Cubes, homotopy and process algebra

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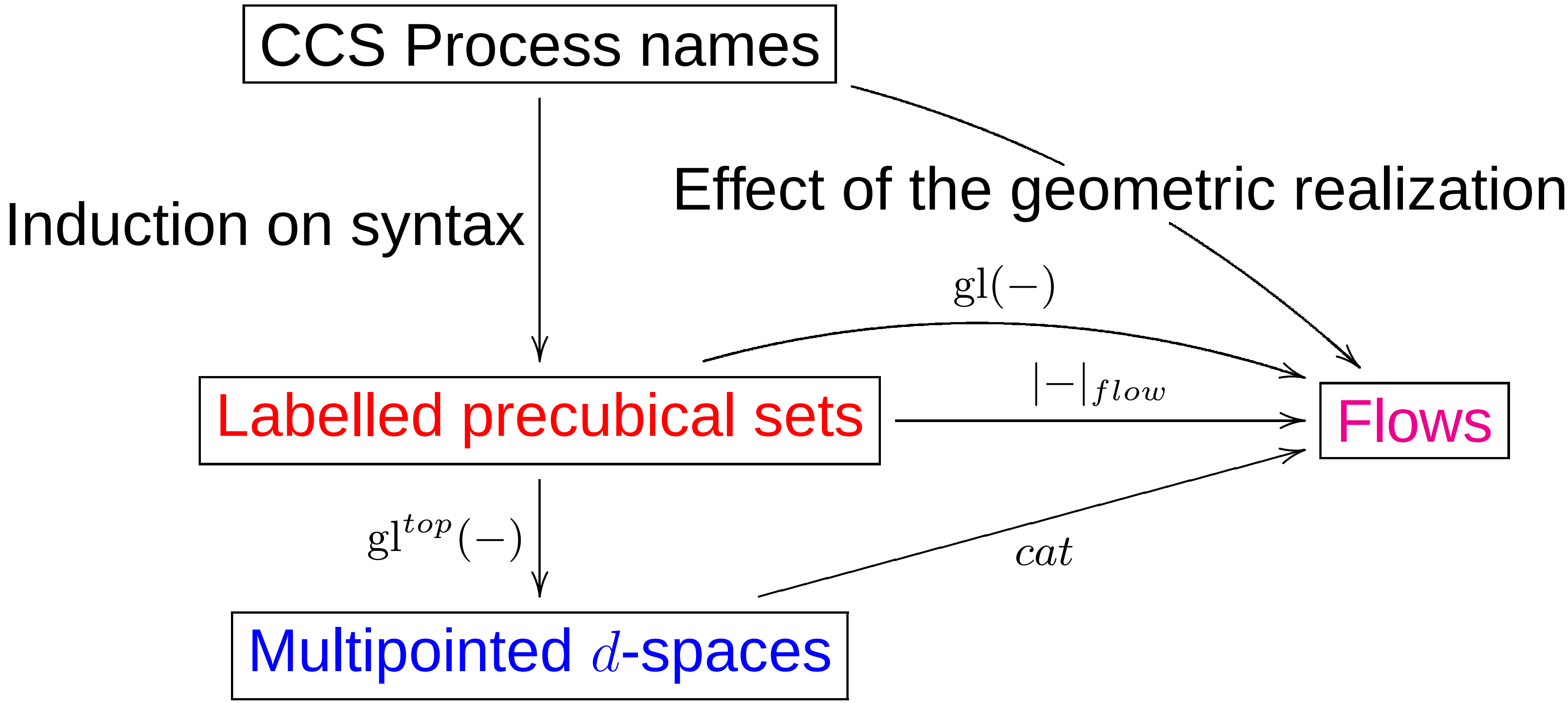
Overview of the talk

- Paradigm of higher dimensional automata (HDA)
- Modelling concurrency combinatorially, topologically, categorically



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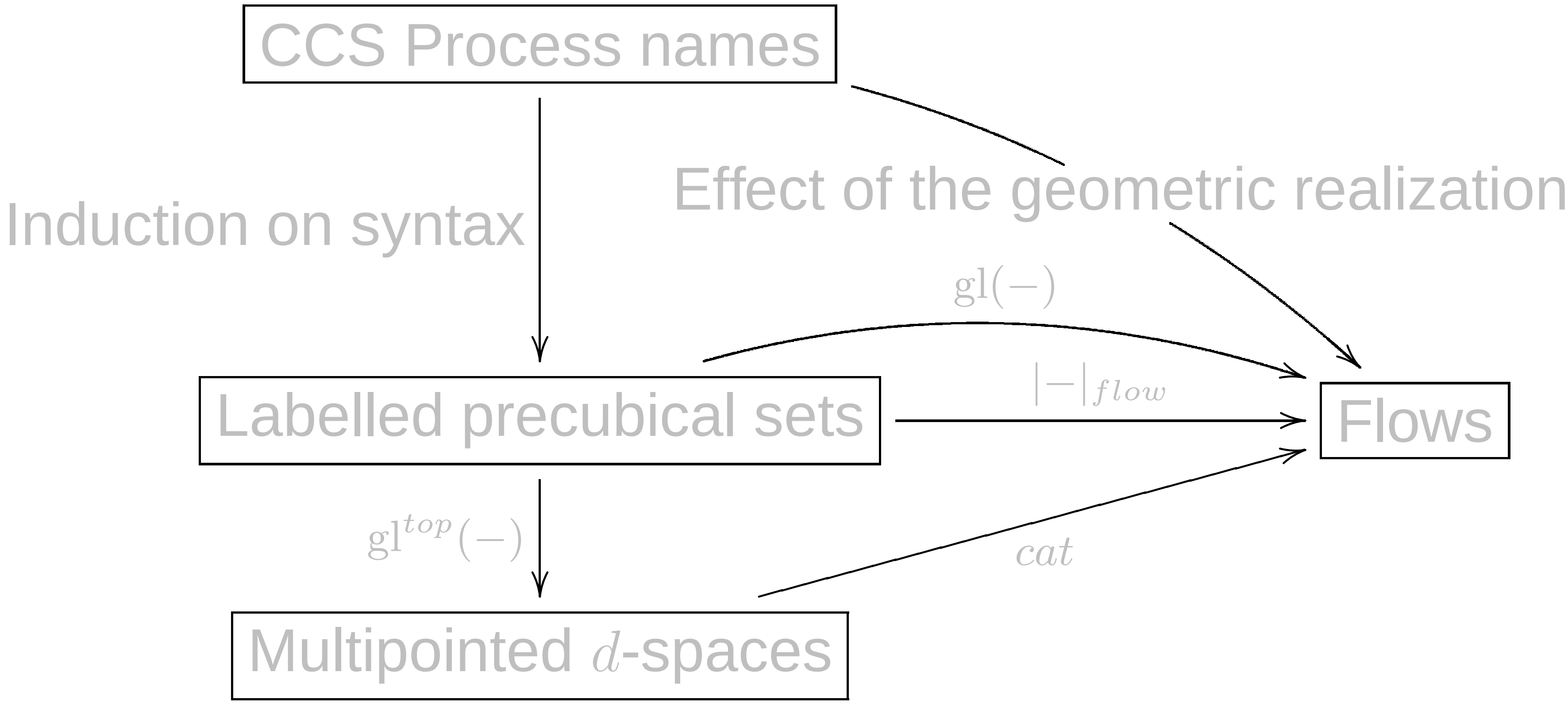


HDA paradigm

- One n-transition (concurrent execution of n actions) corresponding to one full n-cube

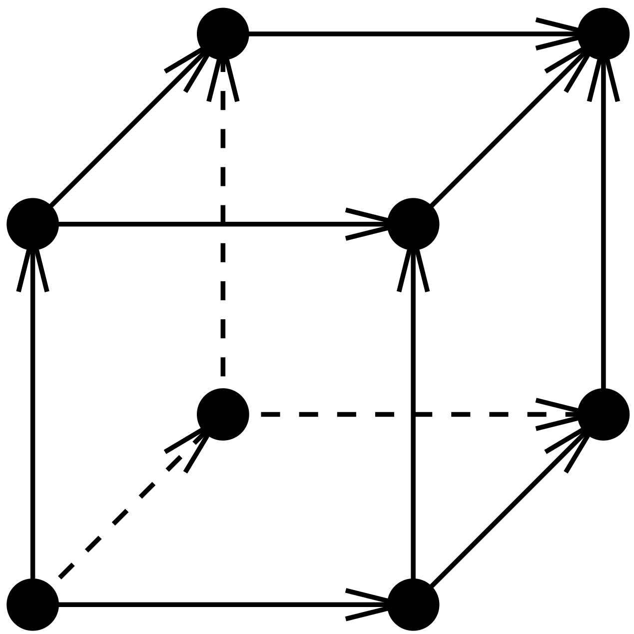
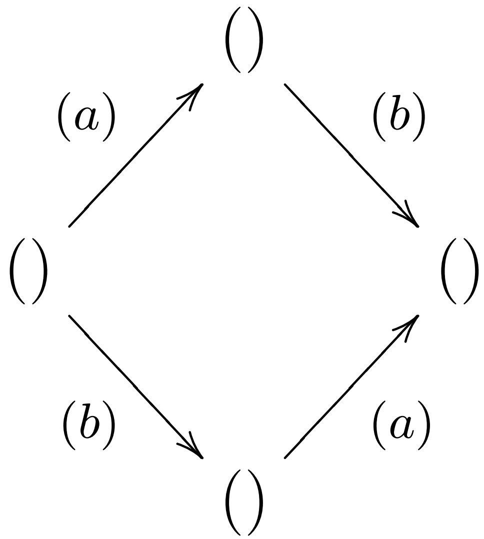


Figure 1: The full 3-cube

Example of impossible HDA

- Start from an empty labelled 2-cube



- Add two squares
- Impossible HDA since either a and b run sequentially (empty case), or a and b run concurrently (full case)

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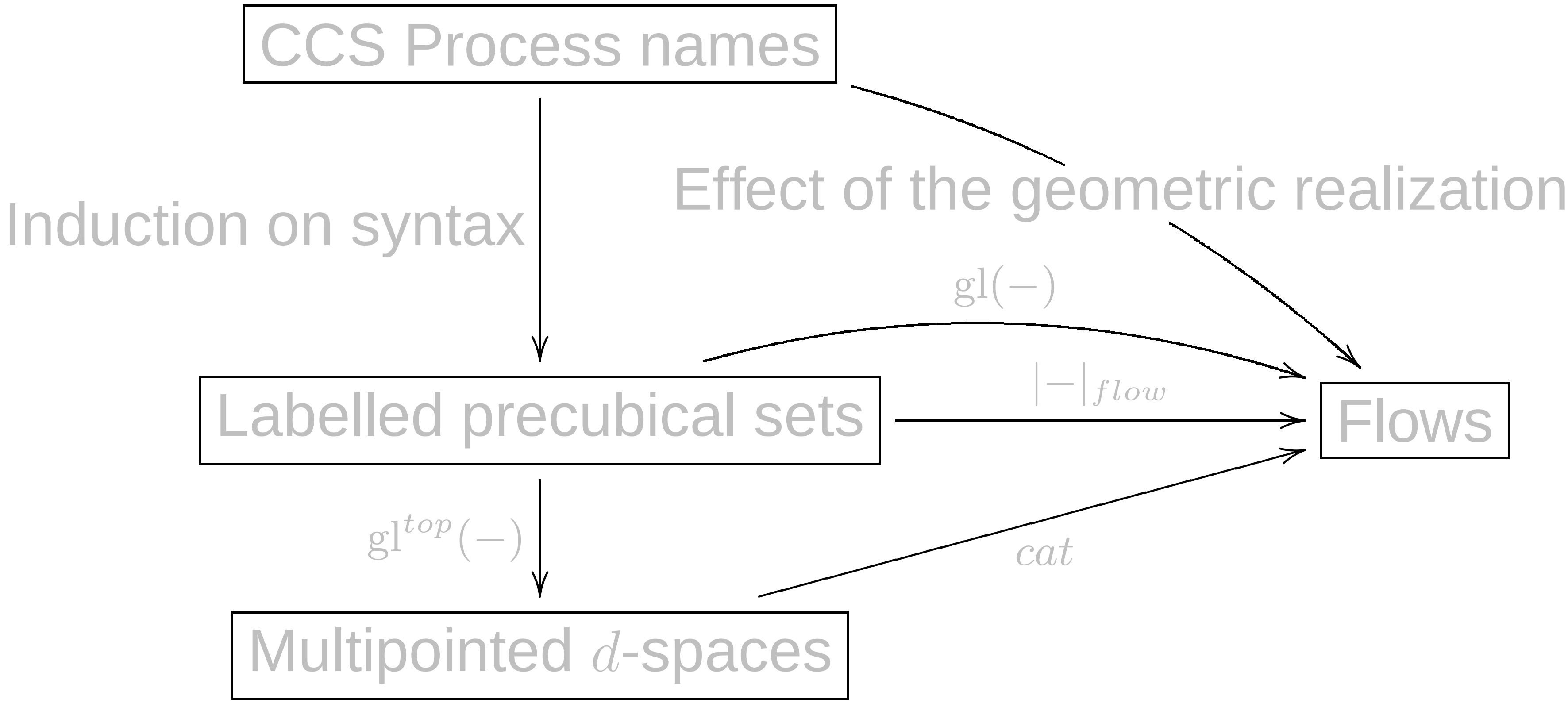

Computer in idle state

- Combinatorics: a point α
- Topology: the one-point topological space
- Category: one object α, no morphisms
- NB: the category with one object α does not model a computer in idle state: $\mathrm{Id}_\alpha^n = \mathrm{Id}_\alpha$ is interpreted as n times the same action executed with n unobservable

The directed segment

- Combinatorics: Directed graph with two vertices α and β and an edge from α to β
- Topology: $[0,1]$, the total ordering modeling time ordering
- Category: two objects α and β and a unique morphism from α to β (no other morphisms)

Modelling concurrency by combinatorics

- $\Sigma = \{a, b, c, \ldots\} \cup \{\overline{a}, \overline{b}, \overline{c}, \ldots\} \cup \{\tau\}$ with $\overline{\overline{a}} = a$

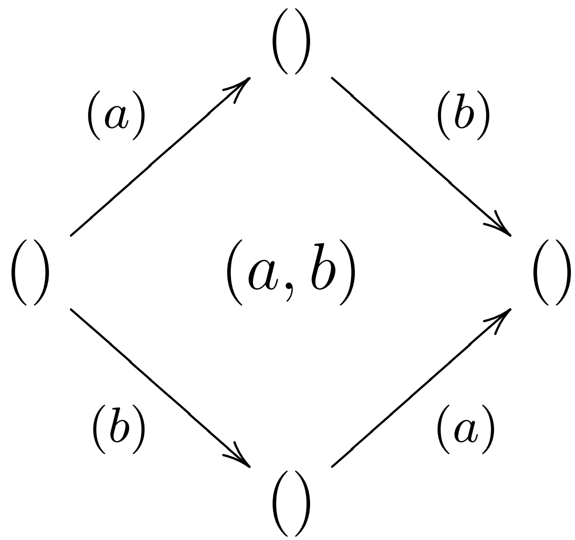

Figure 2: Concurrent execution of a and b

Modelling concurrency by topology

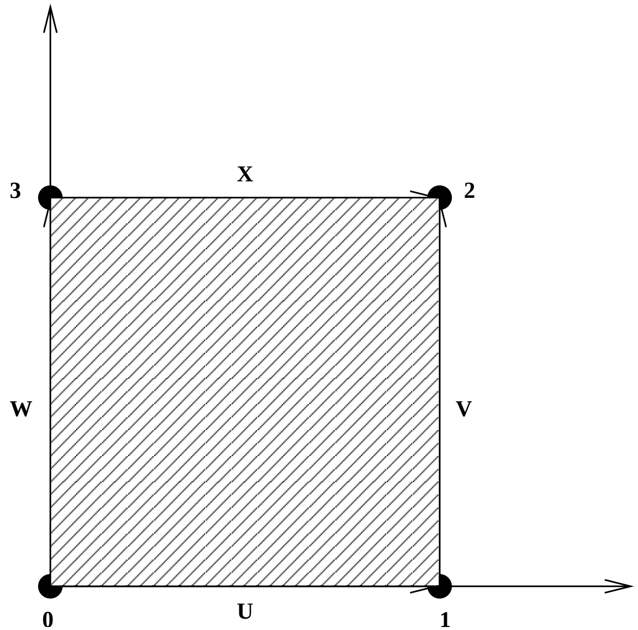

Figure 3: Two concurrent processes

- Each coordinate corresponding to one process
- n processes = n coordinates

Modelling concurrency by category

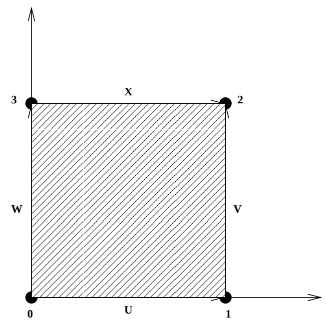

Figure 4: Two concurrent processes

- Four objects $\{0, 1, 2, 3\}$
- Four morphisms $\{U, V, W, X\}$ with two "opposite" morphisms labelled by the same action
- $U * V = W * X$
- The categorical object must not contain any loop

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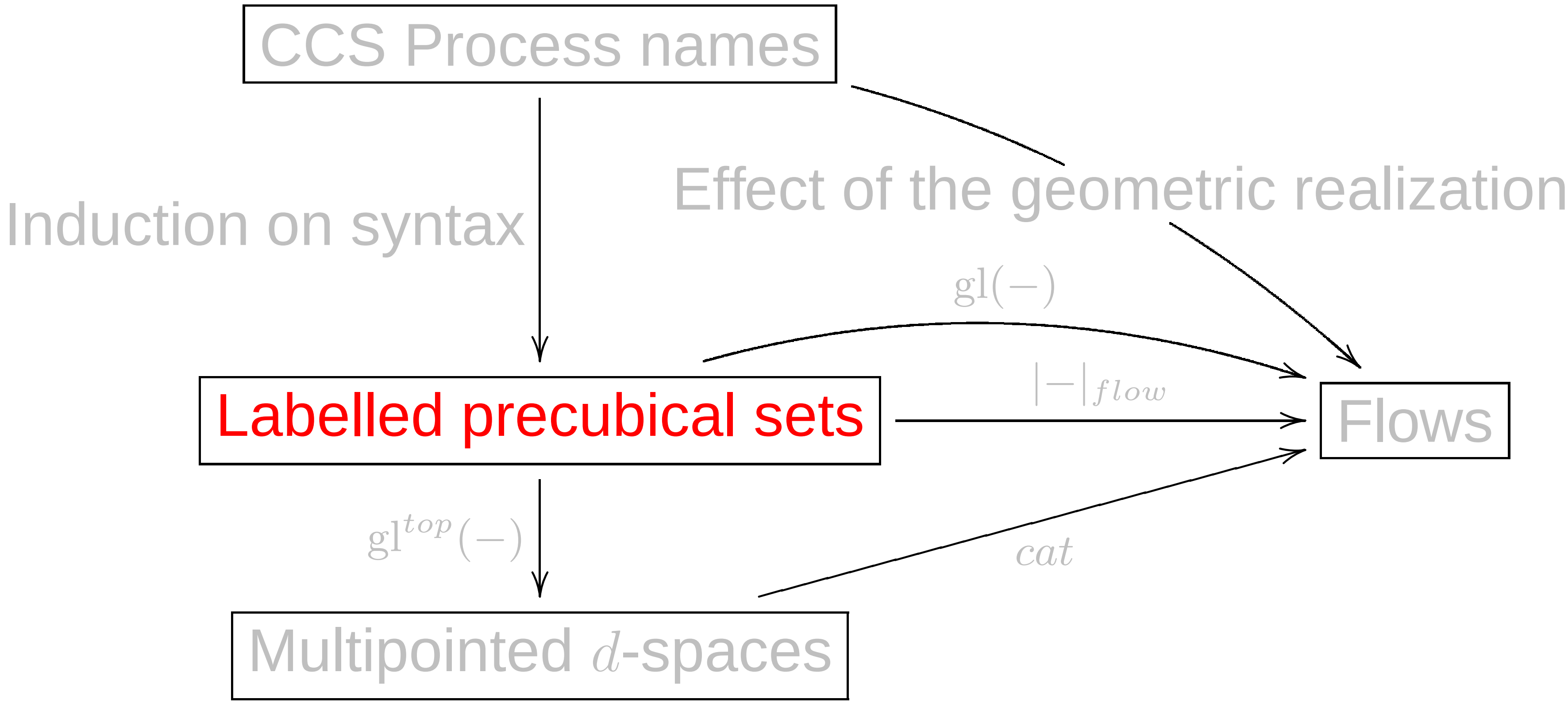

Labelled precubical set

- Precubical set K: cubical set without degeneracy maps : presheaf over $\square$
- The n-cube $\square[n]$
- The boundary of the n-cube $\partial\square[n] := \square_{\leqslant n-1}$
- Precubical set of labels $!\Sigma$
  - $(!\Sigma)_0 = \{()\}$
  - for $n \geqslant 1$, $(!\Sigma)_n = \Sigma^n$
  - $\partial_i^0(a_1, \ldots, a_n) = \partial_i^1(a_1, \ldots, a_n) = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$
- Labelled precubical set: $\ell : K \longrightarrow !\Sigma$
- Two opposite faces have same labelling

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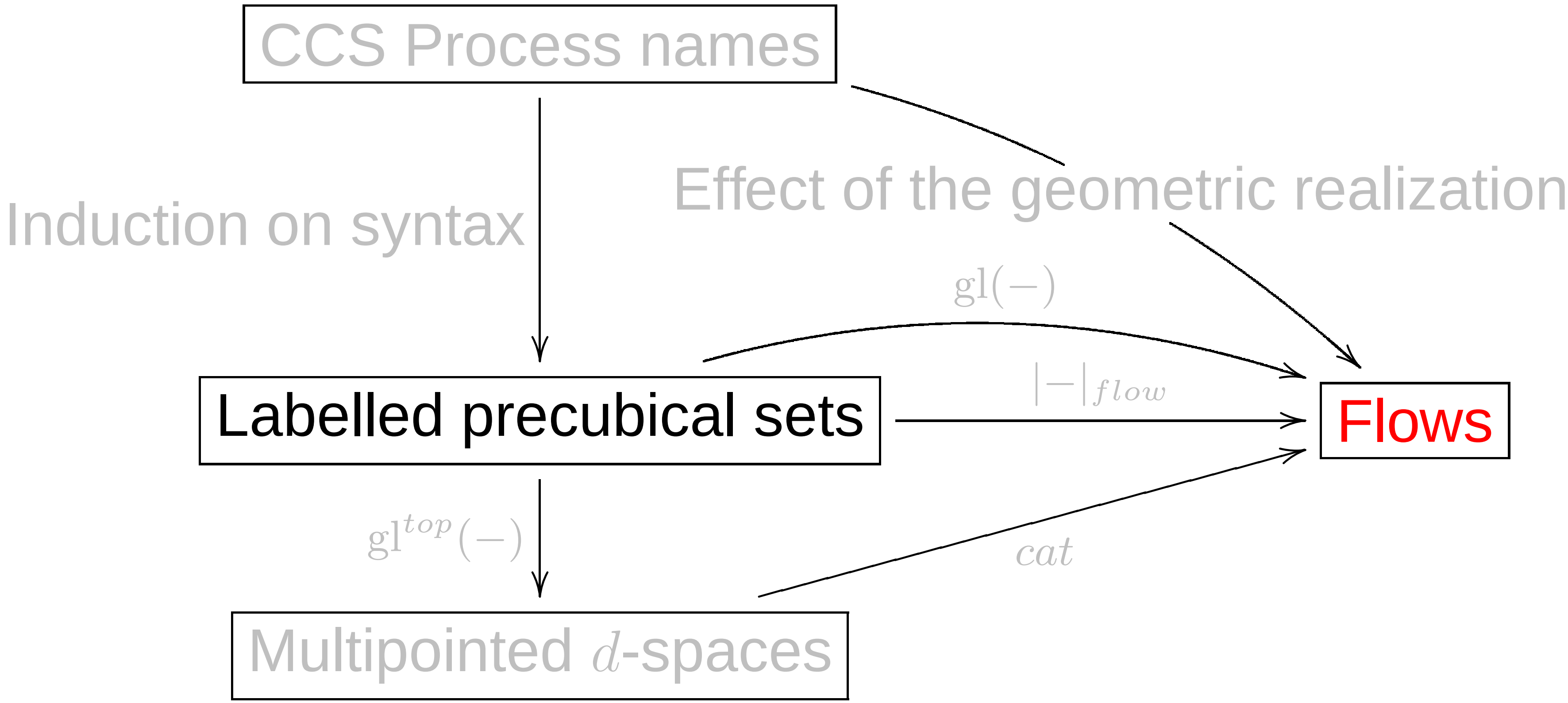


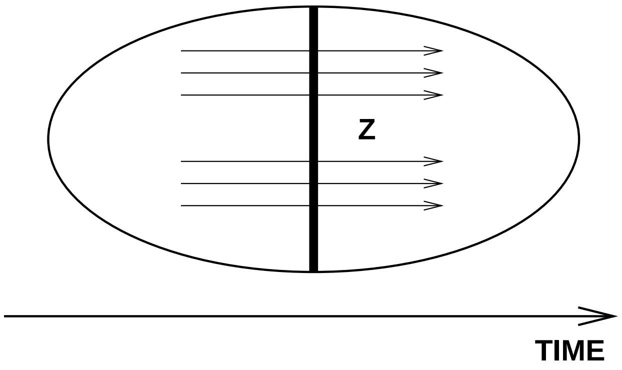
Time flow

- Flow X = small category without identity maps enriched over Δ-generated spaces (colimit of simplices)
- Set of objects X^0 modelling the states of the concurrent system
- Space of morphisms $\mathbb{P}X$ modelling the non-constant execution paths of the concurrent system
- $f : X \to Y$ weak S-homotopy equivalence if $f^0 : X^0 \to Y^0$ bijection and $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ weak homotopy equivalence

Categorical structure of Flow

- Locally presentable
- Tensored and cotensored over Δ-generated spaces with $\mathbf{Flow}(X \otimes K, Y) \cong \mathbf{Top}(K, \mathbf{FLOW}(X, Y)) \cong \mathbf{Flow}(X, Y^K)$
- Combinatorial proper simplicial model category with class of weak equivalences the weak S-homotopy equivalences

Globe of a topological space



- The globe $\mathrm{Glob}(Z)$ of the topological space Z
  - $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$
  - $\mathbb{P}\mathrm{Glob}(Z) = Z$
  - $s = \widehat{0}$
  - $t = \widehat{1}$
  - no composable non-constant execution paths
- The directed segment $\mathrm{Glob}(\{*\}) = \overrightarrow{I}$

Weak S-homotopy model structure

- A set S can viewed as a flow with $S^0 = S$ and $\mathbb{P}S = \varnothing$
- Generating cofibrations:

$$I^{gl}_{+} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}$$

with $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$

- Generating trivial cofibrations:

$$J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \to \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}$$

- $\mathrm{Fib} = \{f : X \to Y$ s.t. $\mathbb{P}f$ Serre fibration$\}$
- Every flow is fibrant

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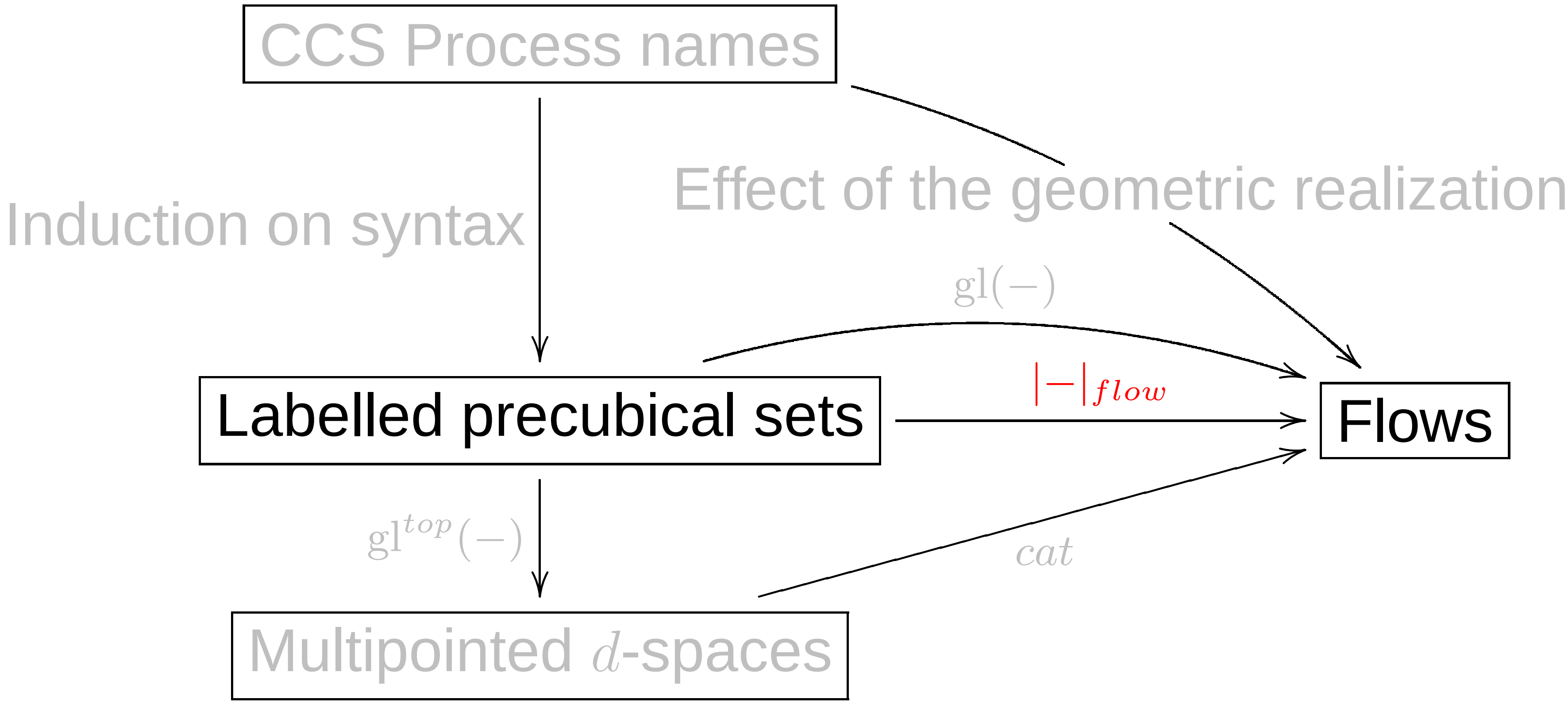


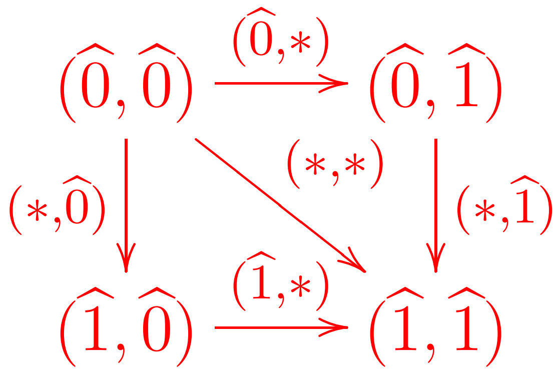
Realizing posets as flows

- $(A, \leqslant)$ poset
- Element of A = State
- Execution path from α to β iff $\alpha < \beta$
- Note $\mathbb{P}_{\alpha,\alpha}A = \varnothing$ for all $\alpha \in A$

Functor $\{$poset+strictly increasing map$\} \to \mathbf{Flow}$

Bad realization of precubical set

- $$|K|_{bad} := \varinjlim_{\square[n]\to K} \{\widehat{0} < \widehat{1}\}^n$$
- The flow $|\square[2]|_{bad} = \{\widehat{0} < \widehat{1}\}^2$:

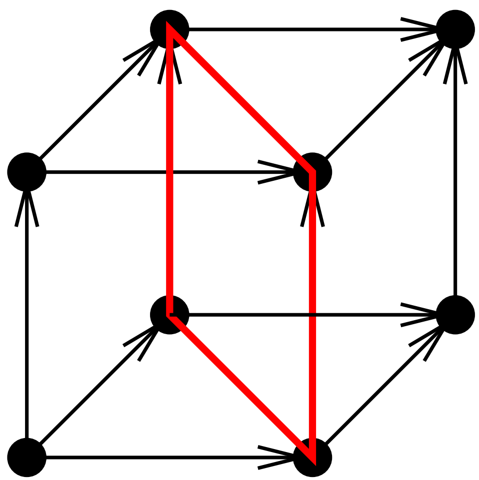


with $(\widehat{0},*) * (*,\widehat{1}) = (*,\widehat{0}) * (\widehat{1},*) = (*,*)$

- $|\partial\square[n]|_{bad} \cong |\square[n]|_{bad}$ for $n \geqslant 3$

Good realization of precubical set

- $$|K|_{flow} := \varinjlim_{\square[n]\to K} (\{\widehat{0} < \widehat{1}\}^n)^{cof}$$
- $$|K|_{flow} \simeq \operatorname{holim}_{\overrightarrow{\square[n]\to K}} \{\widehat{0} < \widehat{1}\}^n$$
- $\mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}|\partial\square[n]|_{flow}$ homotopy equivalent to S^{n-2}



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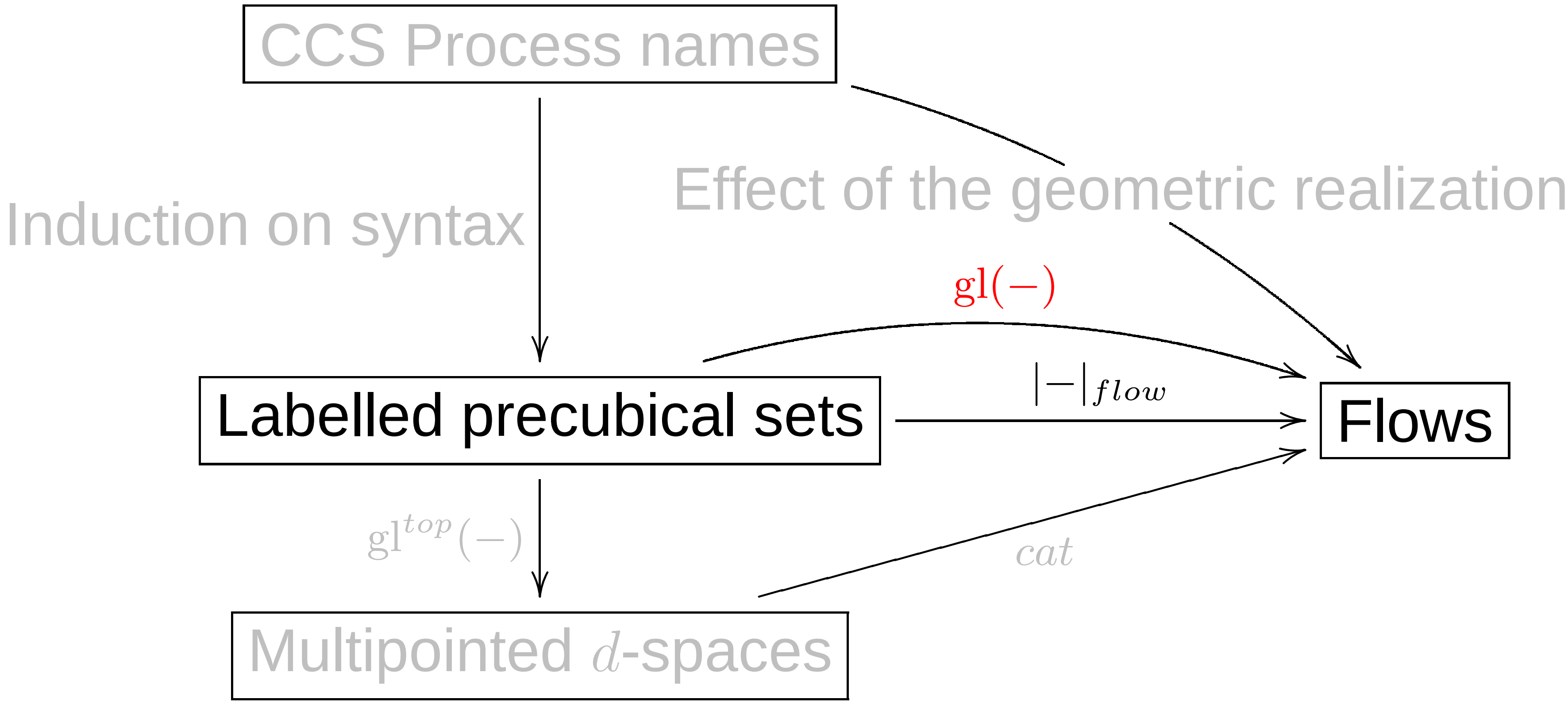

Small realization

Theorem. *There exists a colimit-preserving functor* $K \mapsto \mathrm{gl}(K)$ *such that for all* $n \geqslant 0$, *there is a pushout diagram of flows:*

$$\begin{array}{ccc}
\mathrm{Glob}(\mathbf{S}^{n-1}) & \longrightarrow & \mathrm{gl}(\partial\square[n+1]) \\
\downarrow & & \downarrow \\
\mathrm{Glob}(\mathbf{D}^{n}) & \longrightarrow & \mathrm{gl}(\square[n+1])
\end{array}$$

and such that there exist natural weak S-homotopy equivalences $\mathrm{gl}(K) \xrightarrow{\simeq} |K|_{flow}$ *and* $|K|_{flow} \xrightarrow{\simeq} \mathrm{gl}(K)$.

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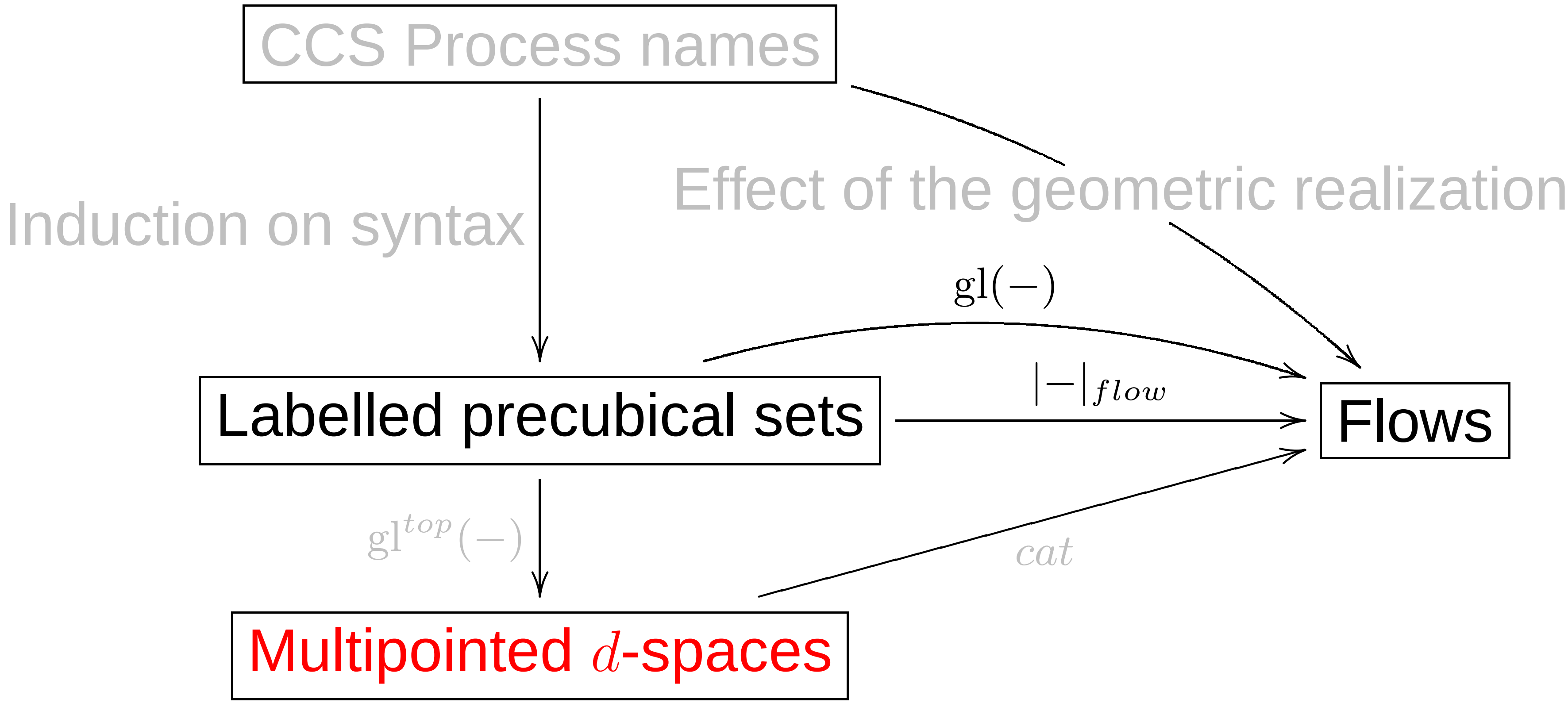


A topological version of flows

- Advantage of a topological version of the category of flows: the full subcategory of colimits of cubes have nice properties (topological, locally presentable, and also complete, cocomplete, etc...)
- The n-cubes model the concurrent execution of n actions: possibility of getting rid of meaningless geometric shapes
- See Fajstrup-Rosický's paper for another (very close) example of such a category convenient for dealing with some problems in directed algebraic topology

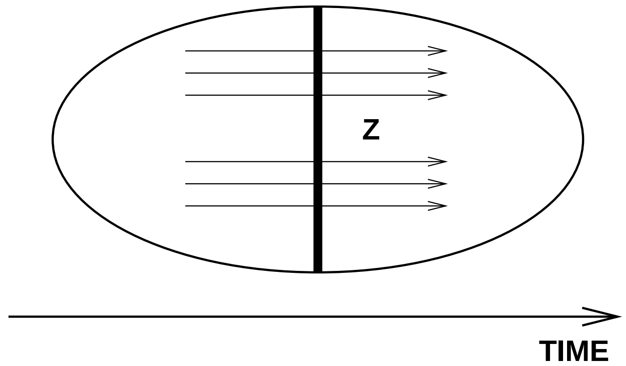
Multipointed d-space

- Multipointed d-space $(|X|, X^0, \mathbb{P}^{top}X)$
  - Δ-generated space $|X|$ together with a subset $X^0 \subset |X|$
  - $\mathbb{P}^{top}X$ set of continuous paths (called execution paths) closed under strictly increasing reparametrization and composition such that $\gamma : [0,1] \to X$ implies $\gamma(0), \gamma(1) \in X^0$
- $f : X \to Y$ map of multipointed d-spaces
  - A continuous map $|f| : |X| \to |Y|$ with $f(X^0) \subset Y^0$
  - $\phi \in \mathbb{P}^{top}X$ implies $\mathbb{P}^{top}f(\phi) := |f| \circ \phi \in \mathbb{P}^{top}Y$
- $f : X \to Y$ weak S-homotopy equivalence if $f^0 : X^0 \to Y^0$ bijection and $\mathbb{P}^{top}f$ weak homotopy equivalence

Categorical structure of MdTop

- Locally presentable
- Tensored and cotensored over Δ-generated spaces with $\mathbf{MdTop}(X \otimes K, Y) \cong \mathbf{Top}(K, \mathbf{MDTOP}(X, Y)) \cong \mathbf{MdTop}(X, Y^K)$
- Combinatorial right proper simplicial model category with class of weak equivalences the weak S-homotopy equivalences
- ($X \mapsto$ underlying set of $|X|$) is topological
- Left properness is still a conjecture

Topological globe of a topological space



- The topological globe $\mathrm{Glob}^{top}(Z)$ of the topological space Z
- $\mathrm{Glob}^{top}(Z)^0 = \{\widehat{0}, \widehat{1}\}$
- $|\mathrm{Glob}^{top}(Z)| = \{\widehat{0}, \widehat{1}\} \sqcup (Z \times [0,1])/\Big((z,0) = (z',0) = \widehat{0}, (z,1) = (z',1) = \widehat{1}\Big)$
- $\mathbb{P}^{top}\mathrm{Glob}^{top}(Z)$ closure by strict increasing reparametrization of $\{t \mapsto (z,t), z \in Z\}$

Weak S-homotopy model structure

- Discrete space S viewed as multipointed d-space with $S^0 = S$ and $\mathbb{P}^{top} S = \varnothing$
- Generating cofibrations:

$$I_+^{gl,top} = \{\mathrm{Glob}^{top}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}^{top}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}$$

  with $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$
- Generating trivial cofibrations:

$$J^{gl,top} = \{\mathrm{Glob}^{top}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}^{top}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}$$

- $\mathrm{Fib} = \{f : X \to Y$ s.t. $\mathbb{P}^{top} f$ Serre fibration$\}$
- Every multipointed d-space is fibrant

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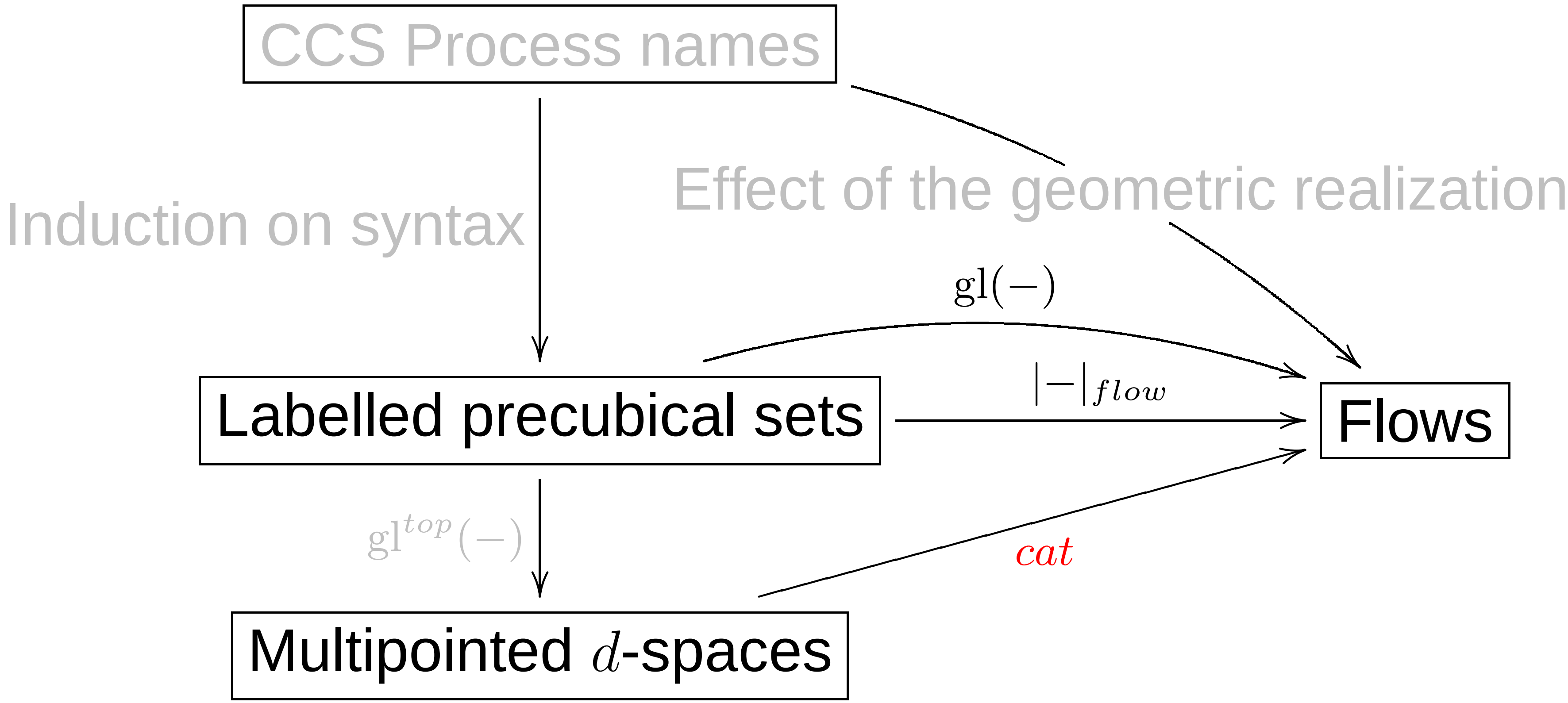

From multipointed d-spaces to flows (I)

- Multipointed d-space $X = (|X|, X^0, \mathbb{P}^{top} X)$
- Flow $cat(X)$ defined as follows:
- $cat(X)^0 := X^0$
- $\mathbb{P}cat(X)$ defined by

$$\frac{\mathbb{P}^{top} X}{\text{strictly increasing reparametrization with fixed extremities}}$$

- $cat : \mathbf{MdTop} \to \mathbf{Flow}$ well-defined functor
- Example : $cat(\mathrm{Glob}^{top}(Z)) \cong \mathrm{Glob}(Z)$

From multipointed d-spaces to flows (II)

- The composite functor

$$\mathbf{MdTop} \xrightarrow{(-)^{cof}} \mathbf{MdTop} \xrightarrow{cat} \mathbf{Flow}$$

induces an equivalence of categories

$$\mathbf{Ho}(\mathbf{MdTop}) \simeq \mathbf{Ho}(\mathbf{Flow}).$$

- The functor $cat : \mathbf{MdTop} \to \mathbf{Flow}$ preserves cofibrations, trivial cofibrations and weak S-homotopy equivalences between cofibrant objects.
- The functor $cat : \mathbf{MdTop} \to \mathbf{Flow}$ is not colimit-preserving

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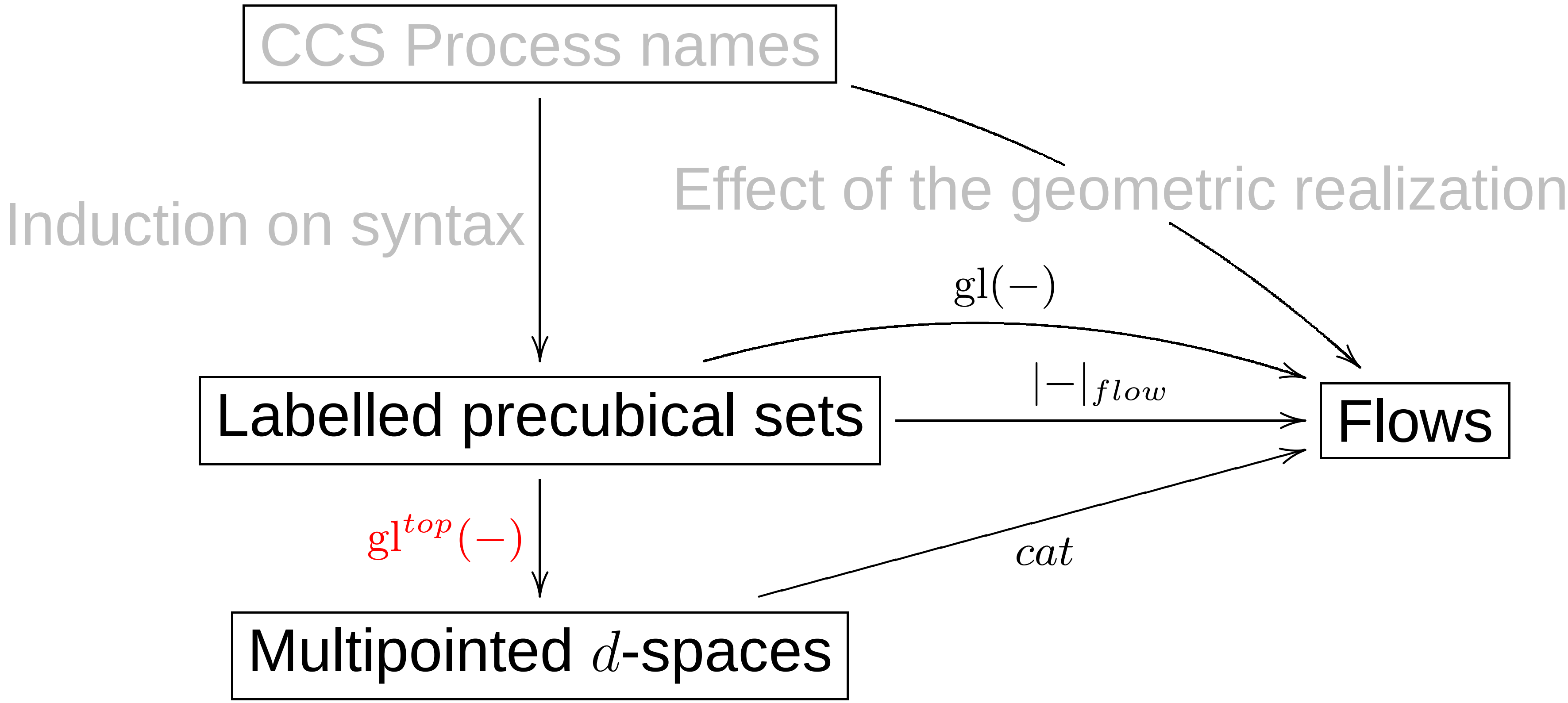

Topological realization

Theorem. *There exists a colimit-preserving functor $K \mapsto \mathrm{gl}^{top}(K)$ such that for all $n \geqslant 0$, there is a pushout diagram of multipointed d-spaces:*

$$\begin{array}{ccc} \mathrm{Glob}^{top}(\mathbf{S}^{n-1}) & \longrightarrow & \mathrm{gl}^{top}(\partial\square[n+1]) \\ \downarrow & & \downarrow \\ \mathrm{Glob}^{top}(\mathbf{D}^{n}) & \longrightarrow & \mathrm{gl}^{top}(\square[n+1]) \end{array}$$

and such that there exist natural weak S-homotopy equivalences $cat(\mathrm{gl}^{top}(K)) \xrightarrow{\simeq} |K|_{flow}$ *and* $|K|_{flow} \xrightarrow{\simeq} cat(\mathrm{gl}^{top}(K))$. *Moreover,*

$$\boxed{|\,\mathrm{gl}^{top}(K)| \simeq \varinjlim_{\square[n]\to K} [0,1]^n}.$$

Euclidian realization

- $\overrightarrow{[0,1]^n} := ([0,1]^n, \{0,1\}^n, \mathbb{P}^{top}\overrightarrow{[0,1]^n})$
- $\mathbb{P}^{top}\overrightarrow{[0,1]^n}$ set of continous maps $\gamma : [0,1] \to [0,1]^n$ (strictly) increasing w. r. t. each coordinate axis with $\gamma(0), \gamma(1) \in \{0,1\}^n$
- $$|K|_{eucl} := \varinjlim_{\square[n]\to K} \overrightarrow{[0,1]^n}$$

Conjecture. *There exists a natural weak S-homotopy equivalence* $\mathrm{gl}^{top}(K) \simeq |K|_{eucl}$ *for every precubical set* K.

- Related problem: constructing a model category structure s.t. $|\partial\square[n]|_{eucl} \subset |\square[n]|_{eucl}$ cofibration for $n \geqslant 0$

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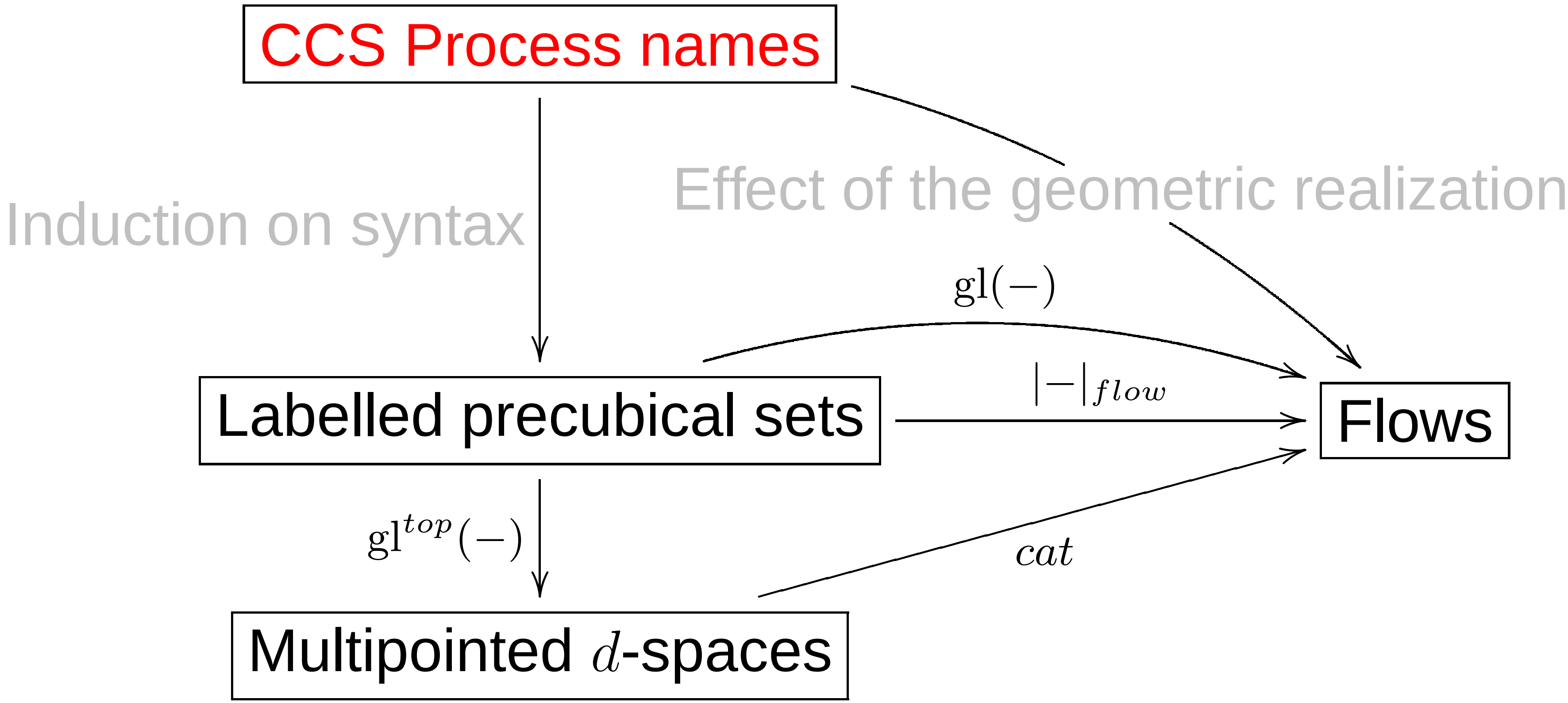


A CCS-like language

(channel, port) names: $a, b, c, \ldots$

co-names: $\overline{a}, \overline{b}, \overline{c}, \ldots$ ($\overline{\overline{a}} = a$)

silent action: τ (synchronized action of a and $\overline{a}$, b and $\overline{b}$, etc...)

actions, prefixes: $\mu ::= a, \overline{a}, \tau$

$P ::= nil$	idle action,
$\mu.P$	prefix,
$P + P$	non-deterministic choice,
$P \mid P$	concurrent execution with synchronization,
$(\nu a)P$	restriction to a local use of a,
$\mathrm{rec}(x)P(x)$	recursion, x guarded variable in $P(x)$
$P \xrightarrow{\mu} Q$	P behaves like Q after executing μ

Operational semantics of CCS

Act $$\frac{}{a.P \xrightarrow{a} P}$$

Res $$\frac{P \xrightarrow{\mu} P' \quad \mu \neq a, \overline{a}}{(\nu a)P \xrightarrow{\mu} (\nu a)P'}$$

Sum1 $$\frac{P \xrightarrow{\mu} P'}{P + Q \xrightarrow{\mu} P'}$$

Sum2 $$\frac{Q \xrightarrow{\mu} Q'}{P + Q \xrightarrow{\mu} Q'}$$

Par1 $$\frac{P \xrightarrow{\mu} P'}{P|Q \xrightarrow{\mu} P'|Q}$$

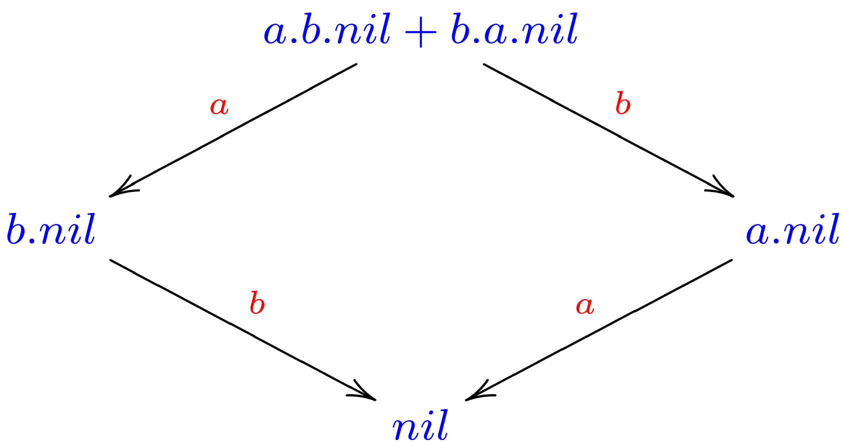
Par2 $$\frac{Q \xrightarrow{\mu} Q'}{P|Q \xrightarrow{\mu} P|Q'}$$

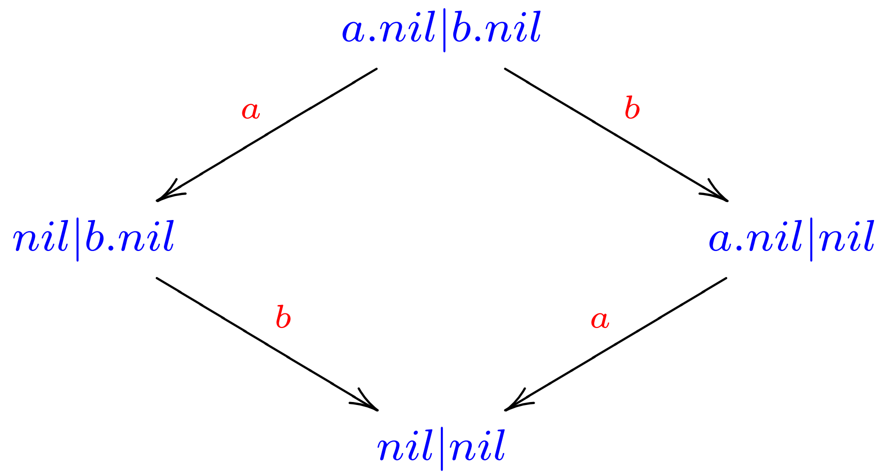
Com $$\frac{P \xrightarrow{a} P', Q \xrightarrow{\overline{a}} Q'}{P|Q \xrightarrow{\tau} P'|Q'}$$

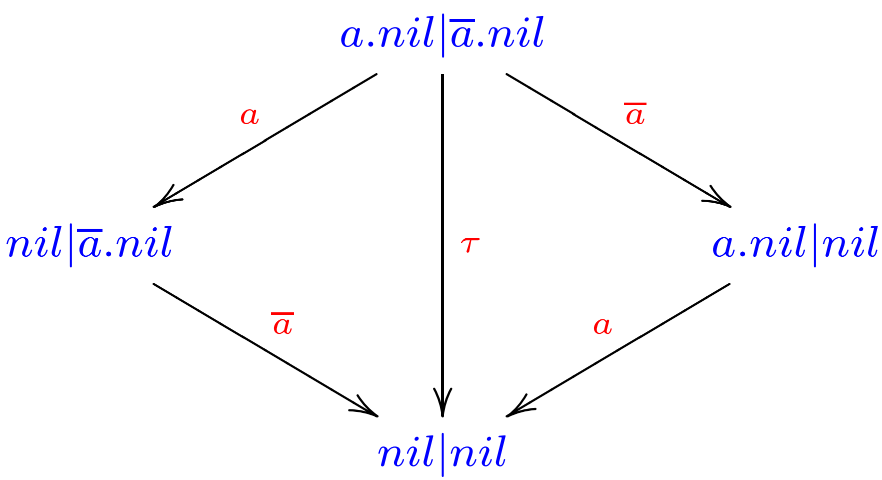
Rec $$\frac{P(\mathrm{rec}(x)P(x)) \xrightarrow{a} P'}{\mathrm{rec}(x)P(x) \xrightarrow{a} P'}$$

1-dimensional semantics of CCS

Labelled precubical sets decorated by CCS terms: the decoration does not belong to the structure of labelled precubical set







With $P(x) = \mu.x$,

$$rec(x)P(x) \xrightarrow{\mu} rec(x)P(x) \xrightarrow{\mu} \ldots$$

$$(\nu a)a.nil|\overline{a}.nil \xrightarrow{\tau} nil|nil$$

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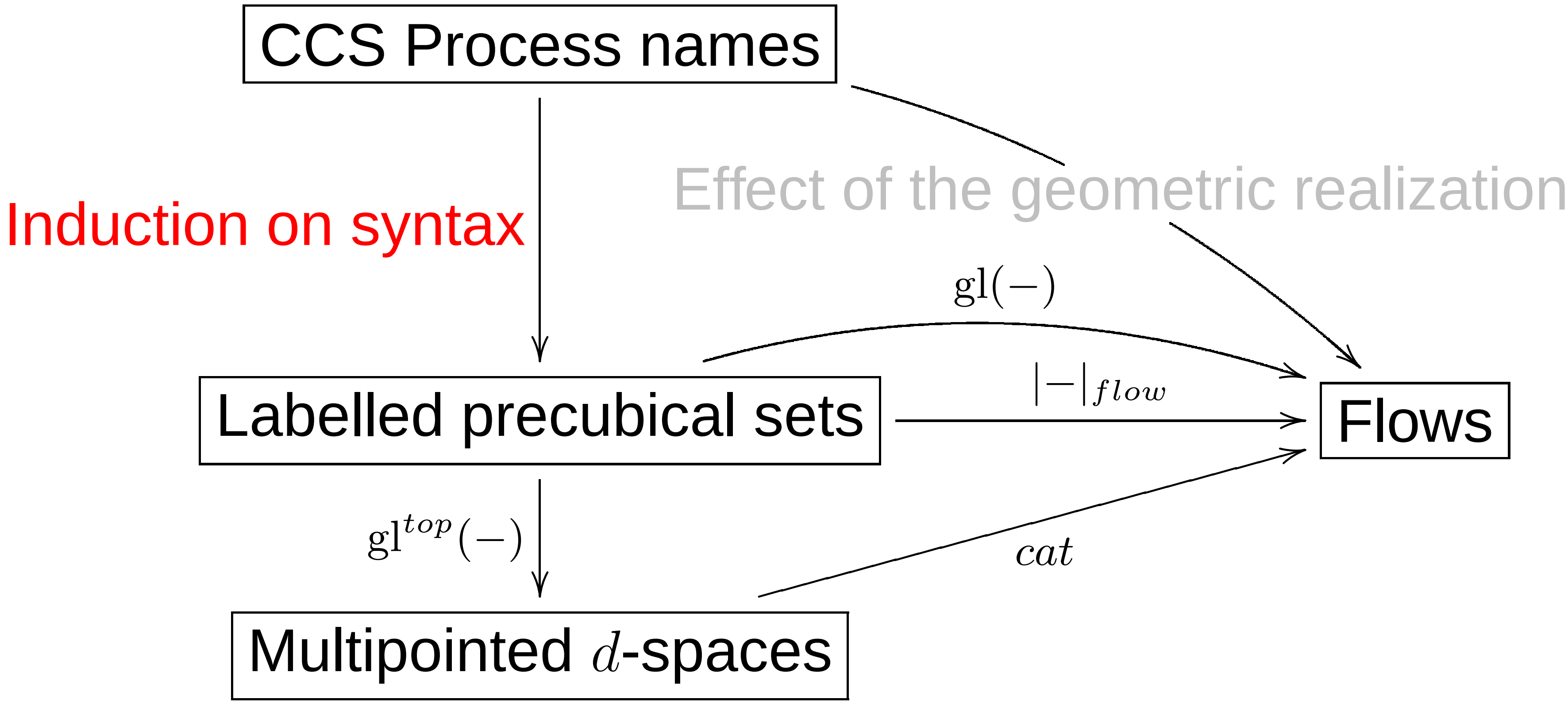


Denotational semantics of CCS

- for every P, every state of $\Box[\![P]\!]$ decorated, and the unique initial one decorated by P
- $\Box[\![nil]\!] = \Box[0]$ and $\Box[\![\mu.nil]\!] = () \xrightarrow{(\mu)} ()$
- $$\begin{array}{ccc} \{0\} & \xrightarrow{0\mapsto nil} & \Box[\![\mu.nil]\!] \\ {\scriptstyle 0\mapsto P}\downarrow & & \downarrow \\ \Box[\![P]\!] & \longrightarrow & \Box[\![\mu.P]\!] \end{array} \qquad \begin{array}{ccc} \Box[\![(\nu a)P]\!] & \longrightarrow & \Box[\![P]\!] \\ \downarrow & & \downarrow \\ !(\Sigma\backslash\{a,\overline{a}\}) & \longrightarrow & !\Sigma \end{array}$$
- $\Box[\![P+Q]\!] = \Box[\![P]\!] \oplus \Box[\![Q]\!]$ where $\oplus$ binary coproduct in the comma category $\{i\}{\downarrow}\Box^{op}\mathbf{Set}{\downarrow}!\Sigma$
- $\Box[\![P|Q]\!] = \Box[\![P]\!] \otimes_\Sigma \Box[\![Q]\!]$
- $\Box[\![\mathrm{rec}(x)P(x)]\!] = \varinjlim_n \Box[\![P^n(nil)]\!]$ (using $nil \rightarrow P(nil) \rightarrow P(P(nil)) \rightarrow \ldots$)

Concurrency and synchronization (I)

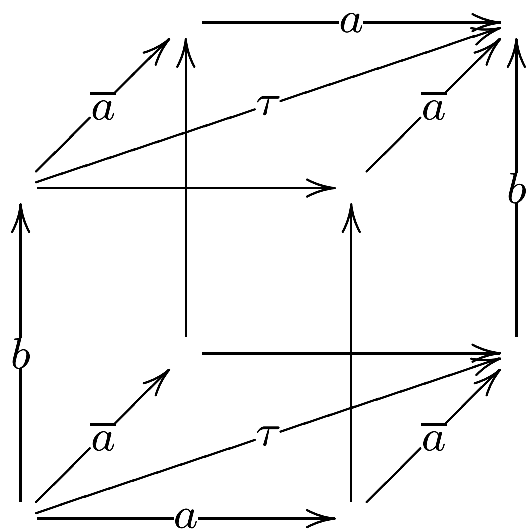


Figure 5: 1-dimensional paths of $(a.nil|b.nil)|\overline{a}.nil$

- The higher dimensional cubes of $\square[3]$
- The concurrent execution of b and τ

Concurrency and synchronization (II)

- K and L two 1-dimensional labelled precubical sets
- $K \times_\Sigma L$ defined by:
  - $(K \times_\Sigma L)_0 = K_0 \times L_0$
  - $(K \times_\Sigma L)_1 = (K_1 \times L_0) \sqcup (K_0 \times L_1) \sqcup \{(x,y) \in K_1 \times L_1, \ell(x) = \overline{\ell(y)}\}$
  - $\partial_1^\alpha(x,y) = (\partial_1^\alpha(x), y)$ for any $(x,y) \in K_1 \times L_0$
  - $\partial_1^\alpha(x,y) = (x, \partial_1^\alpha(y))$ for any $(x,y) \in K_0 \times L_1$
  - $\partial_1^\alpha(x,y) = (\partial_1^\alpha(x), \partial_1^\alpha(y))$ for any $(x,y) \in K_1 \times L_1$
  - $\ell(x,y) = \ell(x)$ for any $(x,y) \in K_1 \times L_0$
  - $\ell(x,y) = \ell(y)$ for any $(x,y) \in K_0 \times L_1$
  - $\ell(x,y) = \tau$ for any $(x,y) \in K_1 \times L_1$ with $\ell(x) = \overline{\ell(y)}$

Concurrency and synchronization (III)

- $(a.nil|b.nil)|\overline{a}.nil$
  - $\square[2]_{\leqslant 1} \times_{\Sigma} \square[1]_{\leqslant 1}$
  - the higher dimensional cubes of $\square[3]$
  - Concurrent execution of b and τ synchronizing a and $\overline{a}$

$$(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_1)$$

  - PROBLEM: the set map

$$(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_2}, \boxed{\epsilon_1}, \epsilon_2)$$

    corresponds to the same execution
  - $(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_1)$ non-twisted

Concurrency and synchronization (IV)

- $Q = (a.nil|b.nil)|(\overline{b}.nil|\overline{a}.nil|c.nil)$
  - $\square[2]_{\leqslant 1} \times_\Sigma \square[3]_{\leqslant 1}$
  - the higher dimensional cubes of $\square[5]$
  - Concurrent execution of a and the action synchronizing b and $\overline{b}$, with the action $\overline{a}$ not yet started and the action c finished
    $(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_2, 0, 1)$
  - Concurrent execution of b, $\overline{b}$ (which do not synchronize here), and the action synchronizing a and $\overline{a}$, with the action c finished
    $(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \boxed{\epsilon_3}, \epsilon_1, 1)$
  - etc...

Concurrency and synchronization (V)

- $R = (a.nil)|(\overline{a}.nil|\overline{a}.nil)$
  - $\square[1]_{\leqslant 1} \times_{\Sigma} \square[2]_{\leqslant 1}$
  - the higher dimensional cubes of $\square[3]$
  - Concurrent execution of the left-hand $\overline{a}$ and the action synchronizing a and the right-hand $\overline{a}$ $(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_1)$
  - etc...

Labelled directed coskeleton

- Adjunction $(-)_{\leqslant 1} : \square^{op}\mathbf{Set}{\downarrow}!\Sigma \leftrightarrows \square_1^{op}\mathbf{Set}{\downarrow}!\Sigma : \mathrm{cosk}_1^{\Sigma}$
- $\Sigma = \{\tau\}$ gives back the usual (i.e. unlabelled) coskeleton functor
- With $K_0 = [p]$, $\overrightarrow{\mathrm{cosk}}^{\Sigma}(K) \subset \mathrm{cosk}_1^{\Sigma}(K)$ subobject defined by $x \in \overrightarrow{\mathrm{cosk}}^{\Sigma}(K)_n$ if and only if $x_0 : \square[n]_0 \to K_0$ non-twisted

Theorem. $\boxed{\overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[n]_{\leqslant 1}) \cong \square[n]}$

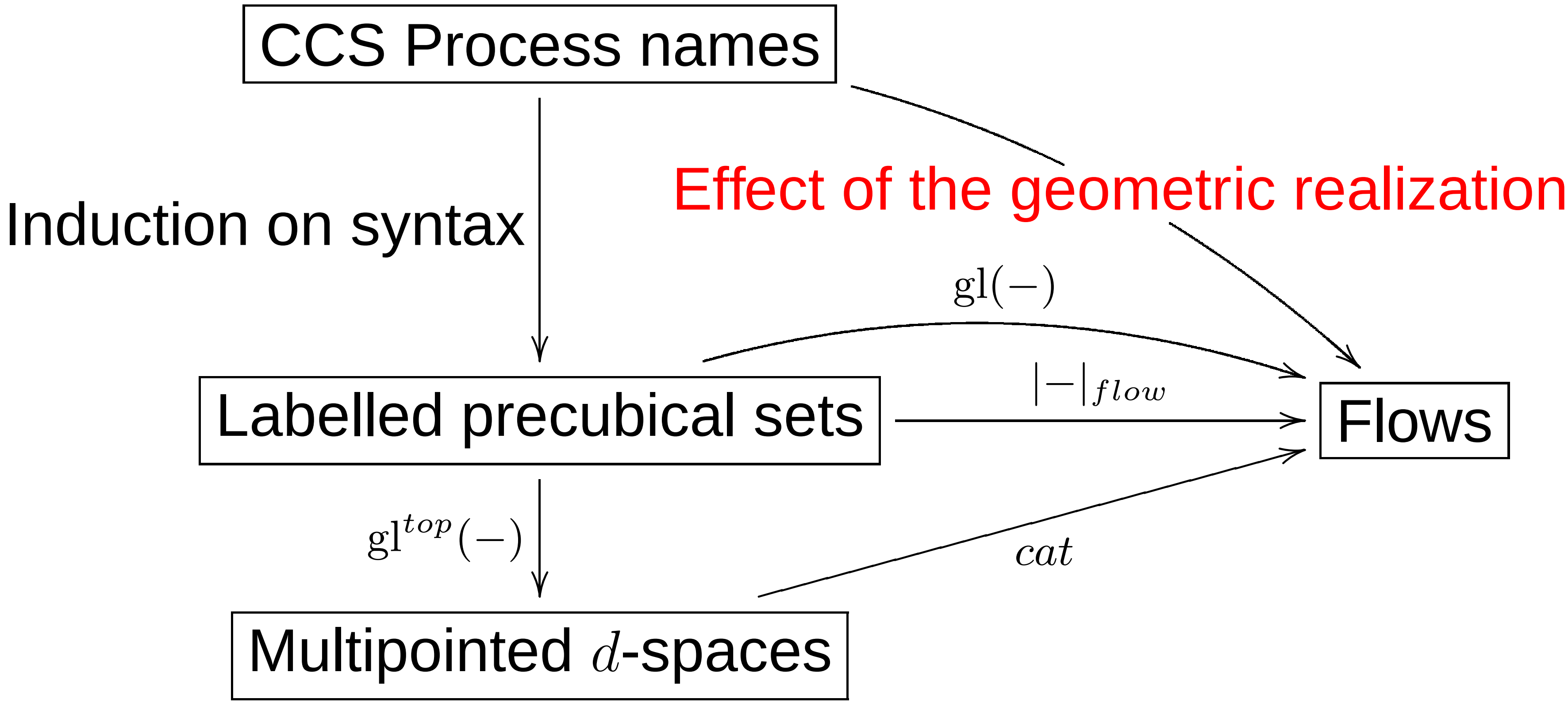
- N.B.: For $n \geqslant 2$, the inclusion of presheaves $\overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[n]_{\leqslant 1}) \subset \mathrm{cosk}_1^{\Sigma}(\square[n]_{\leqslant 1})$ is strict.

Synchronized tensor product

- Two labelled precubical sets K and L
- $$\boxed{K \otimes_\Sigma L := \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[m]_{\leqslant 1} \times_\Sigma \square[n]_{\leqslant 1})}$$
- $K \otimes_\Sigma \square[0] \cong \square[0] \otimes_\Sigma K \cong K$
- $(K \otimes_\Sigma L) \otimes_\Sigma M \cong K \otimes_\Sigma (L \otimes_\Sigma M)$
- $K \otimes_\Sigma -$ and $- \otimes_\Sigma K$ are colimit-preserving
- The underlying precubical sets of $K \otimes_\Sigma L$ and $L \otimes_\Sigma K$ are isomorphic
- Without synchronization, $K \otimes_\Sigma L \cong K \otimes L$
- $(K \otimes_\Sigma L)_{\leqslant 1} \cong K_{\leqslant 1} \times_\Sigma L_{\leqslant 1}$

Overview of the talk

- Paradigm of higher dimensional automata (HDA)
- Modelling concurrency combinatorially, topologically, categorically



Labelled flow

- $\Sigma = \{a, b, c, \dots\} \cup \{\overline{a}, \overline{b}, \overline{c}, \dots\} \cup \{\tau\}$ with $\overline{\overline{a}} = a$
- Flow of labels $?\Sigma$
  - $(?\Sigma)^0 = \{0\}$
  - $\mathbb{P}(?\Sigma)$ freely generated by Σ and the algebraic relations $a * b = b * a$
  - $\boxed{|!\Sigma|_{bad} \cong ?\Sigma}$
- Labelled flow: $\ell : X \to ?\Sigma$
- $\boxed{[\![P]\!] := |\square[\![P]\!]|_{flow}}$

Recursion, prefix, sum

The easy and non-surprising cases first:

- $$[\![\mathrm{rec}(x)P(x)]\!] \simeq \underset{\overrightarrow{n}}{\mathrm{holim}}[\![P^n(nil)]\!]$$
- $$\begin{array}{ccc} \{0\} & \xrightarrow{0 \mapsto nil} & [\![\mu.nil]\!] \\ {\scriptstyle 0\mapsto P}\downarrow & & \downarrow \\ [\![P]\!] & \xrightarrow{\ \ h\ \ } & [\![\mu.P]\!] \end{array}$$
- $$[\![P+Q]\!] \cong [\![P]\!] \oplus [\![Q]\!]$$

Restriction

Pullback and homotopy pullback diagram of flows

$$
\begin{array}{ccc}
[\![(\nu a)P]\!] & \longrightarrow & [\![P]\!] \\
\downarrow & \lrcorner_h & \downarrow \\
?(\Sigma \backslash \{a, \overline{a}\}) & \longrightarrow & ?\Sigma
\end{array}
$$

Parallel composition (local situation)

Theorem. *The map* $(\{\widehat{0} < \widehat{1}\}^*)^{cof} \xrightarrow{\simeq} \{\widehat{0} < \widehat{1}\}^*$ *induces for all* $m, n \geqslant 0$ *a trivial fibration*

$$|\square[m] \otimes_\Sigma \square[n]|_{flow} \xrightarrow{\simeq} |\square[m] \otimes_\Sigma \square[n]|_{bad}$$

Moreover,

$$|\square[m] \otimes_\Sigma \square[n]|_{bad} \cong |(\square[m] \otimes_\Sigma \square[n])_{\leqslant 2}|_{bad}$$

Proof: Adding an algebraic relation is an idempotent operation.

Parallel composition (global situation)

Theorem. *Two CCS processes P and Q. One has:*

$$[\![P|Q]\!] \simeq \underset{\square[m]\to\square[\![P]\!]}{\underrightarrow{\text{holim}}} \ \underset{\square[n]\to\square[\![Q]\!]}{\underrightarrow{\text{holim}}} \ |(\square[m] \otimes_{\Sigma} \square[n])_{\leqslant 2}|_{bad}$$

and

$$[\![P|Q]\!] \simeq \underset{\square[m]\to\square[\![P]\!]}{\underrightarrow{\lim}} \ \underset{\square[n]\to\square[\![Q]\!]}{\underrightarrow{\lim}} \ (|(\square[m] \otimes_{\Sigma} \square[n])_{\leqslant 2}|_{bad})^{cof}$$

- N.B.: the labelled directed coskeleton construction is completely removed

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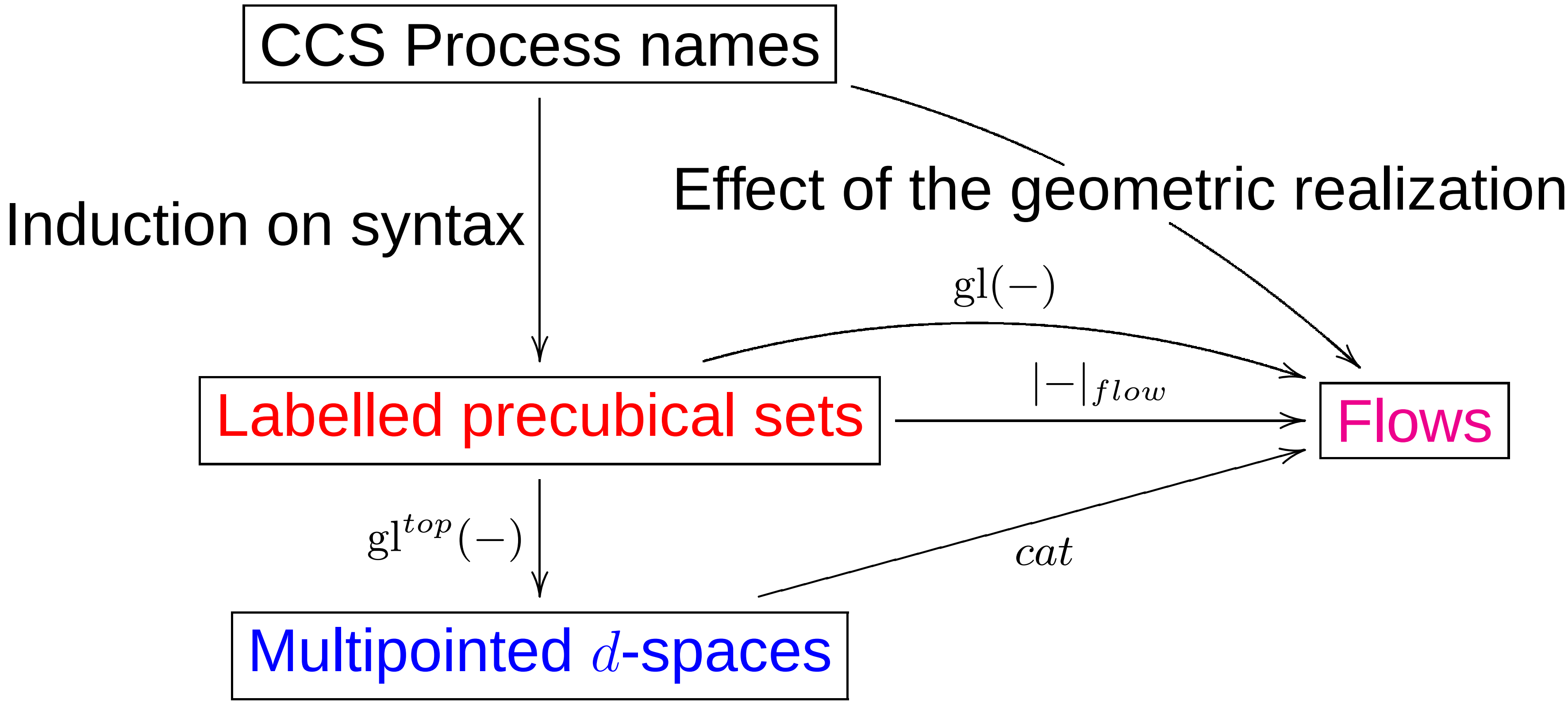

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