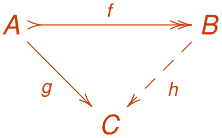
by

$$\begin{array}{rcl} (|A|, \sim_A) & \mapsto & (F(|A|), \Phi(\sim_A)) \\ (f : (|A|, \sim_A) \to (|B|, \sim_B)) & \mapsto & Ff \end{array}$$

Q-Toposes

Coarse objects

- $C \in \mathcal{C}$ is called coarse, if for every $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ which is monic and epic, and for all $g : A \to C$, there exists $h : B \to C$ such that $hf = g$.



- The coarse objects in a q-topos form a reflective subcategory with cartesian rector, which is a topos!
- This allows to recover $\boldsymbol{T}\mathcal{P}$ from $\boldsymbol{F}\mathcal{P}$.
- Functors between q-toposes give rise to functors between toposes by composing with the appropriate components of the reflections.

Embedding toposes into triposes

- We described how to construct toposes from triposes
- Conversely, we can assign triposes to toposes
- Given a topos $\mathcal{E}$, define its presheaf of subobjects

$$\boldsymbol{S}\mathcal{E} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Poset}$$

- For $A \in \mathcal{E}$, $\boldsymbol{S}\mathcal{E}(A)$ is the poset reflection of the preorder of monomorphisms into A
- Morphism part is given by pullback
- $\boldsymbol{S}\mathcal{E}$ is indeed a tripos, and the assignment $\mathcal{E} \mapsto \boldsymbol{S}\mathcal{E}$ is 2-functorial

A universal characterization?

- We want to give a universal characterization of the tripos-to-topos construction.
- Maybe it is left (bi)adjoint to $\boldsymbol{S}$ (the 2-functor from toposes to triposes)?
- The following example shows that it can't be.

Triposes from complete Heyting algebras

- For a complete Heyting algebra A, the functor

$$\mathcal{P}_A \;=\; \mathbf{Set}(-, A)$$

  is a tripos if we equip the sets $\mathbf{Set}(I, A)$ with the pointwise ordering.
- For a meet preserving map $f : A \to A'$ between complete Heyting algebras, the induced natural transformation

$$\mathcal{P}_f = \mathbf{Set}(-, f) \;:\; \mathbf{Set}(-, A) \to \mathbf{Set}(-, A')$$

  is a tripos morphism
- $\boldsymbol{F}\mathcal{P}_A \simeq \mathbf{Sep}(A)$ (separated presheaves on A)
- $\boldsymbol{T}\mathcal{P}_A \simeq \mathbf{Sh}(A)$ (sheaves on A)

Example

- $\mathbb{B}$ is the 2-element Heyting algebra $\mathbb{B} = \{\text{true}, \text{false}\}$ with $\text{false} \leq \text{true}$.
-

$$\mathbb{B} \xrightarrow{\delta} \mathbb{B} \times \mathbb{B} \xrightarrow{\wedge} \mathbb{B}$$

Example

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- $$\mathbb{B} \xrightarrow{\delta} \mathbb{B} \times \mathbb{B} \xrightarrow{\wedge} \mathbb{B}$$
  $$\mathcal{P}_{\mathbb{B}} \xrightarrow{\mathcal{P}_{\delta}} \mathcal{P}_{\mathbb{B} \times \mathbb{B}} \xrightarrow{\mathcal{P}_{\wedge}} \mathcal{P}_{\mathbb{B}}$$

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  $$\mathcal{P}_{\mathbb{B}} \xrightarrow{\mathcal{P}_{\delta}} \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\mathcal{P}_{\wedge}} \mathcal{P}_{\mathbb{B}}$$

  $$\textbf{Sep}(\mathbb{B}) \longrightarrow \textbf{Sep}(\mathbb{B} \times \mathbb{B}) \longrightarrow \textbf{Sep}(\mathbb{B})$$

Example

- $\mathbb{B}$ is the 2-element Heyting algebra $\mathbb{B} = \{\text{true}, \text{false}\}$ with $\text{false} \leq \text{true}$.
- $$\mathbb{B} \xrightarrow{\ \delta\ } \mathbb{B} \times \mathbb{B} \xrightarrow{\ \wedge\ } \mathbb{B}$$

$$\mathcal{P}_{\mathbb{B}} \xrightarrow{\ \mathcal{P}_\delta\ } \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\ \mathcal{P}_\wedge\ } \mathcal{P}_{\mathbb{B}}$$

$$\begin{array}{ccccc}
\mathbf{Sep}(\mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B} \times \mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B}) \\
\downarrow \dashv \uparrow & & \downarrow \dashv \uparrow & & \downarrow \dashv \uparrow \\
\mathbf{Sh}(\mathbb{B}) \simeq \mathbf{Set} & & \mathbf{Sh}(\mathbb{B} \times \mathbb{B}) \simeq \mathbf{Set} \times \mathbf{Set} & & \mathbf{Sh}(\mathbb{B}) \simeq \mathbf{Set}
\end{array}$$

Example

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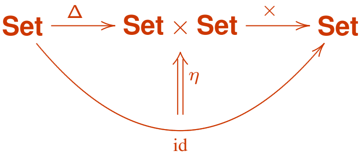
$$\mathbb{B} \xrightarrow{\quad \delta \quad} \mathbb{B} \times \mathbb{B} \xrightarrow{\quad \wedge \quad} \mathbb{B}$$

$$\mathcal{P}_{\mathbb{B}} \xrightarrow{\quad \mathcal{P}_{\delta} \quad} \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \xrightarrow{\quad \mathcal{P}_{\wedge} \quad} \mathcal{P}_{\mathbb{B}}$$

$$\begin{array}{ccccc}
\textbf{Sep}(\mathbb{B}) & \longrightarrow & \textbf{Sep}(\mathbb{B}\times\mathbb{B}) & \longrightarrow & \textbf{Sep}(\mathbb{B}) \\
\downarrow \dashv \uparrow & & \downarrow \dashv \uparrow & & \downarrow \dashv \uparrow \\
\textbf{Sh}(\mathbb{B}) \simeq \textbf{Set} & \xrightarrow[\Delta]{} & \textbf{Sh}(\mathbb{B}\times\mathbb{B}) \simeq \textbf{Set}\times\textbf{Set} & \xrightarrow[\times]{} & \textbf{Sh}(\mathbb{B}) \simeq \textbf{Set}
\end{array}$$

Example

- Comparing the composition of the images of the tripos transformations with the image of the composition we get



- The tripos-to-topos construction seems to be an oplax functor!
- The tripos-to-topos construction can not be a biadjunction (in the usual sense)

Part 2

Dc-categories

Abstract biadjunctions

- Ordinary biadjunctions are biadjunctions in the ‘tricategory’ of 2-categories, and strong functors and transformations
- General lax/oplax functors and transformations do not form a tricategory, we can not define the necessary compositions
- We will define a class of oplax functors and transformations that are as well behaved their strong counterpart, and which admit a notion of ‘oplax’ biadjunction

Dc-categories

Definition

1. A dc-category is given by a 2-category $\mathscr{C}$ together with a designated subclass $\mathscr{C}_r$ of the class of all 1-cells which contains identities and is closed under composition and vertical isomorphisms.
   Elements of $\mathscr{C}_r$ are called regular 1-cells.
   We call a dc-category geometric, if all left adjoints in it are regular.
2. A special functor between dc-categories $\mathscr{C}$ and $\mathscr{D}$ is an oplax functor $F : \mathscr{C} \to \mathscr{D}$ such that Ff is a regular 1-cell whenever f is a regular 1-cell, all identity constraints $FI_A \to I_{FA}$ are invertible, and the composition constraints $F(gf) \to Fg\,Ff$ are invertible whenever g is a regular 1-cell.
3. A special transformation between special functors F, G is an olax natural transformation $\eta : F \to G$ such that all η_A are regular 1-cells and the naturality constraint $\eta_B\,Ff \to Gf\,\eta_A$ is invertible whenever f is a regular 1-cell.

The relevant dc-categories

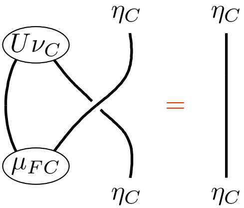
- Toposes form a dc-category, where the 1-cells are the functors that preserve finite limits, and the regular 1-cells are those which additionally preserve epimorphisms
- Triposes form a dc-category, there the 1-cells are the tripos morphisms, and the regular 1-cells are the regular tripos morphisms
- Q-toposes form a dc-category where the 1-cells are the functors that preserve finite limits, and the regular 1-cells additionally preserve epis regular epis.
- All these dc-categories are geometric

Special biadjunctions

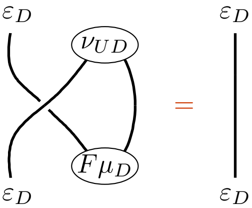
A special biadjunction between dc-categories $\mathscr{C}$ and $\mathscr{D}$ is given by

- special functors $\quad F : \mathscr{C} \to \mathscr{D} \qquad U : \mathscr{D} \to \mathscr{C}$,
- special transformations $\quad \eta : \mathrm{id}_{\mathscr{C}} \to UF \qquad \varepsilon : FU \to \mathrm{id}_{\mathscr{D}}$
- invertible modifications $\quad \mu : \mathrm{id}_U \to U\varepsilon \circ \eta U \qquad \nu : \varepsilon F \circ F\eta \to \mathrm{id}_F$

such that the equalities



and



hold for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

Properties of special biadjunctions

- If they exist, special biadjoints are unique up to equivalence.
- For any special biadjunction $F \dashv U$, the right adjoint U is strong.

Main result

Theorem

The functors $\boldsymbol{T} : \textbf{Trip} \to \textbf{Top}$ *and* $\boldsymbol{S} : \textbf{Top} \to \textbf{Trip}$ *give rise to a special biadjunction*

$$\boldsymbol{T} \dashv \boldsymbol{S}$$

between triposes and toposes.

The proof is lengthy but straightforward.

Part 3

Related concepts

Related concepts

- In his article ‘Fibrations and partial products in a 2-category’ [Johnstone, 1993], Peter Johnstone defines what we call a special transformation.
- In ‘Dualizations and antipodes’ by Day, McCrudden and Street [Day et al., 2003], the authors define what we call a special functor, for the special case of dc-categories where the regular 2-cells are the left adjoints.
- Furthermore, there are the concepts of ‘proarrow equipment’ and ‘framed bicategory’, at which we will have a closer look.

The fine print: In the following, I will not be not be precise about the distinction strict versus strong, and about dualization (there are two ways to dualize 2-categories, the presented ideas were studied in different dualized versions by different authors).

Proarrow equipments

Definition

A proarrow equipment [Wood, 1982] is a 2-functor $E : \mathscr{K} \to \mathscr{M}$ which is locally full and faithful and bijective on objects, such that for every 1-cell f in $\mathscr{K}$, Ef has a right adjoint in $\mathscr{M}$.

- A locally full and faithful 2-functor which is bijective on objects is the same thing as a 2-category with a specified subclass of 1-cells that is closed under composition.
- This means that we can (almost) identify proarrow equipments with dc-categories where every regular 1-cell has a right adjoint
- However, we are more interested in the reverse inclusion (all left adjoints are regular).

Dc-categories as double categories

Definition

A double category $\mathfrak{C}$ is an internal category in **Cat**, represented by a span

$$\mathbb{C}_0 \xleftarrow{L} \mathbb{C}_1 \xrightarrow{R} \mathbb{C}_0$$

with suitable composition and identity functors.

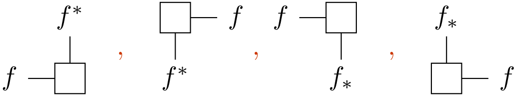
From a dc-category $\mathscr{C}$, we can construct a double category $\tilde{\mathscr{C}}$ as follows:

- $\tilde{\mathscr{C}}$ has the same objects as $\mathscr{C}$
- Horizontal 1-cells of $\tilde{\mathscr{C}}$ arbitrary 1-cells of $\mathscr{C}$
- Vertical 1-cells of $\tilde{\mathscr{C}}$ are regular 1-cells of $\mathscr{C}$
- A 2-cell $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & \searrow^{\alpha} & \downarrow{\scriptstyle j} \\ C & \xrightarrow[g]{} & D \end{array}$ in $\tilde{\mathscr{C}}$ is a 2-cell $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle j} \\ C & \xrightarrow[g]{} & D \end{array}$ in $\mathscr{C}$.

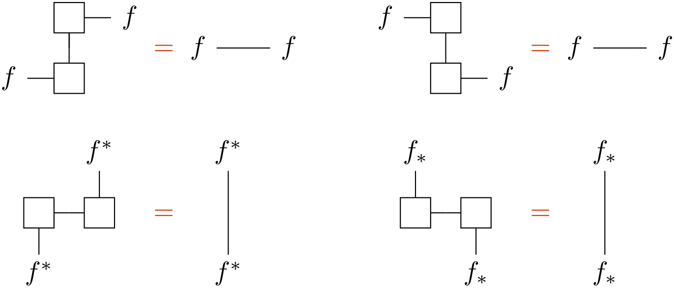
The double categories that are obtained this way from proarrow equipments are precisely Michael Shulman's framed bicategories [Shulman, 2008].

Framed bicategories

A framed bicategory is a double category $\mathscr{D}$ where for each vertical 1-cell $f : A \to B$ there exist horizontal 1-cells $f^* : A \to B$ and $f_* : B \to A$ and 2-cells



such that



General dc-categories as double categories

- The double categories that we obtain as $\tilde{\mathscr{C}}$ from general dc-categories (withoug adjoints) are precisely those, where for every horizontal f there exists f^* (not necessarily f_*), with the associated 2-cells, such that the corresponding equations hold.
- In the following, we will call such a double category a semi-framed bicategory

Dc-categories as double categories

- Why is the double categorical point of view interesting?
- Because the seemingly ad hoc concepts of special functor and special transformation arise naturally in this context.
- Slogan:

  > *Double categories are the natural home of (op)lax functors and transformations.*

- This idea comes from the canadian school of category theory (Paré, Pronk), was further developed and promoted by Shulman.

Double functors and double transformations

- An oplax double functor F between double categories $\mathfrak{C} = \mathbb{C}_0 \xleftarrow{L} \mathbb{C}_1 \xrightarrow{R} \mathbb{C}_0$ and $\mathfrak{D} = \mathbb{D}_0 \xleftarrow{L} \mathbb{D}_1 \xrightarrow{R} \mathbb{D}_0$ is given by a pair of functors

  $$F_0 : \mathbb{C}_0 \to \mathbb{D}_0 \quad \text{and} \quad F_1 : \mathbb{C}_1 \to \mathbb{D}_1$$

  and natural families of 2-cells

  $$\begin{array}{cl} F(g \circ f) \to Fg \circ Ff & f, g \text{ horizontal 1-cells} \\ F(U_C) \to U_{FC} & U_C, U_{FC} \text{ horizontal identities} \end{array}$$

  subject to the usual coherence conditions.
- A double transformation $\eta : F \to G : \mathfrak{C} \to \mathfrak{D}$ between double functors F, G is given by
  - for each $C \in \mathfrak{C}$ a vertical 1-cell $\eta_C : FC \to GC$, and
  - for each horizontal 1-cell $f : C \to D$ in $\mathfrak{C}$ a 2-cell $\begin{array}{ccc} FC & \xrightarrow{Ff} & FD \\ {\scriptstyle \eta_C}\downarrow & \searrow\!\!\searrow^{\eta_f} & \downarrow{\scriptstyle \eta_D} \\ GC & \xrightarrow[Gf]{} & GD \end{array}$,

  subject to coherence conditions.

Special functors as oplax double functors

- Special functors between dc-categories give rise to oplax double functors. betweent the corresponding semi-framed bicategories.
- However, not every oplax double functor between semi-framed bicategories arises this way.
- The special functors correspond to normal oplax double functors.
- Special transformations correspond to double transformations

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