

ABSTRACT HOMOTOPICAL METHODS FOR THEORETICAL COMPUTER SCIENCE

PHILIPPE GAUCHER

ABSTRACT. The purpose of this paper is to collect the homotopical methods used in the development of the theory of flows initialized by author's paper "A model category for the homotopy theory of concurrency". It is presented generalizations of the classical Whitehead theorem inverting weak homotopy equivalences between CW-complexes using weak factorization systems. It is also presented methods of calculation of homotopy limits and homotopy colimits using Quillen adjunctions and Reedy categories.

CONTENTS

1.	Introduction	1
2.	Whitehead theorem and weak factorization system	2
3.	Model category and Quillen adjunction	6
4.	Homotopy limit and homotopy colimit	9
5.	Application of the Reedy theory to the homotopy theory of flows	12
	References	14

1. INTRODUCTION

The purpose of this paper is to collect the homotopical methods used in the development of the theory of flows initialized by author's paper *A model category for the homotopy theory of concurrency* [Gau03c] (see Table 1). An overview of the theory of flows can be found in the two notes [Gau03a] and [Gau03b]. The purpose of this paper is not to give a course in abstract homotopy theory. Indeed, there already exist several good introductions to model category and more generally to abstract homotopy theory. For model category, see the short papers [DS95] [Hes02], or the books [Hov99] and [Hir03]. For the relation between model category and simplicial set, see the book [GJ99]. For cofibration and fibration categories, see the book [Bau89] or the notion of ABC cofibration and fibration categories in [RB06]. For a more general setting allowing the development of the theory of derived functors and homotopy limits and colimits without any model category structure, see for example the books [CS02] [DHKS04]. The original reference for model category is [Qui67] but Quillen's axiomatization is not used in this paper because it is obsolete. The Hovey's book axiomatization is preferred.

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It is also required a good knowledge of basic category theory. Possible references are [ML98] [Bor94a] [Bor94b] and [Bor94c].

The starting point is a complete cocomplete locally small category \mathcal{M} and a class of morphisms \mathcal{W} (the *weak equivalences*) modelling an equivalence relation between these objects. One would like to consider them to be isomorphisms. It is always possible to formally invert the weak equivalences by considering the *categorical localization* $\mathcal{M}[\mathcal{W}^{-1}]$ of the category \mathcal{M} with respect to the morphisms of \mathcal{W} . The categorical localization is equipped with a canonical functor $\mathcal{M} \rightarrow \mathcal{M}[\mathcal{W}^{-1}]$ which is the identity on objects and which is universal for the property of taking all morphisms of \mathcal{W} to isomorphisms. The category $\mathcal{M}[\mathcal{W}^{-1}]$ is difficult to understand since the class of morphisms from X to Y is the quotient of the class of finite zig-zag sequences $X = X_0 \longleftrightarrow X_1 \dots \longleftrightarrow X_n = Y$, where the notation $A \longleftrightarrow B$ means either a map from A to B or a map from B to A , and where all backward maps (i.e. pointing to the left) are weak equivalences, divided by the equivalence relation generated by removing or adding an identity map, and the identifications $A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{g \circ f} C$, $A \xrightarrow{w} B \xleftarrow{w} A = A \xrightarrow{\text{Id}_A} A$ and $B \xleftarrow{w} A \xrightarrow{w} B = B \xrightarrow{\text{Id}_B} B$ [ML98] [GZ67] or [DHKS04]. The class of morphisms in $\mathcal{M}[\mathcal{W}^{-1}]$ between two objects needs even not be a set. Two problems related to the study of $\mathcal{M}[\mathcal{W}^{-1}]$ are treated in this paper.

First of all, it may be useful to construct notions of *cylinder* or *cocylinder* functors related to \mathcal{W} . This can be done by exhibiting a full subcategory \mathcal{M}_{good} of “good” objects of \mathcal{M} such that every object of \mathcal{M} is weakly equivalent to a good object and such that one has an isomorphism of categories $\mathcal{M}_{good}/\sim \cong \mathcal{M}_{good}[\mathcal{W}^{-1}]$ between the quotient of \mathcal{M}_{good} by a congruence \sim generated by a cylinder or a cocylinder functor and the categorical localization $\mathcal{M}_{good}[\mathcal{W}^{-1}]$. Of course, this situation is similar to the usual Whitehead statement inverting weak homotopy equivalences between CW-complexes in classical algebraic topology ([Hat02] Theorem 4.5). The main tool used for this problem is the notion of *weak factorization system*.

It may be also useful to calculate homotopy colimits and homotopy limits with respect to \mathcal{W} . Let \mathcal{B} be a small category. Let $\mathcal{M}^{\mathcal{B}}$ be the category of functors from \mathcal{B} to \mathcal{M} . Let $\mathcal{W}_{\mathcal{B}}$ be the class of morphisms $f : D \rightarrow E$ of $\mathcal{M}^{\mathcal{B}}$ such that for every object b of \mathcal{B} , the map $f_b : D_b \rightarrow E_b$ is a weak equivalence: let us call such a morphism an *objectwise weak equivalence*. The constant diagram functor $\text{Diag} : \mathcal{M} \rightarrow \mathcal{M}^{\mathcal{B}}$ is defined by $\text{Diag}(X)_b = X$ for every object b of \mathcal{B} and $\text{Diag}(X)_{\phi} = \text{Id}_X$ (the identity of X) for every morphism ϕ of \mathcal{B} . It has both a left adjoint and a right adjoint since \mathcal{M} is complete and cocomplete, calculating respectively the colimit functor and the limit functor from $\mathcal{M}^{\mathcal{B}}$ to \mathcal{M} . It also induces a functor $\text{hoDiag} : \mathcal{M}[\mathcal{W}^{-1}] \rightarrow \mathcal{M}^{\mathcal{B}}[\mathcal{W}_{\mathcal{B}}^{-1}]$ between the localized categories because of the universal property satisfied by $\mathcal{M}[\mathcal{W}^{-1}]$. The problem is then to give an explicit description of the left and right adjoints of hoDiag , if they exist. There is a considerable mathematical literature about the subject. Connections between this problem and *model category* theory will be succinctly described. In particular, the Reedy approach will be discussed. The last section is devoted to the detailed description of several examples.

2. WHITEHEAD THEOREM AND WEAK FACTORIZATION SYSTEM

Let $i : A \rightarrow B$ and $p : X \rightarrow Y$ be maps of the category \mathcal{M} . Then i has the left lifting property (LLP) with respect to p (or p has the right lifting property (RLP) with respect

From this paper	Used in ...
Theorem 2.10	[Gau06a] Theorem 3.27 and Theorem 4.6
Proposition 3.5	[Gau05c] Theorem 5.5 ; <i>see also Theorem 4.8,4.9</i>
Proposition 4.1	[Gau05c] Lemma 8.6 and Lemma 8.7
Proposition 4.2	[Gau05e] Theorem 9.3, [Gau07b] Theorem 7.8
Theorem 4.7	[Gau05e] Theorem 8.4 ; <i>see also Theorem 4.8,4.9</i>
Theorem 4.7	[Gau06b] Theorem 8.1, Theorem 8.2 and Theorem 8.3
Theorem 4.8	[Gau05c] Lemma 8.5, [Gau05e] Theorem 7.5
Theorem 4.8	[Gau06b] Corollary 7.4, [Gau07b] Theorem 7.8
Theorem 4.8 + left properness	[Gau05e] Theorem 11.2, [Gau06b] Theorem 9.1
Theorem 4.9	[Gau05a] Theorem IV.3.10 and Theorem IV.3.14
Proposition 5.1	<i>see Theorem 4.8</i>

TABLE 1. Summary of use of homotopical facts in homotopy theory of flows

to *i*) if for every commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & X \\
 \downarrow i & \nearrow g & \downarrow p \\
 B & \xrightarrow{\beta} & Y,
 \end{array}$$

there exists a morphism g called a *lift* making both triangles commutative.

Definition 2.1. [AHRT02] A weak factorization system is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of \mathcal{M} such that the class \mathcal{L} is the class of morphisms having the LLP with respect to \mathcal{R} , such that the class \mathcal{R} is the class of morphisms having the RLP with respect to \mathcal{L} and such that every morphism of \mathcal{M} factors as a composite $r \circ \ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. The weak factorization system is functorial if the factorization $r \circ \ell$ can be made functorial.

If K is a set of morphisms of \mathcal{M} , then the class of morphisms of \mathcal{M} that satisfy the RLP with respect to every morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of \mathcal{M} that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of \mathcal{M} that satisfy the LLP with respect to every morphism of $\mathbf{inj}(K)$.

In a weak factorization system $(\mathcal{L}, \mathcal{R})$, the class \mathcal{L} (resp. \mathcal{R}) is completely determined by \mathcal{R} (resp. \mathcal{L}). The class of morphisms \mathcal{L} is closed under composition, pushout, retract and binary coproduct. In particular, one has the inclusion $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Dually, the class of morphisms \mathcal{R} is closed under composition, pullback, retract and binary product.

Definition 2.2. Let \mathcal{A} be a subcategory of \mathcal{M} . Then an object W is κ -small relative to \mathcal{A} for some regular cardinal κ if for every λ -sequence with $\lambda \geq \kappa$

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots \quad (\beta < \lambda)$$

such that $X_\beta \rightarrow X_{\beta+1}$ is in \mathcal{A} for every ordinal β such that $\beta + 1 < \lambda$, the set map $\varinjlim_{\beta < \lambda} \mathcal{M}(W, X_\beta) \rightarrow \mathcal{M}(W, \varinjlim_{\beta < \lambda} X_\beta)$ is bijective. An object W is small relative to \mathcal{A} if it is κ -small relative to \mathcal{A} for some regular cardinal κ .

A set K of morphisms of \mathcal{M} permits the small object argument if the domains of the morphisms of K are small relative to $\mathbf{cell}(K)$. In such a situation, the pair of classes of morphisms $(\mathbf{cof}(K), \mathbf{inj}(K))$ can be made a functorial weak factorization system using the small object argument ([Hir03] Proposition 10.5.16, [Hov99] Theorem 2.1.14). And moreover, every morphism of $\mathbf{cof}(K)$ is then a retract of a morphism of $\mathbf{cell}(K)$ ([Hov99] Corollary 2.1.15).

Definition 2.3. A functorial weak factorization system $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated if there exists a set K of morphisms of \mathcal{M} permitting the small object argument such that $\mathcal{L} = \mathbf{cof}(K)$ and $\mathcal{R} = \mathbf{inj}(K)$. In this case, one explicitly supposes that the functorial factorization is given by the small object argument described in [Hir03] Proposition 10.5.16 or [Hov99] Theorem 2.1.14, and not by any other method.

Definition 2.3 appears in [Bek00] in the context of locally presentable category in the sense of [AR94] as the notion of *small* weak factorization system. The reason of this terminology is that in a locally presentable category, every set of morphisms permits the small object argument. Indeed, every object of such a category is κ -presentable for a big enough regular cardinal κ , and therefore κ -small relative to the whole class of morphisms. So:

Proposition 2.4. ([Bek00] Proposition 1.3) For every set of morphisms K of a locally presentable category, the pair of classes of morphisms $(\mathbf{cof}(K), \mathbf{inj}(K))$ is a cofibrantly generated weak factorization system.

For the sequel, let us fix a functorial weak factorization system $(\mathcal{L}, \mathcal{R})$.

Definition 2.5. Let X be an object of \mathcal{M} . The cylinder object of X with respect to \mathcal{L} is the functorial factorization

$$X \oplus X \xrightarrow{\alpha(\mathrm{Id}_X \oplus \mathrm{Id}_X)} \mathrm{Cyl}_{\mathcal{L}}(X) \xrightarrow{\beta(\mathrm{Id}_X \oplus \mathrm{Id}_X)} X$$

of the map $\mathrm{Id}_X \oplus \mathrm{Id}_X : X \oplus X \rightarrow X$ by the morphism $\alpha(\mathrm{Id}_X \oplus \mathrm{Id}_X) : X \oplus X \rightarrow \mathrm{Cyl}_{\mathcal{L}}(X)$ of \mathcal{L} composed with the morphism $\beta(\mathrm{Id}_X \oplus \mathrm{Id}_X) : \mathrm{Cyl}_{\mathcal{L}}(X) \rightarrow X$ of \mathcal{R} .

Definition 2.6. An object X of \mathcal{M} is cofibrant with respect to \mathcal{L} if the unique morphism $f_X : \emptyset \rightarrow X$, where \emptyset is the initial object of \mathcal{M} , is an element of \mathcal{L} . Denote by $\mathcal{M}_{\mathbf{cof}}$ the category of cofibrant objects with respect to \mathcal{L} .

Definition 2.7. Let $f, g : X \rightrightarrows Y$ be two morphisms of \mathcal{M} . A left homotopy with respect to \mathcal{L} from f to g is a morphism $H : \mathrm{Cyl}_{\mathcal{L}} X \rightarrow Y$ such that

$$H \circ \alpha(\mathrm{Id}_X \oplus \mathrm{Id}_X) = f \oplus g.$$

This defines a reflexive and symmetric binary relation. The transitive closure is denoted by $\sim_{\mathcal{L}}^l$.

It is clear that the weak factorization system $(\mathcal{L}, \mathcal{R})$ induces a weak factorization system denoted in the same way on the full subcategory of cofibrant objects with respect to \mathcal{L} . One then obtains the:

Theorem 2.8. [KR05] *One has:*

- *The binary relation $\sim_{\mathcal{L}}^l$ does not depend on the choice of the functorial factorization.*
- *The equivalence relation $\sim_{\mathcal{L}}^l$ is a congruence.*
- *Every object of \mathcal{M} is isomorphic in $\mathcal{M}[\mathcal{R}^{-1}]$ to a cofibrant object with respect to \mathcal{L} .*
- *The category $\mathcal{M}_{\text{cof}}/\sim_{\mathcal{L}}^l$ and $\mathcal{M}_{\text{cof}}[\mathcal{R}^{-1}]$ are isomorphic.*
- *The category $\mathcal{M}_{\text{cof}}[\mathcal{R}^{-1}]$ is locally small.*

The class of morphisms \mathcal{R} plays the role of weak equivalences in Theorem 2.8. This is exactly the situation encountered by Y. Lafont and F. Métayer in [Mét03] [LM06] [Laf06] [Mét07] in their study of higher dimensional rewriting systems using ω -categories.

It is possible to dualize these results by working in the opposite category \mathcal{M}^{op} and by considering the weak factorization system $(\mathcal{R}^{op}, \mathcal{L}^{op})$. By definition, a *fibrant object of \mathcal{M} with respect to \mathcal{L}* is a cofibrant object of \mathcal{M}^{op} with respect to \mathcal{R}^{op} . The *path object $\text{Path}_{\mathcal{L}} X$ of X with respect to \mathcal{L}* is the cylinder object $\text{Cyl}_{\mathcal{L}} X$ of X with respect to \mathcal{R}^{op} . Finally, a *right homotopy with respect to \mathcal{L}* between two maps $f, g : X \rightrightarrows Y$ is a left homotopy between $f^{op}, g^{op} : Y \rightrightarrows X$ with respect to \mathcal{R}^{op} . The transitive closure of this binary relation is denoted by $\sim_{\mathcal{L}}^r$. One obtains the following theorem, in which the role of weak equivalences is now played by the morphisms of \mathcal{L} :

Theorem 2.9. [KR05] *One has:*

- *The binary relation $\sim_{\mathcal{L}}^r$ does not depend on the choice of the functorial factorization.*
- *The equivalence relation $\sim_{\mathcal{L}}^r$ is a congruence.*
- *Every object of \mathcal{M} is isomorphic in $\mathcal{M}[\mathcal{L}^{-1}]$ with a fibrant object with respect to \mathcal{L} .*
- *The category $\mathcal{M}_{\text{fib}}/\sim_{\mathcal{L}}^r$ and $\mathcal{M}_{\text{fib}}[\mathcal{L}^{-1}]$ are isomorphic, where \mathcal{M}_{fib} is the full subcategory of fibrant objects with respect to \mathcal{L} .*
- *The category $\mathcal{M}_{\text{fib}}[\mathcal{L}^{-1}]$ is locally small.*

The interest of the dual theorem is that it can be improved as follows:

Theorem 2.10. [Gau06a] *Suppose that the weak factorization system $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated and that every map of \mathcal{L} is a monomorphism. Then the inclusion functor $\mathcal{M}_{\text{fib}} \subset \mathcal{M}$ induces an equivalence of categories $\mathcal{M}_{\text{fib}}/\sim_{\mathcal{L}}^r \simeq \mathcal{M}[\mathcal{L}^{-1}]$. In particular, the category $\mathcal{M}[\mathcal{L}^{-1}]$ is locally small.*

Theorem 2.10 is proved using a fibrant replacement functor $R_{\mathcal{L}} : \mathcal{M} \rightarrow \mathcal{M}_{\text{fib}}$ defined by factoring the natural map $X \rightarrow \mathbf{1}$ from an object X to the terminal object as a composite $X \rightarrow R_{\mathcal{L}}(X) \rightarrow \mathbf{1}$ using the weak factorization system $(\mathcal{L}, \mathcal{R})$. The factorization must be obtained using the small object argument. It is then possible to prove that the functor $R_{\mathcal{L}}$ is inverse to the inclusion functor up to isomorphism of functors.

Theorem 2.10 is actually used in [Gau06a] with \mathcal{M} replaced by the full subcategory of cofibrant objects of a model category in the sense of Definition 3.1. It enables us to prove a Whitehead theorem for the full dihomotopy relation on flows (see [Gau05d] for an informal introduction about dihomotopy of flows). By [Gau05b], the dihomotopy relation

on the category of flows does not correspond to any model category structure in the sense of Definition 3.1 but it can be fully described using the weak S-homotopy model structure constructed in [Gau03c] and an additional weak factorization system modelling refinement of observation.

3. MODEL CATEGORY AND QUILLEN ADJUNCTION

It is introduced in this section the fundamental tool of *model category* and *Quillen derived functor*. The difference with the preceding section is that we now have two weak factorization systems interacting with each other and which are related to the class of weak equivalences.

Definition 3.1. *A model category is a complete and cocomplete category \mathcal{M} equipped with three classes of morphisms $(\text{Cof}, \text{Fib}, \mathcal{W})$ (resp. called the classes of cofibrations, fibrations and weak equivalences) such that:*

- (1) *the class of morphisms \mathcal{W} is closed under retracts and satisfies the 2-out-of-3 property i.e.: if f and g are morphisms of \mathcal{M} such that $g \circ f$ is defined and two of f , g and $g \circ f$ are weak equivalences, then so is the third.*
- (2) *the pairs $(\text{Cof} \cap \mathcal{W}, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap \mathcal{W})$ are both functorial weak factorization systems.*

The triple $(\text{Cof}, \text{Fib}, \mathcal{W})$ is called a model structure. An element of $\text{Cof} \cap \mathcal{W}$ is called a trivial cofibration. An element of $\text{Fib} \cap \mathcal{W}$ is called a trivial fibration. The categorical localization $\mathbf{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$ is called the homotopy category of \mathcal{M} . The model category \mathcal{M} is cofibrantly generated if both weak factorization systems $(\text{Cof} \cap \mathcal{W}, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap \mathcal{W})$ are cofibrantly generated.

It is introduced in [Gau03c] a cofibrantly generated model structure for the study of concurrency. The objects are called *flows*, they are actually categories without identity maps enriched over compactly generated topological spaces (more details for this kind of topological spaces in [Bro88] [May99], the appendix of [Lew78] and also the preliminaries of [Gau03c]). The weak equivalences are the morphisms of flows inducing a bijection between the sets of objects and a weak homotopy equivalence between the spaces of morphisms; the fibrations are the morphisms of flows inducing a fibration on the space of morphisms; finally the cofibrations are determined by the left lifting property with respect to trivial fibrations. Another cofibrantly generated model category relevant for concurrency theory is the model structure constructed by K. Worytkiewicz on cubical sets [Wor06]. The common feature of the two model structures is that the directed segment is not equivalent to a point. It is very important for the preservation of the causal structure of the underlying time flow. See the long introduction of [Gau06a] for further explanations.

An object X of a model category \mathcal{M} is *cofibrant* (resp. *fibrant*) if and only if it is cofibrant with respect to Cof (resp. fibrant with respect to $\text{Cof} \cap \mathcal{W}$). The canonical morphism $\emptyset \rightarrow X$ functorially factors as a composite $\emptyset \rightarrow X^{cof} \rightarrow X$ of a cofibration $\emptyset \rightarrow X^{cof}$ followed by a trivial fibration $X^{cof} \rightarrow X$. Symmetrically, the canonical morphism $X \rightarrow \mathbf{1}$ functorially factors as a composite $X \rightarrow X^{fib} \rightarrow \mathbf{1}$ of a trivial cofibration $X \rightarrow X^{fib}$ followed by a fibration $X^{fib} \rightarrow \mathbf{1}$.

Definition 3.2. *The functor $X \mapsto X^{cof}$ is called the cofibrant replacement functor. The functor $X \mapsto X^{fib}$ is called the fibrant replacement functor.*

Proposition 3.3. ([Hov99] Theorem 1.2.10 and [Hir03] Theorem 8.3.9) *Let \mathcal{M} be a model category. Let $\mathcal{M}_{cof,fib}$ be the full subcategory of cofibrant-fibrant objects of \mathcal{M} . Then the inclusion functor $\mathcal{M}_{cof,fib} \subset \mathcal{M}$ induces an equivalence of categories $\mathcal{M}_{cof,fib}/\sim \simeq \mathbf{Ho}(\mathcal{M})$ where the congruence \sim is left homotopy with respect to \mathbf{Cof} , or equivalently right homotopy with respect to $\mathbf{Cof} \cap \mathcal{W}$ (the two congruences coincide on cofibrant-fibrant objects). In particular, the homotopy category $\mathbf{Ho}(\mathcal{M})$ is locally small.*

The model category \mathcal{M} is *left proper* (resp. *right proper*) if every pushout (resp. pullback) of a weak equivalence along a cofibration (resp. fibration) is a weak equivalence. A model category \mathcal{M} is *proper* if it is both left and right proper.

The following proposition explains the relation between properness, fibrancy and cofibrancy. It can also be viewed as a method of construction of weak equivalences.

Proposition 3.4. (Reedy) ([Hir03] Proposition 13.1.2) *Let \mathcal{M} be a model category. Then*

- *Every pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence.*
- *Every pullback of a weak equivalence between fibrant objects along a fibration is a weak equivalence.*

The two consequences of Proposition 3.4 are:

- A model category where all objects are cofibrant (like the model category of simplicial sets [GJ99]) is left proper.
- A model category where all objects are fibrant (like the model category of compactly generated topological spaces [Hov99] or the model category of flows [Gau03c]) is right proper.

The model categories of simplicial sets, of compactly generated topological spaces and of flows are actually all of them proper, i.e. both left and right proper.

Proposition and Definition 3.5. ([Hir03] Proposition 8.5.3 and Proposition 8.5.4) *A Quillen adjunction is a pair of adjoint functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ between the model categories \mathcal{M} and \mathcal{N} such that one of the following equivalent properties holds:*

- (1) *F preserves both cofibrations and trivial cofibrations.*
- (2) *G preserves both fibrations and trivial fibrations.*
- (3) *F preserves both cofibrations between cofibrant objects and trivial cofibrations (D. Dugger).*
- (4) *G preserves both fibrations between fibrant objects and trivial fibrations (D. Dugger).*

One says that F is a left Quillen functor. One says that G is a right Quillen functor. Moreover, any left Quillen functor preserves weak equivalences between cofibrant objects and any right Quillen functor preserves weak equivalences between fibrant objects.

The branching space functor from the category of flows to the category of compactly generated topological spaces, defined in [Gau05c] and studied in [Gau05e] is an example of left Quillen functor if the category of flows is equipped with the weak S-homotopy model structure constructed in [Gau03c] and if the category of compactly generated topological spaces is equipped with the usual Quillen model structure.

Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen adjunction between the model categories \mathcal{M} and \mathcal{N} . The cofibrant replacement functor takes weak equivalence to weak equivalence by the 2-out-of-3 property. So the functor $F \circ (-)^{cof} : \mathcal{M} \rightarrow \mathcal{N}$ induces a unique functor $\mathbf{L}F : \mathbf{Ho}(\mathcal{M}) \rightarrow \mathbf{Ho}(\mathcal{N})$ making the following diagram commutative

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F \circ (-)^{cof}} & \mathcal{N} \\ i_{\mathcal{M}} \downarrow & & \downarrow i_{\mathcal{N}} \\ \mathbf{Ho}(\mathcal{M}) & \xrightarrow{\mathbf{L}F} & \mathbf{Ho}(\mathcal{N}). \end{array}$$

because of the universal property satisfied by $\mathbf{Ho}(\mathcal{M})$. In the same way, the fibrant replacement functor takes weak equivalence to weak equivalence by the 2-out-of-3 property. So the functor $G \circ (-)^{fib} : \mathcal{N} \rightarrow \mathcal{M}$ induces a unique functor $\mathbf{R}G : \mathbf{Ho}(\mathcal{N}) \rightarrow \mathbf{Ho}(\mathcal{M})$ making the following diagram commutative

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{G \circ (-)^{fib}} & \mathcal{M} \\ i_{\mathcal{N}} \downarrow & & \downarrow i_{\mathcal{M}} \\ \mathbf{Ho}(\mathcal{N}) & \xrightarrow{\mathbf{R}G} & \mathbf{Ho}(\mathcal{M}) \end{array}$$

because of the universal property satisfied by $\mathbf{Ho}(\mathcal{N})$.

Definition 3.6. *The functor $\mathbf{L}F : \mathbf{Ho}(\mathcal{M}) \rightarrow \mathbf{Ho}(\mathcal{N})$ is called the total left derived functor of F . The functor $\mathbf{R}G : \mathbf{Ho}(\mathcal{N}) \rightarrow \mathbf{Ho}(\mathcal{M})$ is called the total right derived functor of G .*

The following theorem is the main tool for the sequel:

Theorem 3.7. *(Quillen) ([Hov99] Lemma 1.3.10 or [Hir03] Theorem 8.5.18) The pair of functors $\mathbf{L}F : \mathbf{Ho}(\mathcal{M}) \rightleftarrows \mathbf{Ho}(\mathcal{N}) : \mathbf{R}G$ is a categorical adjunction.*

The natural transformation $(-)^{cof} \Rightarrow \text{Id}$ induces a natural transformation $\mathbf{L}F \circ i_{\mathcal{M}} = i_{\mathcal{N}} \circ F \circ (-)^{cof} \Rightarrow i_{\mathcal{N}} \circ F$. Of course, if F already preserves weak equivalences, then one has the isomorphism of functors $\mathbf{L}F \circ i_{\mathcal{M}} \cong i_{\mathcal{N}} \circ F$. And the natural transformation $\text{Id} \Rightarrow (-)^{fib}$ induces a natural transformation $i_{\mathcal{M}} \circ G \Rightarrow i_{\mathcal{M}} \circ G \circ (-)^{fib} = \mathbf{R}G \circ i_{\mathcal{N}}$. And if G already preserves weak equivalences, then one has the isomorphism of functors $i_{\mathcal{M}} \circ G \cong \mathbf{R}G \circ i_{\mathcal{N}}$.

In fact, the functor $\mathbf{L}F : \mathbf{Ho}(\mathcal{M}) \rightarrow \mathbf{Ho}(\mathcal{N})$ is the right Kan extension of the functor $i_{\mathcal{N}} \circ F : \mathcal{M} \rightarrow \mathbf{Ho}(\mathcal{N})$ along the canonical functor $i_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbf{Ho}(\mathcal{M})$ and the functor $\mathbf{R}G : \mathbf{Ho}(\mathcal{N}) \rightarrow \mathbf{Ho}(\mathcal{M})$ is the left Kan extension of the functor $i_{\mathcal{M}} \circ G : \mathcal{N} \rightarrow \mathbf{Ho}(\mathcal{M})$ along the canonical functor $i_{\mathcal{N}} : \mathcal{N} \rightarrow \mathbf{Ho}(\mathcal{N})$. So the functor $\mathbf{L}F \circ i_{\mathcal{M}}$ is the closest approximation of F preserving weak equivalences and the functor $\mathbf{R}G \circ i_{\mathcal{N}}$ is the closest approximation of G preserving weak equivalences.

If F_1 and F_2 are two composable left Quillen adjoints, then $F_1 \circ F_2$ is a left Quillen adjoint and the natural transformation $F_1 \circ (-)^{cof} \circ F_2 \circ (-)^{cof} \Rightarrow F_1 \circ F_2 \circ (-)^{cof}$ induces

an isomorphism of functors $\mathbf{L}(F_1) \circ \mathbf{L}(F_2) \cong \mathbf{L}(F_1 \circ F_2)$. Similarly, if G_1 and G_2 are two composable right Quillen adjoints, then $G_1 \circ G_2$ is a right Quillen adjoint and the natural transformation $G_1 \circ G_2 \circ (-)^{fib} \Rightarrow G_1 \circ (-)^{fib} \circ G_2 \circ (-)^{fib}$ induces an isomorphism of functors $\mathbf{R}(G_1 \circ G_2) \cong \mathbf{R}(G_1) \circ \mathbf{R}(G_2)$. See [Hov99] Theorem 1.3.7 for further details.

4. HOMOTOPY LIMIT AND HOMOTOPY COLIMIT

We want to give in this section constructions of homotopy limits and homotopy colimits in the following situations:

- a construction of $\underline{\text{holim}}$ for every cofibrantly generated model category \mathcal{M}
- a construction of $\overleftarrow{\text{holim}}$ for every cofibrantly generated model category \mathcal{M} such that \mathcal{M} is locally presentable¹
- a construction of $\underline{\text{holim}}$ when \mathcal{B} is a Reedy category with fibrant constants
- a construction of $\overleftarrow{\text{holim}}$ when \mathcal{B} is a Reedy category with cofibrant constants.

It is not always possible to get such a situation. Hence the interest of other approaches like [CS02] [DHKS04], the simplicial technique of [Hir03] Chapter 18 and the technique of frames of [Hir03] Chapter 19.

Before going further, it may be useful to point out that the homotopy category of any model category is weakly complete and weakly cocomplete. Weak limit and weak colimit satisfy the same property as limit and colimit except the unicity. Weak small (co)products coincide with small (co)products. Weak (co)limits are constructed using small (co)products and weak (co)equalizers in the same way as (co)limits are constructed by small (co)products and (co)equalizers ([ML98] Theorem 1 p109). And a weak coequalizer

$$A \begin{matrix} \xrightarrow{f,g} \\ \rightrightarrows \end{matrix} B \xrightarrow{h} D$$

is given by a weak pushout

$$\begin{array}{ccc} B & \xrightarrow{h} & D \\ \uparrow (f, \text{Id}_B) & & \uparrow h \\ A \sqcup B & \xrightarrow{(g, \text{Id}_B)} & B \end{array}$$

And finally, weak pushouts (resp. weak pullbacks) are given by homotopy pushouts (resp. homotopy pullbacks) by [Ros05] Remark 4.1. See also [Hel88] Chapter III.

Let us come back now to homotopy limits and colimits. The principle, exposed in this section, for calculating $\underline{\text{holim}}$ is the construction of a model structure on the category of

¹ The construction requires that the class of weak equivalences satisfies the solution set condition. J. Rosický has a proof that every locally presentable cofibrantly generated model category has an accessible class of weak equivalences. Thus in particular, it satisfies the solution set condition: these two conditions are not equivalent; a large cardinal axiom is needed for the converse [RT95]. If \mathcal{M} is left proper and simplicial, then [Hov01] Proposition 3.2 provides an accessible functor from \mathcal{M} to simplicial sets detecting weak equivalences. So by [AR94] Corollary 2.45, the class of weak equivalences satisfies the solution set condition since the class of weak equivalences of simplicial sets is accessible ([Bek00] Example 3.1 and [Ill71] [Ill72]).

diagrams $\mathcal{M}^{\mathcal{B}}$ such that the colimit functor $\varinjlim : \mathcal{M}^{\mathcal{B}} \rightarrow \mathcal{M}$ becomes a left Quillen functor. Symmetrically, the principle for calculating \varprojlim is the construction of a model structure on the category of diagrams $\mathcal{M}^{\mathcal{B}}$ such that the limit functor $\varprojlim : \mathcal{M}^{\mathcal{B}} \rightarrow \mathcal{M}$ becomes a right Quillen functor. Indeed, if the categorical adjunction $\varinjlim : \mathcal{M}^{\mathcal{B}} \rightleftarrows \mathcal{M} : \text{Diag}$ is a Quillen adjunction, then the natural transformation $\text{hoDiag} \Rightarrow \text{hoDiag} \circ \mathbf{Ho}((-)^{fib}) = \mathbf{RDiag}$ is an isomorphism of functors. So the left adjoint of hoDiag exists by Theorem 3.7 and is the homotopy colimit. And if the categorical adjunction $\text{Diag} : \mathcal{M} \rightleftarrows \mathcal{M}^{\mathcal{B}} : \varprojlim$ is a Quillen adjunction, then the natural transformation $\mathbf{LDiag} = \text{hoDiag} \circ \mathbf{Ho}((-)^{cof}) \Rightarrow \text{hoDiag}$ is an isomorphism of functors as well. So the right adjoint of hoDiag exists by Theorem 3.7 and is the homotopy limit. A straightforward consequence is:

Proposition 4.1. *A total left derived functor commutes with homotopy colimits. A total right derived functor commutes with homotopy limits.*

Proposition 4.1 is used in the proofs of [Gau05c] Lemma 8.6 and [Gau05c] Lemma 8.7. The following proposition gives an example of calculation of homotopy colimit, used in [Gau05e] Theorem 9.3 and in [Gau07b] Theorem 7.8:

Proposition 4.2. ([Hir03] *Proposition 18.1.6 and Proposition 14.3.13*) *The homotopy colimit of a diagram of contractible topological spaces over \mathcal{B} is homotopy equivalent to the classifying space of \mathcal{B} ([Hir03] Chapter 14 or [Seg68] [Qui73] for a definition of the classifying space). In particular, if \mathcal{B} has an initial or a terminal object, then this homotopy colimit is contractible as well.*

There exist two general theorems providing model structures on $\mathcal{M}^{\mathcal{B}}$ such that the colimit functor (resp. the limit functor) is a left (resp. right) Quillen functor. The following theorem ensures the existence of homotopy colimit for any cofibrantly generated model category \mathcal{M} and for any small category \mathcal{B} :

Theorem 4.3. ([Hir03] *Theorem 11.6.1, Theorem 11.6.8 and Theorem 13.1.14*) *Let us suppose \mathcal{M} equipped with a cofibrantly generated model structure. Then there exists a unique model structure on $\mathcal{M}^{\mathcal{B}}$ such that the fibrations are the objectwise fibrations and the weak equivalences the objectwise weak equivalences. Moreover, one has:*

- *Every cofibration of this model structure is an objectwise cofibration.*
- *This model structure is cofibrantly generated.*
- *The colimit functor $\varinjlim : \mathcal{M}^{\mathcal{B}} \rightarrow \mathcal{M}$ is a left Quillen functor.*
- *If \mathcal{M} is left proper (resp. right proper, proper), then so is $\mathcal{M}^{\mathcal{B}}$.*

The following theorem ensures the existence of homotopy limit for any cofibrantly generated model category \mathcal{M} with \mathcal{M} locally presentable with a class of weak equivalences satisfying the solution set condition and for any small category \mathcal{B} :

Theorem 4.4. (*Unknown reference*) *Let us suppose \mathcal{M} locally presentable and cofibrantly generated. Then there exists a unique model structure on $\mathcal{M}^{\mathcal{B}}$ such that the cofibrations are the objectwise cofibrations and the weak equivalences the objectwise weak equivalences. Moreover, one has:*

- *Every fibration of this model structure is an objectwise fibration.*

- *This model structure is cofibrantly generated.*
- *The limit functor $\varprojlim : \mathcal{M}^{\mathcal{B}} \rightarrow \mathcal{M}$ is a right Quillen functor.*

Theorem 4.4 is a direct consequence of a theorem due to J. Smith and exposed in [Bek00] Theorem 1.7. Theorem 4.4 is very close to the statement of [Lur06] Proposition A.3.3.3. Theorem 4.3 and Theorem 4.4 can be applied to the case of category of simplicial presheaves [MLM94]. The model structure of Theorem 4.3 is then known as the Bousfield-Kan model structure and the model structure of Theorem 4.4 is then known as the Heller model structure [Hel88].

These two model structures are complicated to use since their respectively cofibrant and fibrant replacement functors are not easy to understand. The Reedy theory that is going to be exposed now is much simpler. This new approach allows to work with model categories which are not necessarily cofibrantly generated: it is useful for example for [Gau05a] Theorem IV.3.10 where the model category \mathcal{M} is the Strøm model category structure² of compactly generated topological spaces with the homotopy equivalences as weak equivalences [Str66] [Str68] [Str72] [Col06]. But it requires a particular structure on the small category \mathcal{B} :

Definition 4.5. *Let \mathcal{B} be a small category. A Reedy structure on \mathcal{B} consists of two subcategories \mathcal{B}_- and \mathcal{B}_+ , a function $d : \mathcal{B} \rightarrow \lambda$ called the degree function for some ordinal λ , such that every non-identity map in \mathcal{B}_+ raises degree, every non-identity map in \mathcal{B}_- lowers degree, and every map $f \in \mathcal{B}$ can be factored uniquely as $f = g \circ h$ with $h \in \mathcal{B}_-$ and $g \in \mathcal{B}_+$. A small category together with a Reedy structure is called a Reedy category.*

Let \mathcal{B} be a Reedy category. Let b be an object of \mathcal{B} . The *latching category* $\partial(\mathcal{B}_+ \downarrow b)$ at b is the full subcategory of the comma category $\mathcal{B}_+ \downarrow b$ containing all the objects except the identity map of b . The *matching category* $\partial(b \downarrow \mathcal{B}_-)$ at b is the full subcategory of the comma category $b \downarrow \mathcal{B}_-$ containing all the objects except the identity map of b .

Definition 4.6. *Let \mathcal{M} be a complete and cocomplete category. Let \mathcal{B} be a Reedy category. Let b be an object of \mathcal{B} . The latching space functor is the composite $L_b : \mathcal{M}^{\mathcal{B}} \rightarrow \mathcal{M}^{\partial(\mathcal{B}_+ \downarrow b)} \rightarrow \mathcal{M}$ where the latter functor is the colimit functor. The matching space functor is the composite $M_b : \mathcal{M}^{\mathcal{B}} \rightarrow \mathcal{M}^{\partial(b \downarrow \mathcal{B}_-)} \rightarrow \mathcal{M}$ where the latter functor is the limit functor.*

The *Reedy model structure* of $\mathcal{M}^{\mathcal{B}}$ is then constructed as follows:

- The *Reedy weak equivalences* are the objectwise weak equivalences.
- The *Reedy cofibrations* are the morphisms of diagrams from X to Y such that for every object b of \mathcal{B} the morphism $X_b \sqcup_{L_b X} L_b Y \rightarrow Y_b$ is a cofibration of \mathcal{M} .
- The *Reedy fibrations* are the morphisms of diagrams from X to Y such that for every object b of \mathcal{B} the morphism $X_b \rightarrow Y_b \times_{M_b Y} M_b X$ is a fibration of \mathcal{M} .

Theorem 4.7. *(D.M. Kan) ([Hir03] Theorem 15.3.4, Theorem 15.3.15 and Theorem 15.6.-27) Let \mathcal{B} be a Reedy category. Let \mathcal{M} be a model category. The objectwise weak equivalences together with the Reedy cofibrations and the Reedy fibrations assemble to a structure of model category. Moreover, one has:*

²The Strøm model category is conjecturally not cofibrantly generated.

- If \mathcal{M} is a cofibrantly generated for which there are a set I of generating cofibrations whose domains and codomains are small relative to I and a set J of generating trivial cofibrations whose domains and codomains are small relative to J , then the Reedy model structure is cofibrantly generated.
- If \mathcal{M} is left proper (resp. right proper, proper), then so is $\mathcal{M}^{\mathcal{B}}$.
- A morphism of diagrams from X to Y is a trivial Reedy cofibration if and only if for every object b of \mathcal{B} the morphism $X_b \sqcup_{L_b X} L_b Y \longrightarrow Y_b$ is a trivial cofibration of \mathcal{M} .
- A morphism of diagrams from X to Y is a trivial Reedy cofibration if and only if for every object b of \mathcal{B} the morphism $X_b \longrightarrow Y_b \times_{M_b Y} M_b X$ is a trivial fibration of \mathcal{M} .

Theorem 4.7 is used in [Gau05e] Theorem 8.4 and in [Gau06b] Theorem 8.1, Theorem 8.2 and Theorem 8.3.

The calculation of homotopy colimits is then possible in the following situation:

Theorem 4.8. ([Hir03] Proposition 15.10.2 and Theorem 15.10.8) *Let \mathcal{B} be a Reedy category. Then the following conditions are equivalent:*

- For every fibrant object X of every model category \mathcal{M} , the diagram $\text{Diag}(X)$ is Reedy fibrant (one says that the category \mathcal{B} has fibrant constants).
- For every object b of \mathcal{B} , the matching category $\partial(b \downarrow \mathcal{B}_-)$ is either empty or connected.
- For every model category \mathcal{M} , the categorical adjunction $\varinjlim : \mathcal{M}^{\mathcal{B}} \rightleftarrows \mathcal{M} : \text{Diag}$ is a Quillen adjunction.

The statements of [Gau05e] Theorem 7.5 and of [Gau06b] Corollary 7.4 exhibit non-trivial examples of small categories having fibrant constants: the Reedy categories are constructed from the category of simplices, i.e. from the order complex [Qui78], of a locally finite poset. These constructions are reused in the proof of [Gau07b] Theorem 7.8.

The dual statement allows the calculation of homotopy limits:

Theorem 4.9. ([Hir03] Proposition 15.10.2 and Theorem 15.10.8) *Let \mathcal{B} be a Reedy category. Then the following conditions are equivalent:*

- For every cofibrant object X of every model category \mathcal{M} , the diagram $\text{Diag}(X)$ is Reedy cofibrant (one says that the category \mathcal{B} has cofibrant constants).
- For every object b of \mathcal{B} , the latching category $\partial(\mathcal{B}_+ \downarrow b)$ is either empty or connected.
- For every model category \mathcal{M} , the categorical adjunction $\text{Diag} : \mathcal{M} \rightleftarrows \mathcal{M}^{\mathcal{B}} : \varprojlim$ is a Quillen adjunction.

5. APPLICATION OF THE REEDY THEORY TO THE HOMOTOPY THEORY OF FLOWS

The following situations are applications of Theorem 4.8 and of Theorem 4.9 to the homotopy theory of flows.

Homotopy pushout. Let A , B and C be three cofibrant objects of \mathcal{M} . The colimit of the diagram $A \longleftarrow B \longrightarrow C$ is a homotopy colimit as soon as one of the map $B \rightarrow A$ or $B \rightarrow C$ is a cofibration. In particular, consider an objectwise weak equivalence f from a diagram $X_1 : A_1 \longleftarrow B_1 \longrightarrow C_1$ to a diagram $X_2 : A_2 \longleftarrow B_2 \longrightarrow C_2$ such that the objects $A_1, B_1, C_1, A_2, B_2, C_2$ are cofibrant and such that the maps $B_i \rightarrow C_i$ are cofibrations for

$i = 1, 2$. Then $\varinjlim f : \varinjlim X_1 \rightarrow \varinjlim X_2$ is a weak equivalence of \mathcal{M} ([Hov99] Lemma 5.2.6 called the cube lemma by M. Hovey). The proof consists of considering the Reedy category $\mathcal{I} : 0 \leftarrow 1 \rightarrow 2$ with the degree equal to the corresponding object. In particular, the small category \mathcal{I} has fibrant constants. By Theorem 4.8, the colimit functor $\varinjlim : \mathcal{M}^{\mathcal{I}} \rightarrow \mathcal{M}$ is therefore a left Quillen adjoint. One has the equalities

- $M_0X = M_2X = \mathbf{1}$ and $M_1X = X_0$
- $L_0X = L_1X = \emptyset$ and $L_2X = X_1$
- $M_0Y = M_2Y = \mathbf{1}$ and $M_1Y = Y_0$
- $L_0Y = L_1Y = \emptyset$ and $L_2Y = Y_1$.

Thus, a map of diagrams $f : X \rightarrow Y$ is a Reedy cofibration if and only if $f_0 : X_0 \rightarrow Y_0$ and $f_1 : X_1 \rightarrow Y_1$ are cofibrations and the map $X_2 \sqcup_{X_1} Y_1 \rightarrow Y_2$ is a cofibration. So with the hypothesis above, the diagrams X and Y are both Reedy cofibrant. Thus, the colimit functor takes the objectwise weak equivalence f to a weak equivalence $\varinjlim f$ and the colimits of the diagrams X and Y give their homotopy colimit. This technique is used in the proof of [Gau05c] Lemma 8.5.

Homotopy colimit of tower of cofibrations between cofibrant objects. Let $(A_n \rightarrow A_{n+1})_{n \geq 0}$ be a family of cofibrations between cofibrant objects. Then the colimit $\varinjlim A_n$ is a homotopy colimit. It suffices to consider the Reedy category $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots$. All matching categories are empty so this category has fibrant constants. And $L_0A = \emptyset$, $L_nA = A_{n-1}$ for any $n \geq 1$. So the tower is Reedy cofibrant. Hence the colimit gives the homotopy colimit.

Homotopy colimit of tower of cofibrations in a left proper model category. Let $(A_n \rightarrow A_{n+1})_{n \geq 0}$ be a family of cofibrations. Assume \mathcal{M} left proper. Then the colimit $\varinjlim A_n$ is a homotopy colimit again. We do not suppose anymore the objects A_i cofibrant but we must suppose \mathcal{M} left proper. The situation is very close to the preceding situation. In fact, one is reduced to working in the comma category $A_0 \downarrow \mathcal{M}$ after a clever argument due to D. M. Kan using left properness (see [Hir03] Proposition 17.9.3). This technique is used in the proof of [Gau05e] Theorem 11.2 and in the proof of [Gau06b] Theorem 9.1. What matters is the left properness of the category of flows which is proved in [Gau07a] Theorem 7.4.

Homotopy pullback. Let A, B and C be three fibrant objects of \mathcal{M} . The limit of the diagram $A \rightarrow B \leftarrow C$ is a homotopy limit as soon as one of the maps $A \rightarrow B$ or $B \leftarrow C$ is a fibration. This situation is used for the proof of [Gau05a] Theorem IV.3.14 where the model category \mathcal{M} is again the Strøm model category structure of compactly generated topological spaces with the homotopy equivalences as weak equivalences [Str66] [Str68] [Str72].

Homotopy limit of tower of fibrations between fibrant objects. Let $(A_{n+1} \rightarrow A_n)_{n \geq 0}$ be a family of fibrations between fibrant objects. Then the limit $\varprojlim A_n$ is a homotopy limit. This situation is used for the proof of [Gau05a] Theorem IV.3.10.

Homotopy limit of tower of fibrations in a right proper model category. Let $(A_{n+1} \rightarrow A_n)_{n \geq 0}$ be a family of fibrations. Assume \mathcal{M} right proper. Then the limit $\varprojlim A_n$ is a homotopy limit.

Category of simplices and of cubes and homotopy colimit. Let K be a simplicial set [GJ99]. Consider the category ΔK of simplices of K defined as follows ($\Delta[n]$ being the n -simplex): the objects are the maps $\Delta[n] \rightarrow K$ and the morphisms are the commutative diagrams of simplicial sets

$$\begin{array}{ccc} \Delta[m] & \longrightarrow & \Delta[n] \\ & \searrow & \swarrow \\ & K & \end{array}$$

In other terms, ΔK is the comma category $\Delta \downarrow K$ where Δ is the small category such that the presheaves over Δ are the simplicial sets. Then

Proposition 5.1. ([Hir03] *Proposition 15.10.4*) *Let K be a simplicial set. The category of simplices ΔK is a Reedy category which has fibrant constants.*

Similarly, let K be a precubical set, that is a cubical set without degeneracy maps [BH81] [Gau07b]. Let \square be the small category such that the presheaves over \square are the precubical sets. Then the category of cubes $\square \downarrow K$ of the precubical set K is a Reedy category which has fibrant constants. In particular, this implies (with a little work) that the geometric realization of a precubical set as a flow $|K| := \varinjlim_{\square[n] \rightarrow K} (\widehat{0} < \widehat{1})^n$ as defined in [Gau07b] is actually an homotopy colimit. So one has the weak S-homotopy equivalence $|K| \simeq \underline{\text{holim}}_{\square[n] \rightarrow K} \widehat{0} < \widehat{1}$.

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LABORATOIRE PPS (CNRS UMR 7126), UNIVERSITÉ PARIS 7–DENIS DIDEROT, CASE 7014, 75205
PARIS CEDEX 13, FRANCE

E-mail address: gaucher@pps.jussieu.fr

URL: <http://www.pps.jussieu.fr/~gaucher/>