# A topological model for studying branching and merging homologies of time flows

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# Organization of the talk

1. The combinatorial model category of flows
2. The topological version: the combinatorial model category of multipointed $d$-spaces
3. The link between the two combinatorial model categories
4. The question

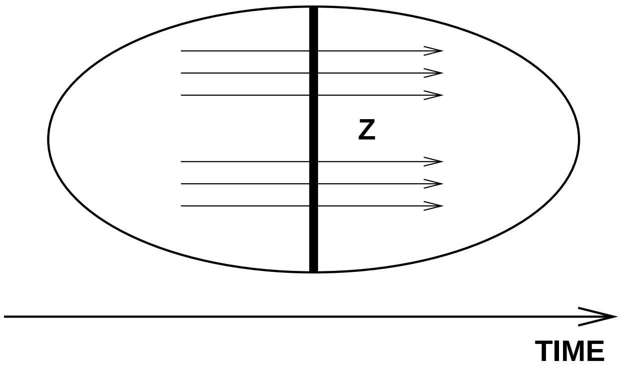
# Time flow

- Flow $X$ = small category without identity maps enriched over $\Delta$-generated spaces (colimit of simplices)
- Set of objects $X^0$ modelling the states of the concurrent system
- Space of morphisms $\mathbb{P}X$ modelling the non-constant execution paths of the concurrent system
- $f : X \to Y$ weak S-homotopy equivalence if $f^0 : X^0 \to Y^0$ bijection and $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ weak homotopy equivalence

# Categorical structure of Flow

- Locally presentable
- Tensored and cotensored over $\Delta$-generated spaces with $\mathbf{Flow}(X \otimes K, Y) \cong \mathbf{Top}(K, \mathbf{FLOW}(X,Y)) \cong \mathbf{Flow}(X, Y^K)$
- Combinatorial proper simplicial model category with class of weak equivalences the weak S-homotopy equivalences
- $X \mapsto X^0 \sqcup \mathbb{P}X$ is not topological
- $X \mapsto \mathbb{P}X$ is functorial: key point for studying branching and merging homologies

# Globe of a topological space




- The globe $\mathrm{Glob}(Z)$ of the topological space $Z$
  - $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$
  - $\mathbb{P}\mathrm{Glob}(Z) = Z$
  - $s = \widehat{0}$
  - $t = \widehat{1}$
  - no composable non-constant execution paths
- The directed segment $\mathrm{Glob}(\{*\}) = \overrightarrow{I}$

# Weak S-homotopy model structure

- A set $S$ can viewed as flow with $S^0 = S$ and $\mathbb{P}S = \varnothing$
- Generating cofibrations:

$$\boxed{I_+^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}}$$

with $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$

- Generating trivial cofibrations:

$$\boxed{J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}}$$

- $\mathrm{Fib} = \{f : X \to Y$ s.t. $\mathbb{P}f$ Serre fibration$\}$
- Every flow is fibrant

# A topological version of flows ?

- Advantage of a topological version of the category of flows: the full subcategory of colimits of cubes have nice properties (topological, locally presentable, and also complete, cocomplete, etc...)
- The $n$-cubes model the concurrent execution of $n$ actions: possibility of getting rid of meaningless geometric shapes
- See Fajstrup-Rosický's paper for an example of such a category convenient for dealing with some problems in directed algebraic topology

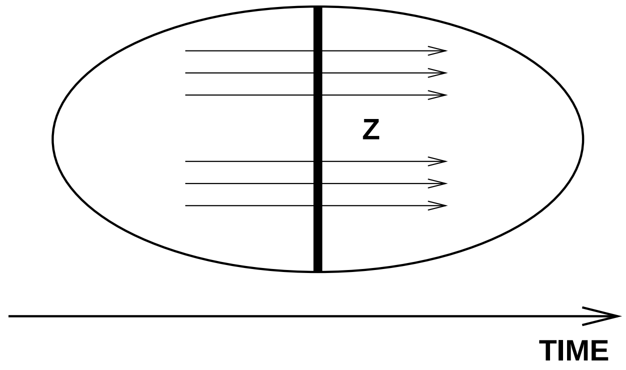
# Multipointed $d$-space

- Multipointed $d$-space $(|X|, X^0, \mathbb{P}^{top}X)$
  - $\Delta$-generated space $|X|$ together with a subset $X^0 \subset |X|$
  - $\mathbb{P}^{top}X$ set of continuous paths closed under strictly increasing reparametrization and composition such that $\gamma : [0,1] \to X$ implies $\gamma(0), \gamma(1) \in X^0$
- $f : X \to Y$ map of multipointed $d$-spaces
  - A continuous map $|f| : |X| \to |Y|$ with $f(X^0) \subset Y^0$
  - $\phi \in \mathbb{P}^{top}X$ implies $\mathbb{P}^{top}f(\phi) := |f| \circ \phi \in \mathbb{P}^{top}Y$
- $f : X \to Y$ weak S-homotopy equivalence if $f^0 : X^0 \to Y^0$ bijection and $\mathbb{P}^{top}f$ weak homotopy equivalence

# Categorical structure of MdTop

- Locally presentable
- Tensored and cotensored over $\Delta$-generated spaces with $\mathbf{MdTop}(X \otimes K, Y) \cong \mathbf{Top}(K, \mathbf{MDTOP}(X,Y)) \cong \mathbf{MdTop}(X, Y^K)$
- Combinatorial right proper simplicial model category with class of weak equivalences the weak S-homotopy equivalences
- ($X \mapsto$ underlying set of $|X|$) is topological
- And of course $X \mapsto \mathbb{P}^{top}X$ is functorial
- Left properness is still a conjecture

# Topological globe of a topological space




- The topological globe $\mathrm{Glob}^{top}(Z)$ of the topological space $Z$
- $\mathrm{Glob}^{top}(Z)^0 = \{\widehat{0}, \widehat{1}\}$
- $|\mathrm{Glob}^{top}(Z)| = \{\widehat{0}, \widehat{1}\} \sqcup (Z \times [0,1])/\left((z,0) = (z',0) = \widehat{0}, (z,1) = (z',1) = \widehat{1}\right)$
- $\mathbb{P}^{top}\mathrm{Glob}^{top}(Z)$ closure by strict increasing reparametrization of $\{t \mapsto (z,t), z \in Z\}$

# Weak S-homotopy model structure

- Discrete space $S$ viewed as multipointed $d$-space with $S^0 = S$ and $\mathbb{P}^{top} S = \varnothing$
- Generating cofibrations:

$$\boxed{I_+^{gl,top} = \{\mathrm{Glob}^{top}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}^{top}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}}$$

  with $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$
- Generating trivial cofibrations:

$$\boxed{J^{gl,top} = \{\mathrm{Glob}^{top}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}^{top}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}}$$

- $\mathrm{Fib} = \{f : X \to Y$ s.t. $\mathbb{P}^{top} f$ Serre fibration$\}$
- Every multipointed $d$-space is fibrant

# From multipointed $d$-spaces to flows (I)

- Multipointed $d$-space $X = (|X|, X^0, \mathbb{P}^{top}X)$
- Flow $cat(X)$ defined as follows:
- $cat(X)^0 := X^0$
- $\mathbb{P}cat(X)$ defined by

$$\frac{\mathbb{P}^{top}X}{\text{strictly increasing reparametrization with fixed extremities}}$$

- $cat : \mathbf{MdTop} \to \mathbf{Flow}$ well-defined functor
- Example : $cat(\mathrm{Glob}^{top}(Z)) \cong \mathrm{Glob}(Z)$

# From multipointed $d$-spaces to flows (II)

- The composite functor

$$\mathbf{MdTop} \xrightarrow{(-)^{cof}} \mathbf{MdTop} \xrightarrow{cat} \mathbf{Flow}$$

induces an equivalence of categories

$$\mathbf{Ho}(\mathbf{MdTop}) \simeq \mathbf{Ho}(\mathbf{Flow}).$$

- The functor $cat : \mathbf{MdTop} \to \mathbf{Flow}$ preserves cofibrations, trivial cofibrations and weak S-homotopy equivalences between cofibrant objects.
- The functor $cat : \mathbf{MdTop} \to \mathbf{Flow}$ is not colimit-preserving

# Question

- IF: $F : \mathcal{M} \to \mathcal{N}$ functor between two combinatorial model categories preserving cofibrations, trivial cofibrations, weak equivalences between cofibrant objects, not colimit-preserving, $\mathcal{M}$ topological over $\mathbf{Set}$ and $F \circ (-)^{cof} : \mathbf{Ho}(\mathcal{M}) \to \mathbf{Ho}(\mathcal{N})$ equivalence of categories
- THEN: are $\mathcal{M}$ and $\mathcal{N}$ Quillen equivalent ?