Globular realization and cubical underlying homotopy type

(talk in French, slides in English)

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Precubical set

- Precubical set K = presheaf over $\square$: cubical set without degeneracy maps
- n-cube $\square[n] = \square(-, [n])$
- Boundary of the n-cube $\partial\square[n]$

Modeling time flow

- Flow X: small category without identities enriched over compactly generated topological spaces
- Set X^0 of states/objects of X
- Compactly generated space $\mathbb{P}_{\alpha,\beta}X$ of morphisms/non-constant execution paths from α to β of X^0

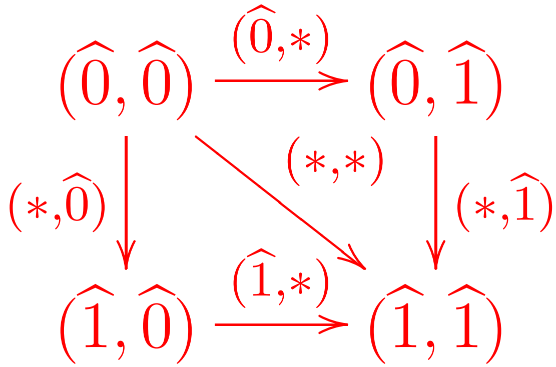
Realizing poset

- $(A, \leqslant)$ poset
- Element of A = State
- Execution path from α to β iff $\alpha < \beta$
- Note $\mathbb{P}_{\alpha,\alpha}A = \varnothing$ for all $\alpha \in A$

Functor $\{$poset+strictly increasing map$\} \to \mathbf{Flow}$

Bad realization of precubical set

- K precubical set : $$|K|_{bad} := \varinjlim_{\square[n]\to K} \{\widehat{0} < \widehat{1}\}^n$$
- The flow $|\square[2]|_{bad} = \{\widehat{0} < \widehat{1}\}^2$:



with $(\widehat{0},*) * (*,\widehat{1}) = (*,\widehat{0}) * (\widehat{1},*) = (*,*)$

- $|\partial\square[n]|_{bad} \cong |\square[n]|_{bad}$ for $n \geqslant 3$

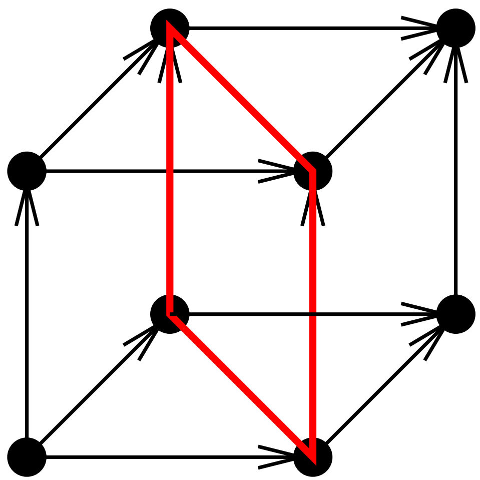
Homotopy colimit and weak S-homotopy

- $f : X \xrightarrow{\simeq} Y$ weak S-homotopy iff f^0 bijection and $\mathbb{P}f$ weak homotopy equivalence
- $\mathbf{Ho}(\mathbf{Flow}^I)$ homotopy category of $\mathbf{Flow}^I$ inverting objectwise weak S-homotopy equivalences, I small category
- Constant diagram functor $\mathbf{Ho}(\mathbf{Flow}) \rightarrow \mathbf{Ho}(\mathbf{Flow}^I)$

$$\begin{array}{ccc} \mathbf{Flow}^I & \xrightarrow{\overrightarrow{\mathrm{holim}}} & \mathbf{Flow} \\ \downarrow & & \downarrow \\ \mathbf{Ho}(\mathbf{Flow}^I) & \xrightarrow{\text{left adjoint}} & \mathbf{Ho}(\mathbf{Flow}) \end{array}$$

Good realization of precubical set

- K precubical set : $$|K|_{good} := \varinjlim_{\square[n]\to K} \{\widehat{0} < \widehat{1}\}^n$$
- $\mathbb{P}_{\widehat{0}\dots\widehat{0},\widehat{1}\dots\widehat{1}}|\partial\square[n]|_{good}$ homotopy equivalent to $\mathbf{S}^{n-2}$
- Underlying state space of $|\partial\square[n]|_{good}$ homotopy equivalent to $\mathbf{S}^{n-1}$



Weak S-homotopy model category

Theorem. *There exists exactly one model structure* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *on* **Flow** *such that*

- $\mathrm{Fib} = \{f : X \to Y$ *s.t.* $\mathbb{P}f$ *Serre fibration*$\}$
- $\mathcal{W} = \{$*weak S-homotopy*$\}$ *(*$\Rightarrow R : \{0,1\} \to \{0\} \in \mathrm{Cof}$*)*

It is cofibrantly generated, proper, simplicial, not cellular, not topological. Every flow is fibrant.

- K precubical set :

$$|K|_{good} \simeq \boxed{\varinjlim_{\square[n] \to K} (\{\widehat{0} < \widehat{1}\}^n)^{cof} =: |K|_{flow}}$$

- Advantage : it is a left adjoint
- Drawback : not very tractable, because of use of transfinite constructions

Globe of a topological space

- The globe $\mathrm{Glob}(Z)$ of the topological space Z
  - $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$
  - $\mathbb{P}\mathrm{Glob}(Z) = Z$
  - $s = \widehat{0}$
  - $t = \widehat{1}$
  - no composable non-constant execution paths
- The directed segment $\mathrm{Glob}(\{*\}) = \overrightarrow{I}$

Axiomatizing realization functors

- A functor $F : \square^{op}\mathbf{Set} \to \mathbf{Flow}$ is a realization functor if:
  - It is colimit-preserving
  - There is an objectwise trivial fibration of cocubical flows $F(\square[*]) \xrightarrow{\simeq} \{\widehat{0} < \widehat{1}\}^*$
  - For all $n \geqslant 0$, $F(\partial\square[n]) \to F(\square[n])$ cofibration
- The functor $K \mapsto |K|_{flow}$ is a realization functor
- Moreover there is a homotopy pushout diagram of flows

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^{n-1}) & \longrightarrow & F(\partial\square[n+1]) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^{n}) & \longrightarrow & F(\square[n+1]) \end{array} \quad (h)$$

Small realizations

Theorem. *There exists a realization functor* $K \mapsto \mathrm{gl}(K)$ *such that for all* $n \geqslant 0$*, there is a pushout diagram of flows:*

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^{n-1}) & \longrightarrow & \mathrm{gl}(\partial\square[n+1]) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^n) & \longrightarrow & \mathrm{gl}(\square[n+1]). \end{array}$$

Moreover, the pushout above is a homotopy pushout. And there exist natural (weak) S-homotopy equivalences

$$\mu_K : \mathrm{gl}(K) \xrightarrow{\simeq} |K|_{flow}$$

and

$$\nu_K : |K|_{flow} \xrightarrow{\simeq} \mathrm{gl}(K).$$

Construction of $\mathrm{gl}(-)$

- Construction of $\mathrm{gl}(K)$ by induction on the dimension of K
- $\mathrm{gl}(\square[0]) = \{0\}$
- Choice of a map $s_n : \mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{gl}(\partial\square[n+1])$ inducing a homotopy equivalence

$$\mathbb{P}_{\widehat{0},\widehat{1}} s_n : \mathbf{S}^{n-1} \longrightarrow \mathbb{P}_{\widehat{0}\ldots\widehat{0},\widehat{1}\ldots\widehat{1}}\, \mathrm{gl}(\partial\square[n+1])$$

-

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^{n-1}) & \xrightarrow{s_n} & \mathrm{gl}(\partial\square[n+1]) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^n) & \longrightarrow & \mathrm{gl}(\square[n+1]) \end{array}$$

Uniqueness of $\mathrm{gl}(-)$

- Colimit-preserving functor $\mathrm{gl} : \square^{op}\mathbf{Set} \to \mathbf{Flow}$ characterized by cocubical flow $\mathrm{gl}(\square[*])$
- $\mathrm{gl}_1, \mathrm{gl}_2$ two realization functors
- $\mathcal{F}_0$ set of natural transformations μ

$$\begin{array}{ccc} \mathrm{gl}_1(\square[*]) & \xrightarrow{\quad\mu\quad} & \mathrm{gl}_2(\square[*]) \\ & \searrow \quad \swarrow & \\ & \{\widehat{0} < \widehat{1}\}^* & \end{array}$$

- The space of μ's is contractible

$$\mathrm{gl}_1(K) \simeq \underset{\square[n]\to K}{\underrightarrow{\mathrm{holim}}}\ \mathrm{gl}_1(\square[n]) \simeq \underset{\square[n]\to K}{\underrightarrow{\mathrm{holim}}}\ \mathrm{gl}_2(\square[n]) \simeq \mathrm{gl}_2(K)$$

Contractibility of the space of μ's

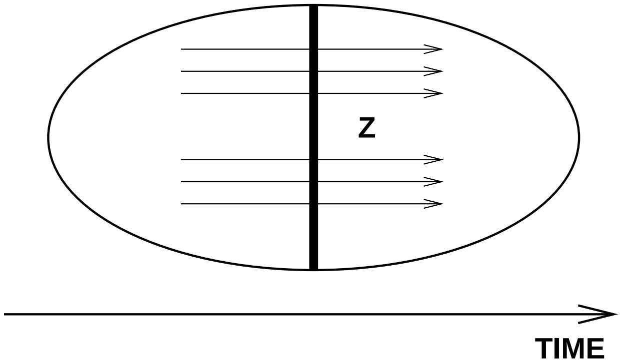
- Small category $\square$ direct Reedy category
- Simplicial Reedy model structure of cocubical flows
- Pullback diagram of simplicial sets:

$$\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathrm{Map}(\mathrm{gl}_1(\square[*]), \mathrm{gl}_2(\square[*])) \\
\downarrow \simeq & & \downarrow \simeq \\
\left\{\mathrm{gl}_1(\square[*]) \to \{\widehat{0} < \widehat{1}\}^*\right\} & \longrightarrow & \mathrm{Map}(\mathrm{gl}_1(\square[*]), \{\widehat{0} < \widehat{1}\}^*)
\end{array}$$

Multipointed d-space

- Multipointed space (X, X^0): $X^0 \to X$ of $\mathbf{Set}{\downarrow}\mathbf{Top}$
- Multipointed d-space $(X, \mathbb{P}^{top}X, X^0)$
  - Multipointed space (X, X^0)
  - $\mathbb{P}^{top}X$ set of continuous paths closed under strictly increasing reparametrization and composition such that $\gamma : [0,1] \to X$ implies $\gamma(0), \gamma(1) \in X^0$
- A set S gives rise to the multipointed d-spaces $(S, \varnothing, S)$, (S, S, S), $(S, \varnothing, \varnothing)$

Globular complex



- $\left(\frac{\{\widehat{0},\widehat{1}\}\sqcup(Z\times[0,1])}{(z,0)=(z',0)=\widehat{0},(z,1)=(z',1)=\widehat{1}}, \mathbb{P}^{top}\mathrm{Glob}^{top}(Z), \{\widehat{0},\widehat{1}\}\right)$
- $\mathbb{P}^{top}\mathrm{Glob}^{top}(Z)$ closure by strict increasing reparametrization (and composition !) of the set of continuous maps $\{t \mapsto (z,t), z \in Z\}$.
- $I^{gl,top} := \{\mathrm{Glob}^{top}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}^{top}(\mathbf{D}^n), n \geqslant 0\}$
- Globular complex $(X, \mathbb{P}^{top}X, X^0)$: $(X^0, \varnothing, X^0) \to (X, \mathbb{P}^{top}X, X^0) \in \mathbf{cell}(I^{gl,top})$

Underlying homotopy type of a flow

The underlying homotopy type $\Omega(X)$ of a flow X

- Write $\varnothing \to X^{cof}$ as a transfinite composition of pushouts of $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$, and of $\mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{Glob}(\mathbf{D}^n)$ with $n \geqslant 0$
- Replace this transfinite composition by a transfinite composition of pushouts of $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$ and of $\mathrm{Glob}^{top}(\mathbf{S}^{n-1}) \to \mathrm{Glob}^{top}(\mathbf{D}^n)$ with $n \geqslant 0$
- Take the underlying topological spaces
- One obtains $\varnothing \to \Omega(X)$
- $\Omega : \mathbf{Flow} \to \mathbf{Ho}(\mathbf{Top})$

Cubical underlying homotopy type

Theorem. *For every precubical set K, there is a natural homotopy equivalence*

$$\Omega(|K|_{flow}) \simeq \varinjlim_{\square[n]\to K} [0,1]^n.$$

- Construction of a functor $\mathrm{gl}^{top} : \square^{op}\mathbf{Set} \to \mathbf{glTop}$ by replacing in the construction of $\mathrm{gl}(K)$ each $\mathrm{Glob}(Z)$ by a $\mathrm{Glob}^{top}(Z)$
- Comparison of the two cocubical topological spaces $|\,\mathrm{gl}^{top}(\square[*])|$ and $[0,1]^*$ using an appropriate Reedy model structure on cocubical topological spaces.

Euclidian ordering (work in progress)

- $\overrightarrow{[0,1]^n}$ multipointed d-space $([0,1]^n, \mathbb{P}^{top}\overrightarrow{[0,1]^n}, \{0,1\}^n)$
- $\mathbb{P}^{top}\overrightarrow{[0,1]^n}$ closure of the set of continous maps $\gamma : [0,1] \to [0,1]^n$ (strictly) increasing w. r. t. each coordinate axis
- $$|K|_{cub} := \varinjlim_{\square[n]\to K} \overrightarrow{[0,1]^n}$$
- Natural S-homotopy equivalence between $|K|_{cub}$ and $\mathrm{gl}^{top}(K)$

Flows colimit-generated by cubes

- $\mathbf{Flow}_{\Delta}$ category of flows over Δ-generated Δ-separated topological spaces
- That $\mathbf{Flow}_{\Delta}$ is locally presentable is a consequence of a recent work of J. Rosický
- Does there exist a model structure such that the set of generating cofibrations is:

$$\{\mathrm{gl}(\partial\square[n] \subset \square[n]), n \geqslant 0\} \cup \{R : \{0,1\} \longrightarrow \{0\}, C : \varnothing \longrightarrow \{0\}\}$$