An Invitation to Toposes

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Categories

A category $\mathcal{C}$ is given by

- a collection $\mathrm{Obj}(\mathcal{C})$ of *objects*
- for each pair $A, B \in \mathrm{Obj}(\mathcal{C})$ of objects, a set $\mathcal{C}(A, B)$ of *morphisms from A to B*
- for each object $A \in \mathrm{Obj}(\mathcal{C})$, a specified *identity morphism*

$$\mathrm{id}_A \in \mathcal{C}(A, A)$$

- for all triples $A, B, C \in \mathrm{Obj}(\mathcal{C})$ of objects, a *composition operation*

$$\begin{aligned} (- \circ -) : \mathcal{C}(A, B) \times \mathcal{C}(B, C) &\to \mathcal{C}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

Categories

... such that

- For morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, we have

$$(h \circ g) \circ f = h \circ (g \circ f)$$

- For each morphism $A \xrightarrow{f} B$, we have

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$$

Examples of Categories

The following classes of mathematical structures form categories in a natural way:

- Groups and group homomorphisms
- Topological spaces and continuous functions
- Sets and functions

Examples of Categories — continued

Observations

- Size issues — Categories seem to be very “large” structures
- Last example seems boring — there is no structure
- However, the category of sets interests us, because it is the

mathematical universe

Motivation for Definition of Topos

- We will study the category of sets categorically
- What makes this category special?
- Which properties are necessary to justify calling it a mathematical universe?
- Can we express the constructions performed in mathematics (e.g. Analysis) categorically?
- Answers to these questions will give axiomatization of topos

Motivation for Definition of Topos

Concept of topos:
"Category theoretical abstraction of the necessary structure that makes the category of sets a mathematical universe"

A topos is a category that may be viewed as a mathematical universe, just like the category of sets.

What are the basic constructions used in mathematics?

Cartesian Product of Sets

Given two sets M, N, define their *cartesian product*

$$M \times N := \{(m, n) \mid m \in M, n \in N\},$$

where

$$(m, n) := \{\{m, n\}, n\}.$$

Associated to $M \times N$ are the projections

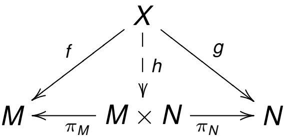
$$\begin{aligned} \pi_M : M \times N &\to M \\ (m, n) &\mapsto m \end{aligned} \qquad \begin{aligned} \pi_N : M \times N &\to N \\ (m, n) &\mapsto n \end{aligned}$$

Categorical characterization of cartesian product

- Important about the cartesian product $M \times N$ is the following property:

Universal property of the cartesian product

Given a set X, and functions $f : X \to M$ and $g : X \to N$, there exists *a unique* function $h : X \to M \times N$ such that $\pi_M \circ h = f$ and $\pi_N \circ h = g$.



Definition

If a cone $A \xleftarrow{\pi} P \xrightarrow{\pi'} B$ in a category $\mathcal{C}$ has the above universal property, then we call P a *product* of A and B.

- Observe principle of description by universal property, typical for category theory

Comparison

- Two approaches to cartesian product:
  - set theoretic description
  - universal characterization
- Connection to computer science: implementation vs specification
- Philosophical: atomistic approach vs holistic approach

The terminal object

- Set $\{*\}$ with one element needed to talk about elements categorically
- What is the right universal characterization of the one-element set?

The terminal object

Universal property of the one-element set

Given an arbitrary set M, there exists a unique function

$$!_M : M \to \{*\}$$

Definition, notation

If an object A of a category $\mathcal{C}$ has the above universal property, we call it a *terminal object*, and denote it (ambiguously) by **1**.

The natural numbers

Definitions

Let $\mathbb{N}$ denote the set $\{0, 1, 2, 3, ...\}$ of natural numbers with 0 and define the functions

$$\begin{aligned} \text{zero} : \mathbf{1} &\to \mathbb{N} \\ * &\mapsto 0 \end{aligned} \qquad\qquad \begin{aligned} \text{succ} : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n + 1 \end{aligned}$$

- What are the fundamental properties that characterize the set of natural numbers?
- Induction and (primitive) recursion!
- This is the idea behind the following universal characterization:

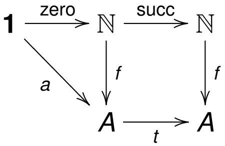
The natural numbers

Universal characterization of the natural numbers

Given a set A, and functions

$$\mathbf{1} \xrightarrow{a} A \xrightarrow{t} A,$$

there exists a unique $f : \mathbb{N} \to A$ such that $f \circ \text{zero} = a$ and $f \circ \text{succ} = \text{succ} \circ t$.



Definition

Let $\mathcal{C}$ be a category with terminal object. If object $\mathbb{N}$ of $\mathcal{C}$, together with morphisms zero and succ as above, satisfies the above universal property, then it is called a *natural number object* (often abbreviated NNO).

Sets of functions

- Many mathematical constructions, e.g. in analysis, involve sets of functions

Definitions

Given sets A, B, we define

$$B^A := \{f \mid f \text{ is a function from } A \text{ to } B\} \qquad \text{and}$$

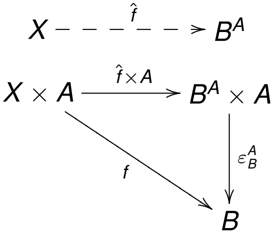
$$\begin{aligned} \varepsilon_B^A : B^A \times A &\to B \\ (f, a) &\mapsto f(a) \end{aligned}$$

- So what is the universal property of function spaces?
- It's related to currying (known from functional programming)!

Sets of functions

The universal property of sets of functions

For each set X and function $f : X \times A \to B$, there exists a unique function $\hat{f} : X \to B^A$ such that $\varepsilon_B^A \circ (\hat{f} \times A) = f$



Definition

If in a category $\mathcal{C}$, for given objects A, B there exists an object B^A and a morhpism ε_B^A with the universal property above, then we call B^A an *exponential object* of A and B, and and we call ε_B^A the *evaluation morphism*.

Subsets and predicates

- Subsets are *extensions* of predicates:
  Given a predicate $\varphi(x)$ on a set M, we may form the subset

  $$\{x \in M \mid \varphi(x)\}$$

  of M (Cantor's comprehension principle).
- Lots of mathematical constructions involve subsets — how can we characterize this algebraically in the language of categories?
- In the category of sets, subsets correspond to injective functions:

Subsets and predicates

Correspondence between subsets and injective functions

- Given a set M and a subset $U \subseteq M$, there is a canonical injective function

$$\begin{aligned} \iota_U : U &\to M \\ x &\mapsto x \end{aligned}$$

- Conversely, any injective function

$$m : U \rightarrowtail M$$

  gives rise to a subset

$$\{y \in M \mid \exists x \in U \,.\, m(x) = y\}$$

  of M

Subsets and predicates

- A categorical generalization of “injective function” is the notion of *monomorphism*.

Definition

A morphism $m : A \to B$ in a category $\mathcal{C}$ is called a *monomorphism* if for all objects X of $\mathcal{C}$ and all pairs of morphisms $f, g : X \to A$, $m \circ f = m \circ g$ already implies $f = g$.

$$X \underset{g}{\overset{f}{\rightrightarrows}} A \overset{m}{\rightarrowtail} B$$

Note: For monomorphisms, we use the arrow symbol $\rightarrowtail$.

Solving equations

- Classical mathematical problem:
  Solve an equation, i.e. given two functions $f, g : M \to N$, determine the subset U of M, where f and g coincide.

$$U = \{x \in M \mid f(x) = g(x)\}$$

- With arrows, this looks as follows:

$$U \overset{\iota}{\rightarrowtail} M \underset{g}{\overset{f}{\rightrightarrows}} N$$

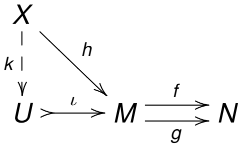
- Can we characterize this subset inclusion categorically?
- Yes, we can!

Solving equations

Universal property of solution sets

With the notation of the previous slide, we have

1. $f \circ \iota = g \circ \iota$,   and
2. for any set X and any function $h : X \to M$, if $f \circ h = g \circ h$, then there exists a unique function $k : X \to U$ with $\iota \circ k = h$.



Definition

In a category $\mathcal{C}$, given a parallel pair $f, g : A \to B$ of morphisms, a morphism ι with the above universal property is called an *equalizer* of f and g.

Note: We do not require ι to be a monomorphism; rather, this fact can be *deduced* from the universal property.

Pullbacks

- Cartesian products, terminal objects and equalizers are instances of the more general concept of *finite limit*.
- *Pullbacks* are another important type of finite limit.

Pullbacks

Definition

Given a diagram of shape $\begin{array}{ccc} & & A \\ & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$ in a category $\mathcal{C}$, a *pullback* is given by an object P and two morphisms $p : P \to A, \quad q : P \to B$ such that

1. $\begin{array}{ccc} P & \xrightarrow{p} & A \\ q\downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$ commutes, and
2. given any pair h, k of morphisms such that $\begin{array}{cccc} X & & \xrightarrow{h} & \\ & \searrow^{k} & & A \\ & & & \downarrow f \\ & B & \xrightarrow{g} & C \end{array}$, there exists a *unique* $m : X \to P$ such that $p \circ m = h$ and $q \circ m = k$, as indicated in the following diagram.

$$\begin{array}{ccccc} X & & \xrightarrow{h} & & \\ & \searrow^{m} & & \searrow & \\ & & P & \xrightarrow{p} & A \\ & \searrow^{k} & \downarrow q & & \downarrow f \\ & & B & \xrightarrow{g} & C \end{array}$$

Pullbacks

- Pullbacks can be described in terms of products and equalizers
- In set, we have $P = \{(a, b) \in A \times B \mid f(a) = g(b)\}$ (referring to the diagram of the previous slide)
- Different viewpoints on pullbacks $\Rightarrow$ different notations:
  - $P = B \times_C A$ (P is the fibred product of B and A over C)
  - $q = g^*(f)$ (q is the pullback of f along g)
  - $p = f^*(g)$ (p is the pullback of g along f)
- Pullbacks of monomorphisms (along arbitrary morphisms) are monomorphisms
- In the category of sets, we can represent monomorphisms by subsets, and for $f : M \to N$ and $U \subseteq N$, we have

$$f^*(U) = \{x \in M \mid f(x) \in U\}$$

Truth values and characteristic functions

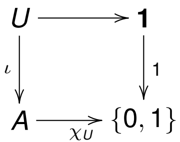
- Given a set M and a subset $U \subseteq M$, we can construct U's *characteristic function*

$$\chi_U : M \to \{0, 1\}$$
$$m \mapsto \begin{cases} 1 & m \in U \\ 0 & \text{else} \end{cases}$$

- $\{0, 1\}$ can be thought as the set of *truth values*, 1 meaning 'true', and 0 meaning 'false'.
- $\chi_U(m)$ says whether it is true that m is in U.
- We can recover U from χ_U:

$$U = \chi_U^*(\{1\})$$

- As a diagram:



Truth values and characteristic functions

Definition

Let $\mathcal{C}$ be a category with pullbacks and a terminal object.
A *subobject classifier* in $\mathcal{C}$ is given by a object Ω together with a morphism $\top : \mathbf{1} \to \Omega$ such that for all monomorphisms $m : U \rightarrowtail A$, there exists a *unique* morphism $\chi_m : A \to \Omega$ with $\chi_m^*(\top) \cong m$.

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{1} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \top} \\ A & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

- Intuitively, Ω is the set of truth values, and $\top$ denotes the truth value ‘true’.
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Toposes

Definition

A *topos* is a category $\mathcal{C}$ that has

1. all binary products,
2. a terminal object,
3. all exponentials, and
4. a subobject classifier

- From the viewpoint of ordinary mathematics, the concept of subobject classifier seems a bit weird, whereas we would like to have power sets. These however can be defined as

$$\mathbf{P}(A) = \Omega^A$$

The internal logic

- The internal ‘predicates’ of a topos are the subobjects
- Logical operations ($\wedge, \vee, \Rightarrow, \top, \bot, \exists, \forall$) can be made explicit in a topos as algebraic operations on the subobjects characterized by universal properties!
- Formally, this means that we can define a semantic for a language of higher order logic in any topos
- Surprise:
- The internal logic is not classical, but constructive!

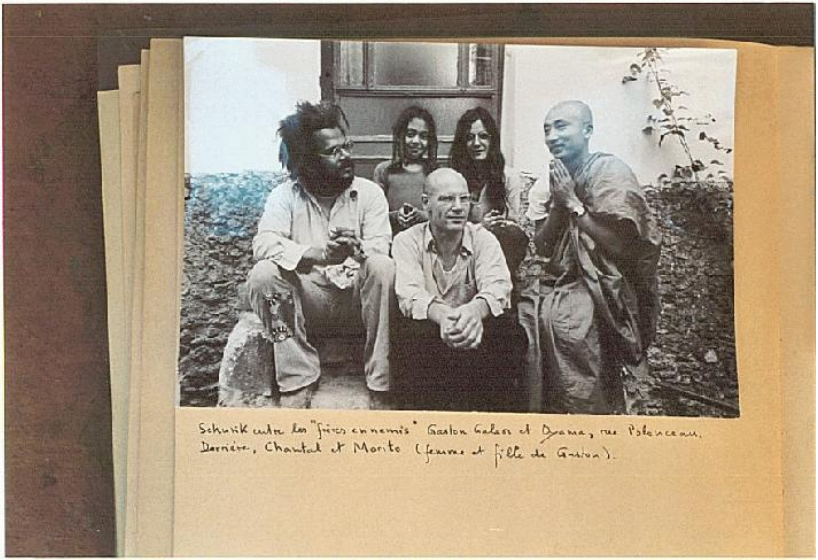

Figure: Alexander Grothendieck (middle) with some of his PhD students