Combinatorics of labelling in higher dimensional automata

(talk in French, slides in English)

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Modelling concurrency

- $\Sigma = \{a, b, c, \dots\} \cup \{\overline{a}, \overline{b}, \overline{c}, \dots\} \cup \{\tau\}$ with $\overline{\overline{a}} = a$

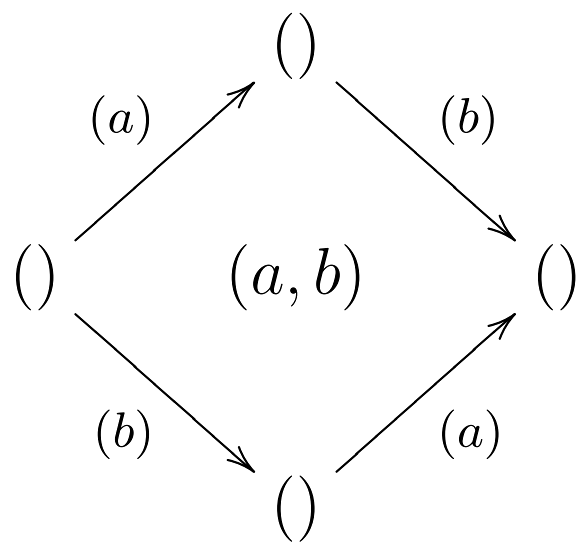
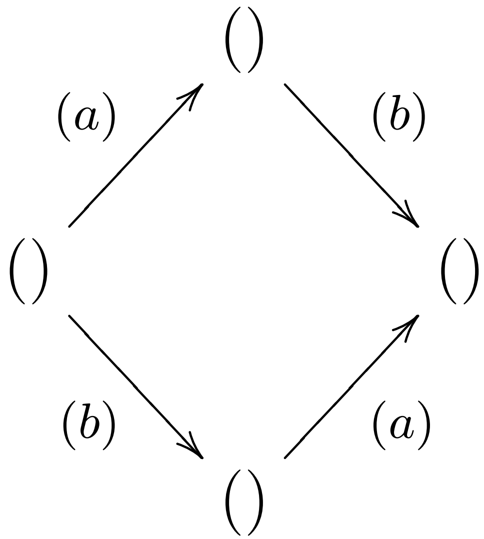


Figure 1: Concurrent execution of a and b

Example of impossible HDA

- Start from an empty labelled 2-cube



- Add two squares
- Impossible HDA since either a and b run sequentially (empty case), or a and b run concurrently (full case)

Reduced box category $\square$

- $[n] = \{0 < 1\}^n$, $\{0 < 1\}^0 = [0] = \{()\}$
- $\delta_i^\alpha : [n-1] \to [n]$ defined for $1 \leqslant i \leqslant n$ and $\alpha \in \{0,1\}$ by $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$
- $\mathbf{PoSet}$ category of posets together with the strictly increasing maps
- $\square \subset \mathbf{PoSet}$ subcategory generated by the δ_i^α
- $\square$ is an example of category of cubes (see the definition later)
- $\square_n \subset \square$ full subcategory whose set of objects is $\{[p], p \leqslant n\}$

Labelled precubical set

- Precubical set K: presheaf over $\square$
- The n-cube $\square[n] = \square(-, [n])$
- The boundary of the n-cube $\partial\square[n] := \square_{\leqslant n-1}$
- Precubical set of labels $!\Sigma$
  - $(!\Sigma)_0 = \{()\}$
  - for $n \geqslant 1$, $(!\Sigma)_n = \Sigma^n$
  - $\partial_i^0(a_1, \dots, a_n) = \partial_i^1(a_1, \dots, a_n) = (a_1, \dots, \widehat{a_i}, \dots, a_n)$
- Labelled precubical set: $\ell : K \longrightarrow !\Sigma$
- Two opposite faces have same labelling

Concurrency and synchronization (I)

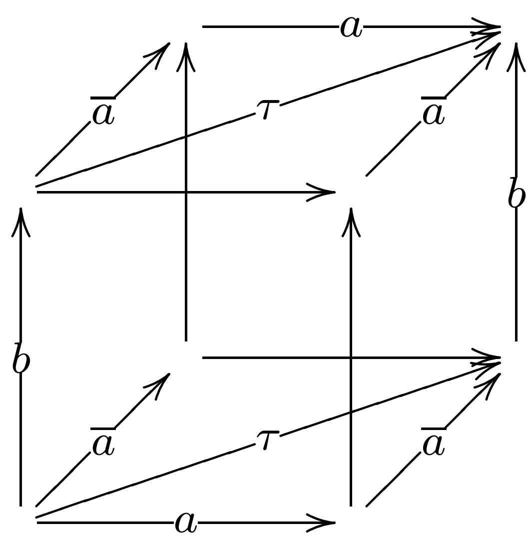


Figure 2: 1-dimensional paths of $(a.nil|b.nil)|\overline{a}.nil$

- The higher dimensional cubes of $\square[3]$
- The concurrent execution of b and τ

Concurrency and synchronization (II)

- K and L two 1-dimensional labelled precubical sets
- $K \times_\Sigma L$ defined by:
  - $(K \times_\Sigma L)_0 = K_0 \times L_0$
  - $(K \times_\Sigma L)_1 = (K_1 \times L_0) \sqcup (K_0 \times L_1) \sqcup \{(x,y) \in K_1 \times L_1, \ell(x) = \overline{\ell(y)}\}$
  - $\partial_1^\alpha(x,y) = (\partial_1^\alpha(x), y)$ for any $(x,y) \in K_1 \times L_0$
  - $\partial_1^\alpha(x,y) = (x, \partial_1^\alpha(y))$ for any $(x,y) \in K_0 \times L_1$
  - $\partial_1^\alpha(x,y) = (\partial_1^\alpha(x), \partial_1^\alpha(y))$ for any $(x,y) \in K_1 \times L_1$
  - $\ell(x,y) = \ell(x)$ for any $(x,y) \in K_1 \times L_0$
  - $\ell(x,y) = \ell(y)$ for any $(x,y) \in K_0 \times L_1$
  - $\ell(x,y) = \tau$ for any $(x,y) \in K_1 \times L_1$ with $\ell(x) = \overline{\ell(y)}$

Concurrency and synchronization (III)

- $(a.nil|b.nil)|\overline{a}.nil$
  - $\square[2]_{\leqslant 1} \times_\Sigma \square[1]_{\leqslant 1}$
  - the higher dimensional cubes of $\square[3]$
  - Concurrent execution of b and τ synchronizing a and $\overline{a}$

$$(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_1)$$

  - PROBLEM: the set map

$$(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_2}, \boxed{\epsilon_1}, \epsilon_2)$$

    corresponds to the same execution
  - $(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_1)$ non-twisted

Concurrency and synchronization (IV)

- $Q = (a.nil|b.nil)|(\overline{b}.nil|\overline{a}.nil|c.nil)$
  - $\square[2]_{\leqslant 1} \times_{\Sigma} \square[3]_{\leqslant 1}$
  - the higher dimensional cubes of $\square[5]$
  - Concurrent execution of a and the action synchronizing b and $\overline{b}$, with the action $\overline{a}$ not yet started and the action c finished
    $(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_2, 0, 1)$
  - Concurrent execution of b, $\overline{b}$ (which do not synchronize here), and the action synchronizing a and $\overline{a}$, with the action c finished
    $(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \boxed{\epsilon_3}, \epsilon_1, 1)$
  - etc...

Concurrency and synchronization (V)

- $R = (a.nil)|(\overline{a}.nil|\overline{a}.nil)$
  - $\square[1]_{\leqslant 1} \times_{\Sigma} \square[2]_{\leqslant 1}$
  - the higher dimensional cubes of $\square[3]$
  - Concurrent execution of the left-hand $\overline{a}$ and the action synchronizing a and the right-hand $\overline{a}$ $(\epsilon_1, \epsilon_2) \mapsto (\boxed{\epsilon_1}, \boxed{\epsilon_2}, \epsilon_1)$
  - etc...

Labelled directed coskeleton

- Adjunction $(-)_{\leqslant 1} : \square^{op}\mathbf{Set}{\downarrow}!\Sigma \leftrightarrows \square_1^{op}\mathbf{Set}{\downarrow}!\Sigma : \mathrm{cosk}_1^{\Sigma}$
- $\Sigma = \{\tau\}$ gives back us the usual (i.e. unlabelled) coskeleton functor
- With $K_0 = [p]$, $\overrightarrow{\mathrm{cosk}}^{\Sigma}(K) \subset \mathrm{cosk}_1^{\Sigma}(K)$ subobject defined by $x \in \overrightarrow{\mathrm{cosk}}^{\Sigma}(K)_n$ if and only if $x_0 : \square[n]_0 \to K_0$ non-twisted

Theorem. $\boxed{\overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[n]_{\leqslant 1}) \cong \square[n]}$

- N.B.: For $n \geqslant 2$, the inclusion of presheaves $\overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[n]_{\leqslant 1}) \subset \mathrm{cosk}_1^{\Sigma}(\square[n]_{\leqslant 1})$ is strict

Synchronized tensor product (I)

- Two labelled precubical sets K and L
- $$K \otimes_\Sigma L := \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[m]_{\leqslant 1} \times_\Sigma \square[n]_{\leqslant 1})$$
- $K \otimes_\Sigma \square[0] \cong \square[0] \otimes_\Sigma K \cong K$
- $(K \otimes_\Sigma L) \otimes_\Sigma M \cong K \otimes_\Sigma (L \otimes_\Sigma M)$
- $K \otimes_\Sigma -$ and $- \otimes_\Sigma K$ are colimit-preserving
- The underlying precubical sets of $K \otimes_\Sigma L$ and $L \otimes_\Sigma K$ are isomorphic
- Without synchronization, $K \otimes_\Sigma L \cong K \otimes L$
- $(K \otimes_\Sigma L)_{\leqslant 1} \cong K_{\leqslant 1} \times_\Sigma L_{\leqslant 1}$

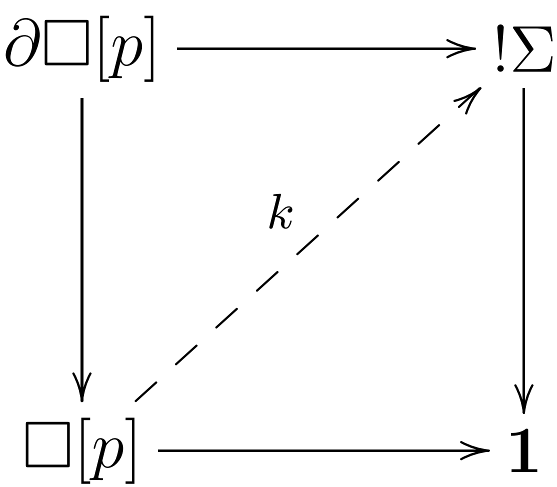
Category of cubes

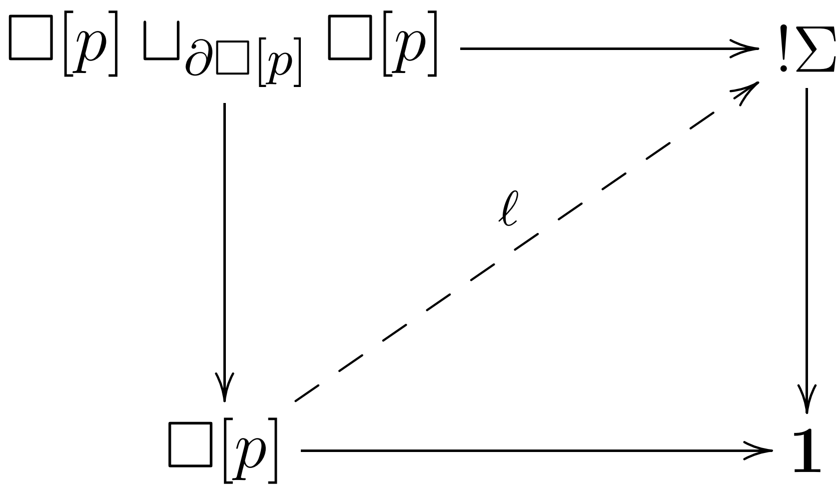
- $d((\epsilon_1, \ldots, \epsilon_n), (\epsilon'_1, \ldots, \epsilon'_n)) = \sum_{i=1}^{i=n} |\epsilon_i - \epsilon'_i|$
- $f : [m] \rightarrow [n]$ strictly increasing is adjacency-preserving if $d((\epsilon_1, \ldots, \epsilon_m), (\epsilon'_1, \ldots, \epsilon'_m)) = 1$ implies $d(f(\epsilon_1, \ldots, \epsilon_m), f(\epsilon'_1, \ldots, \epsilon'_m)) = 1$
- Category of cubes $\mathcal{A}$
  - set of objects $\{[n], n \geqslant 0\}$
  - $\square \subset \mathcal{A} \subset \mathbf{PoSet}$
  - every morphism of $\mathcal{A}$ is adjacency-preserving
- A $\mathcal{A}$-set is a presheaf over $\mathcal{A}$
- Category of $\mathcal{A}$-sets denoted by $\mathcal{A}^{op}\mathbf{Set}$
- $\mathcal{A}_n \subset \mathcal{A}$ full subcategory whose set of objects is $\{[p], p \leqslant n\}$

$\mathcal{A}$-set of labels (I)

- Observation: $\mathbf{1} \cong !\{\tau\}$
- Adjunction $\mathcal{L}_{\mathcal{A}} : \square^{op}\mathbf{Set} \leftrightarrows \mathcal{A}^{op}\mathbf{Set} : \omega_{\mathcal{A}}$
- Inclusion of presheaves $K \subset \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(K)$
- Bad candidate for the $\mathcal{A}$-set of labels: $\mathcal{L}_{\mathcal{A}}(!\Sigma)$
  - The inclusion $\mathbf{1} \cong !\{\tau\} \subset \omega_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(!\{\tau\})$ is strict as soon as there exists $f \in \mathcal{A}([n],[n])\backslash\{\mathrm{Id}_{[n]}\}$ for some $n \geqslant 2$
  - An n-cube of the $\mathcal{A}$-set of labels must be determined by its boundary
  - When $\Sigma = \{\tau\}$, the notion of labelled $\mathcal{A}$-set must coincide with the notion of unlabelled $\mathcal{A}$-set

$\mathcal{A}$-set of labels (II)





- For every $p \geqslant 2$, there exists at most one lift k
- For every $p \geqslant 2$, there exists exactly one lift ℓ
- $!\Sigma \in \{\square[p] \sqcup_{\partial\square[p]} \square[p] \to \square[p], p \geqslant 2\}^{\perp}$, full reflective locally presentable subcategory of $\square^{op}\mathbf{Set}$

Labelled $\mathcal{A}$-set

- $\mathcal{A}[p] = \mathcal{L}_{\mathcal{A}}(\Box[p])$, $\partial\mathcal{A}[p] = \mathcal{L}_{\mathcal{A}}(\partial\Box[p])$
- $\mathrm{Sh}_{\mathcal{A}} : \mathcal{A}^{op}\mathbf{Set} \to \{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}^{\perp}$ left adjoint to the inclusion
- For any category of cubes $\mathcal{A}$, $\mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\{\tau\}) \cong \mathbf{1}$
- Labelled $\mathcal{A}$-set: $\ell : K \to \mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma)$
- A labelled $\mathcal{A}$-set over $\{\tau\}$ is an unlabelled $\mathcal{A}$-set
- The canonical map $\mathcal{L}_{\mathcal{A}}(!\Sigma) \to \mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma)$ induces a functor

$$\mathcal{L}_{\mathcal{A}} : \Box^{op}\mathbf{Set}{\downarrow}!\Sigma \to \mathcal{A}^{op}\mathbf{Set}{\downarrow}\,\mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma)$$

Labelled coskeleton over $\mathcal{A}$

- For any adjunction $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ where $\mathcal{C}$ has pullbacks, for any object $A \in \mathcal{C}$, the functor $L_A : \mathcal{C}{\downarrow}A \to \mathcal{D}{\downarrow}L(A)$ defined by $L_A(X \to A) := L(X) \to L(A)$ has a right adjoint R_A defined by :

$$\begin{array}{ccc} R_A(Y) & \longrightarrow & R(Y) \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & R(L(A)) \end{array}$$

- Adjunction $(-)_{\leqslant n} : \mathcal{A}^{op}\mathbf{Set}{\downarrow}\,\mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma) \leftrightarrows \mathcal{A}_n^{op}\mathbf{Set}{\downarrow}\,\mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma) : \mathrm{cosk}_n^{\mathcal{A},\Sigma}$ for any $n \geqslant 0$

Description of $\mathrm{cosk}_n^{\mathcal{A},\Sigma}$

- Adjunction $(-)_{\leqslant n} : \mathcal{A}_{n+1}^{op}\mathbf{Set}{\downarrow}\,\mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma) \leftrightarrows \mathcal{A}_{n}^{op}\mathbf{Set}{\downarrow}\,\mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma) : \mathrm{cosk}_{n,n+1}^{\mathcal{A},\Sigma}$ for any $n \geqslant 0$
- The $(n+1)$-cubes of $\mathrm{cosk}_{n,n+1}^{\mathcal{A},\Sigma}(K)$ are in one-to-one correspondence with the commutative squares of the form

$$\begin{array}{ccc} \partial\mathcal{A}[n+1] & \longrightarrow & K \\ \downarrow & & \downarrow \\ \mathcal{A}[n+1] & \longrightarrow & \mathrm{Sh}_{\mathcal{A}}\,\mathcal{L}_{\mathcal{A}}(!\Sigma) \end{array}$$

- $\mathrm{cosk}_n^{\mathcal{A},\Sigma} \cong \cdots \circ \mathrm{cosk}_{n+2,n+3}^{\mathcal{A},\Sigma} \circ \mathrm{cosk}_{n+1,n+2}^{\mathcal{A},\Sigma} \circ \mathrm{cosk}_{n,n+1}^{\mathcal{A},\Sigma}$

Shell-complete category of cubes

- There is always an inclusion of presheaves $\mathcal{A}[p] \subset \mathrm{cosk}_1^{\mathcal{A}}(\mathcal{A}[p]_{\leqslant 1})$ for any $p \geqslant 0$ (equality for $p = 0, 1$)
- A category of cubes $\mathcal{A}$ is shell-complete if for any $p \geqslant 0$, $\mathcal{A}[p] \cong \mathrm{cosk}_1^{\mathcal{A}}(\mathcal{A}[p]_{\leqslant 1})$
- The category $\square$ is not shell-complete
- The whole category $\widehat{\square}$ of $\mathbf{PoSet}$ with set of objects $\{[n], n \geqslant 0\}$ and with morphisms all adjacency-preserving maps is the unique shell-complete category of cubes ($\widehat{\square}$ is the maximal category of cubes)
- A transverse symmetric precubical set is a presheaf over $\widehat{\square}$

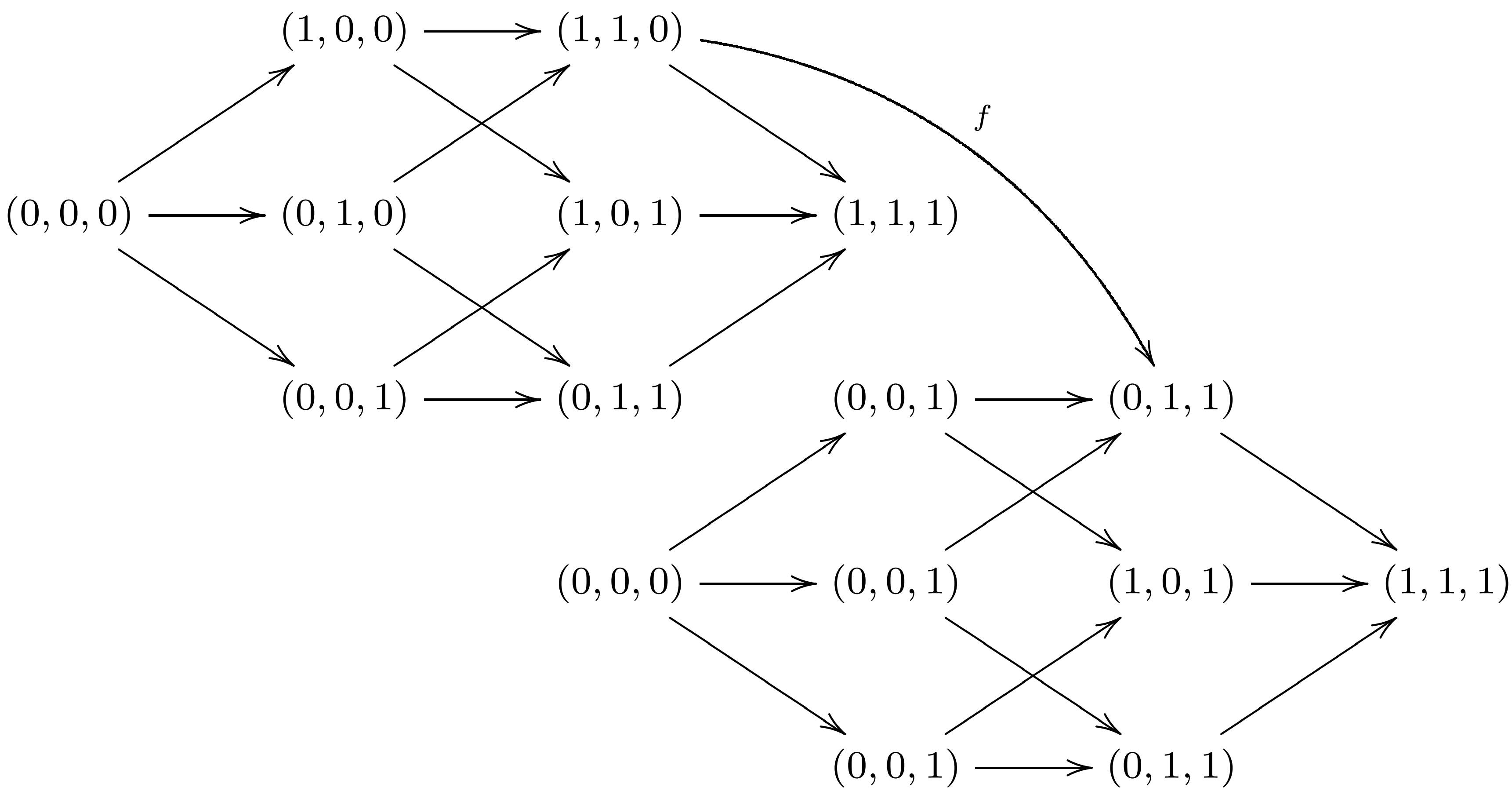
Negative result about $\widehat{\square}$ (I)

- Let $\sigma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \dots, \epsilon_n)$ and $\gamma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \max(\epsilon_i, \epsilon_{i+1}), \min(\epsilon_i, \epsilon_{i+1}), \epsilon_{i+2}, \dots, \epsilon_n)$
- The σ_i maps are the well-know symmetry operators
- A transverse degeneracy map is a adjacency-preserving map $f : [n] \to [n]$ which is not bijective
- The γ_i maps are particular cases of transverse degeneracy maps; Question: is there other transverse degeneracy maps ? Answer: yes
- $\widehat{\square}$ is not generated by $\square$ and the maps σ_i and γ_i

Negative result about $\widehat{\square}$ (II)

- Algebraic relations satisfied by these maps:
  - $\delta_j^\beta \delta_i^\alpha = \delta_i^\alpha \delta_{j-1}^\beta$ for $i < j$ and for all $(\alpha, \beta) \in \{0,1\}^2$
  - $\sigma_i \delta_j^\alpha = \delta_j^\alpha \sigma_{i-1}$ for $j < i$, $\sigma_i \delta_j^\alpha = \delta_{i+1}^\alpha$ for $j = i$, $\sigma_i \delta_j^\alpha = \delta_i^\alpha$ for $j = i+1$ and $\sigma_i \delta_j^\alpha = \delta_j^\alpha \sigma_i$ for $j > i+1$
  - $\gamma_j \delta_i^\alpha = \delta_i^\alpha \gamma_j$ for $j < i-1$, $\gamma_j \delta_i^\alpha = \delta_i^\alpha \gamma_{j-1}$ for $j \geqslant i+1$, $\gamma_j \delta_i^\alpha = \delta_{i-\alpha}^\alpha$ for $j = i-1$ and $\gamma_j \delta_i^\alpha = \delta_{i+1-\alpha}^\alpha$ for $j = i$
  - $\sigma_i \sigma_i = \mathrm{Id}$, $\gamma_i \gamma_i = \gamma_i$, $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ and $\gamma_i \gamma_j \gamma_i = \gamma_j \gamma_i \gamma_j$ for $i = j-1$, and $\sigma_i \sigma_j = \sigma_j \sigma_i$ and $\gamma_i \gamma_j = \gamma_j \gamma_i$ for $i < j-1$
  - $\gamma_j \sigma_i = \sigma_i \gamma_j$ for $j > i+1$ and $j < i-1$, $\gamma_i \sigma_i = \gamma_i$, $\sigma_{i+1} \gamma_i \sigma_{i+1} = \sigma_i \gamma_{i+1} \sigma_i$
- These relations conjecturally give a presentation of the small category generated by $\square$, σ_i and γ_i

Negative result about $\widehat{\square}$ (III)



$f\delta_1^0 = \delta_1^0\sigma_1\gamma_1$ and $f\delta_1^1 = \delta_3^1\sigma_1$ implies that f is not a composite of σ_i and δ_i

$\mathrm{cosk}_1^{\widehat{\square},\overline{\Sigma}}$ is well-behaved (I)

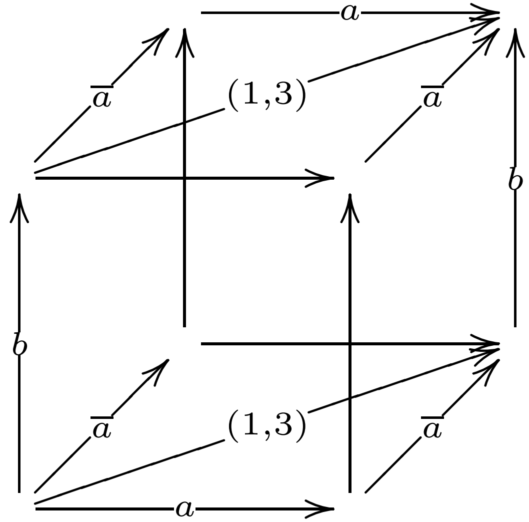
- Inclusion of presheaves

$$\mathcal{L}_{\widehat{\square}}\left(\overrightarrow{\mathrm{cosk}}^{\Sigma}(\square[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma \square[a_{m+1},\dots,a_{m+n}]_{\leqslant 1})\right)$$
$$\subset \mathrm{cosk}_1^{\widehat{\square},\Sigma}\left(\widehat{\square}[a_1,\dots,a_m]_{\leqslant 1}\times_\Sigma \widehat{\square}[a_{m+1},\dots,a_{m+n}]_{\leqslant 1}\right)$$

- Equality when $m=0$ or $n=0$
- Strict inclusion for $m=2$, $a_1=a_2\neq\tau$
- $\mathrm{cosk}_1^{\widehat{\square},\Sigma}(\widehat{\square}[a,a]_{\leqslant 1}\times_\Sigma \widehat{\square}[\overline{a},\overline{a}]_{\leqslant 1})$ contains a 2-cube x such that $x_0(0,0)=(0,0,0,0)$, $x_0(0,1)=(1,0,0,1)$, $x_0(1,0)=(1,0,1,0)$ and $x_0(1,1)=(1,1,1,1)$ since all 1-cubes of x are labelled by τ

$\mathrm{cosk}_1^{\widehat{\square},\overline{\Sigma}}$ is well-behaved (II)

- Representation of $\square[a,b]_{\leqslant 1}\overline{\times}_{\Sigma}\square[\overline{a}]$, labelled over $\overline{\Sigma} = \Sigma \sqcup (\mathbb{N}^* \times \mathbb{N}^*)$:



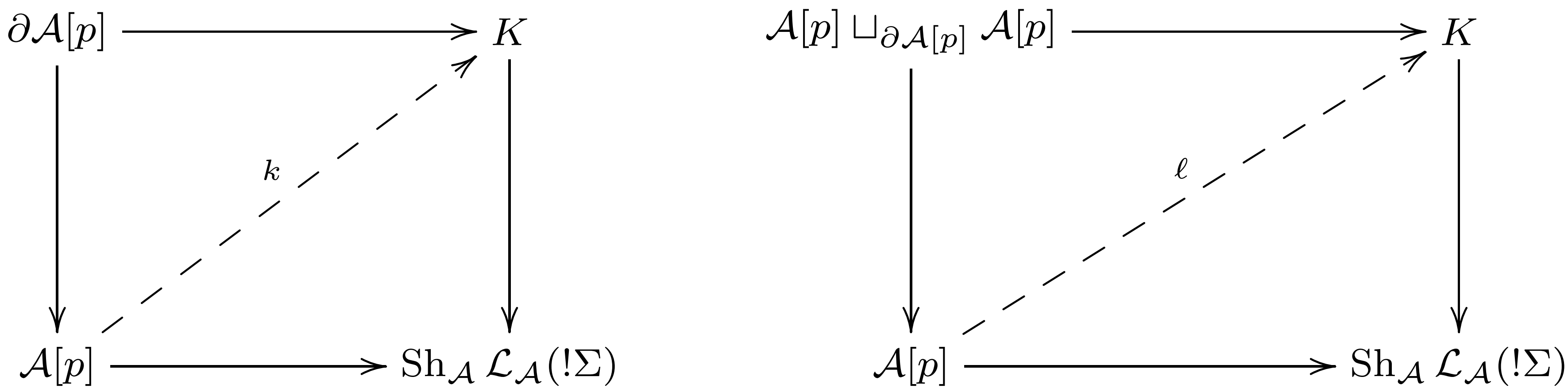
$$\mathcal{L}_{\widehat{\square}}\left(\overrightarrow{\mathrm{cosk}}^{\Sigma}\left(\square[a_1,\ldots,a_m]_{\leqslant 1}\times_{\Sigma}\square[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)\right)$$

$$\cong \mathrm{cosk}_1^{\widehat{\square},\overline{\Sigma}}\left(\widehat{\square}[a_1,\ldots,a_m]_{\leqslant 1}\overline{\times}_{\Sigma}\widehat{\square}[a_{m+1},\ldots,a_{m+n}]_{\leqslant 1}\right)$$

Synchronized tensor product (II)

- Two labelled transverse symmetric precubical sets K and L
- $$K \otimes_\Sigma L := \varinjlim_{\widehat{\square}[m]\to K} \varinjlim_{\widehat{\square}[n]\to L} \operatorname{cosk}_1^{\widehat{\square},\overline{\Sigma}}(\widehat{\square}[m]_{\leqslant 1} \overline{\times}_\Sigma \widehat{\square}[n]_{\leqslant 1})$$
- $K \otimes_\Sigma \widehat{\square}[0] \cong \widehat{\square}[0] \otimes_\Sigma K \cong K$
- $(K \otimes_\Sigma L) \otimes_\Sigma M \cong K \otimes_\Sigma (L \otimes_\Sigma M)$
- $K \otimes_\Sigma -$ and $- \otimes_\Sigma K$ are colimit-preserving
- $K \otimes_\Sigma L \cong L \otimes_\Sigma K$
- $\mathcal{L}_{\widehat{\square}}(K \otimes_\Sigma L) \cong \mathcal{L}_{\widehat{\square}}(K) \otimes_\Sigma \mathcal{L}_{\widehat{\square}}[L)$

HDA paradigm for $\mathcal{A}$-sets



- Labelled $\mathcal{A}$-set K satisfying the HDA paradigm
- For every $p \geqslant 2$, there exists at most one lift k
- For every $p \geqslant 2$, there exists exactly one lift ℓ
- $K \in \{\mathcal{A}[p] \sqcup_{\partial\mathcal{A}[p]} \mathcal{A}[p] \to \mathcal{A}[p], p \geqslant 2\}^{\perp}$, full reflective locally presentable subcategory of $\mathcal{A}^{op}\mathbf{Set}$

HDA paradigm for $\widehat{\square}^{op}\mathbf{Set}$

- The HDA paradigm is meaningful only for labelled transverse symmetric precubical sets: for any category of cubes $\mathcal{A} \subsetneq \widehat{\square}$, the underlying $\mathcal{A}$-set of $\widehat{\square}[n]$ satisfies the HDA paradigm for $\mathcal{A}$-set whereas the geometric realization as $\mathcal{A}$-set is not contractible
- The category of labelled transverse symmetric precubical sets satisfying the HDA paradigm is equivalent to a concrete category axiomatized by a set of limit sentences $\forall \underline{x}, (\phi(\underline{x}) \Rightarrow (\exists ! \underline{y}\ \psi(\underline{x}, \underline{y})))$ where ϕ and ψ are conjunctions of atomic formulas
- Work in progress: this set of axioms is Cattani-Sassone's notion of higher dimensional transition system