Time flows up to homotopy

Between concurrent processes and model categories

(talk in French, slides in English)

Philippe Gaucher

http://www.pps.jussieu.fr/~gaucher

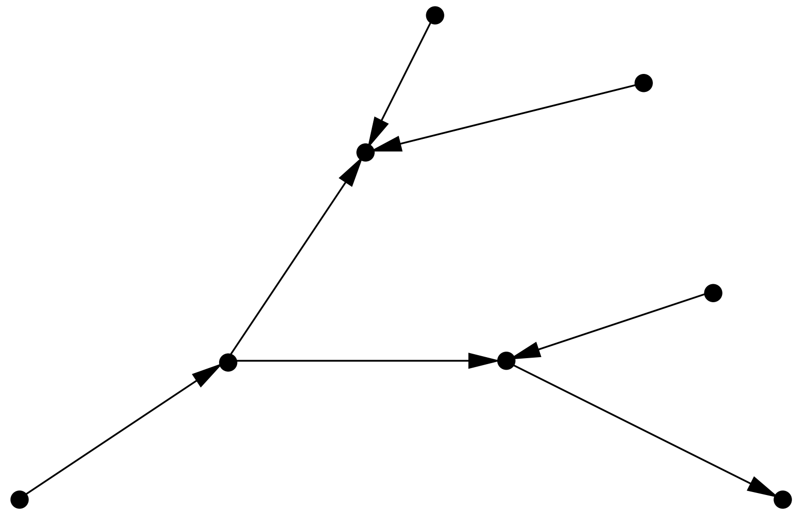
Preuves, Programmes et Systèmes, CNRS UMR 7126 et Paris 7

Part I

- Geometry of time flows
  - Time flow, weak S-homotopy, weak dihomotopy, weak quasidihomotopy, branching homology, merging homology, underlying homotopy type
- CCS (Calculus of the Communicating Systems)
  - Very Short Course about CCS (VSC)
  - Semantics of process algebra using CCS precubical sets (modification of Fahrenberg's Goubault's and Worytkiewicz's constructions)
  - Semantics of process algebra using CCS flows
- Bisimulation, weak flow
  - Cubical bisimulation and Bousfield bisimulation
  - Segal flow and Rezk model category

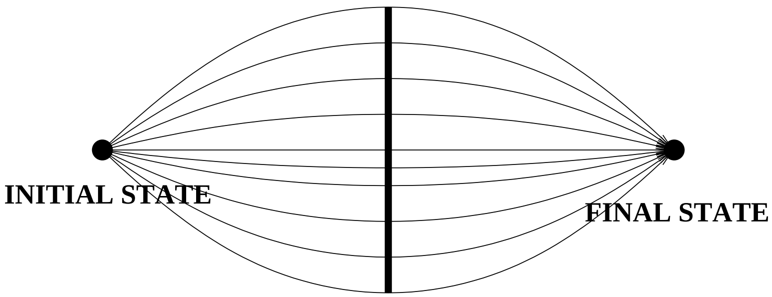
The causal structure of a time flow

- A good notion of homotopy must preserve initial states, final states, non-deterministic branching and merging areas, and more generally the causal structure



Tree-like time flow

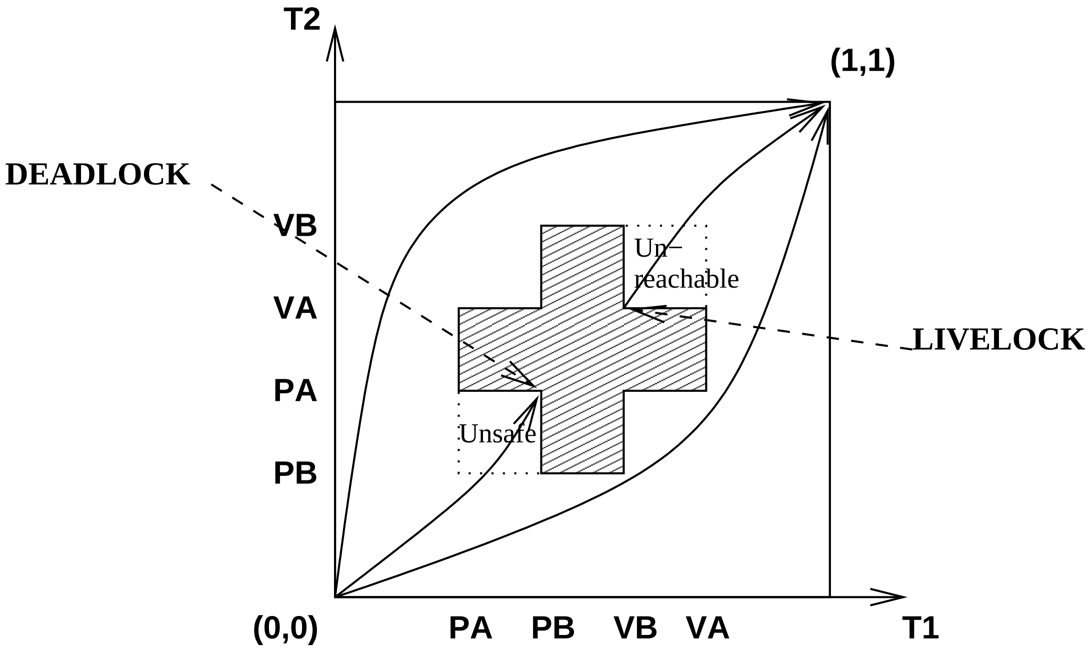
- Forbidden contraction of the directed segment



Full 2-dimensional globe $\mathrm{Glob}(\mathbf{D}^1)$

- Allowed contraction of the achronal segment

The Swiss Flag example

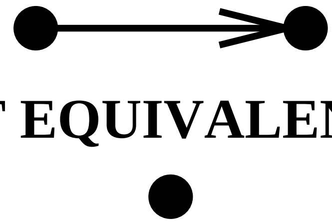


- Two ressources A and B used by at most one process at the same time
- PA: take A, VA: release A, PB: take B, VB: release B
- Shaded area represents impossible states corresponding to mutual exclusion

Flow and weak S-homotopy

- Flow = small category without identities enriched over **Top** (compactly generated topological space)
  - set X^0 of states of X = the objects
  - space $\mathbb{P}_{\alpha,\beta}X$ of non-constant execution paths from α to β of X^0 = the morphisms from α to β
  - Strictly associative composition of execution paths: the composition law $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X$
  - $f : X \to Y$ consists of a set map $f^0 : X^0 \to Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ compatible with s, t and $*$
- X loopless iff $\mathbb{P}_{\alpha,\alpha}X = \varnothing$ for every α
- $f : X \to Y$ weak S-homotopy iff f^0 bijection and $\mathbb{P}f$ weak homotopy equivalence

A good theory of directed homotopy

- 

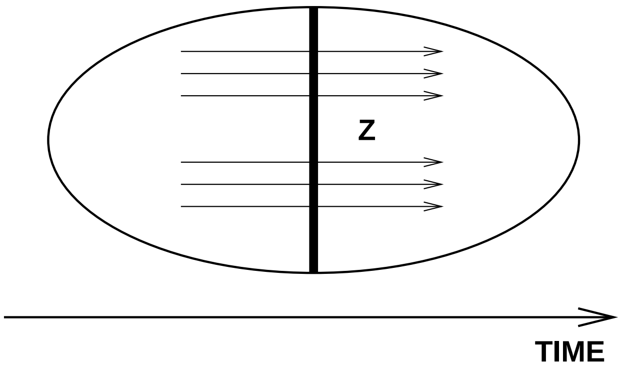
- The functoriality

  $$X \mapsto \{\text{non-constant execution paths of } X\}$$

- Expressive enough to contain all examples of concurrency

- Weak S-homotopy does not identify the directed segment and the point
- The path space functor $X \mapsto \mathbb{P}X = \bigsqcup_{(\alpha,\beta)} \mathbb{P}_{\alpha,\beta}X$

Globe of a topological space



$$|\mathrm{Glob}(Z)| = \frac{Z \times [0,1]}{(z,0) = \widehat{0}, (z,1) = \widehat{1}}$$

(the underlying space of $\mathrm{Glob}(Z)$ for $Z \neq \varnothing$)

- The globe $\mathrm{Glob}(Z)$ of the topological space Z
  - $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$
  - $\mathbb{P}\mathrm{Glob}(Z) = Z$
  - $s = \widehat{0}$
  - $t = \widehat{1}$
  - no composable non-constant execution paths
- The directed segment $\mathrm{Glob}(\{*\}) = \overrightarrow{I}$

Weak S-homotopy model category

Theorem. *There exists exactly one model structure* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *on* **Flow** *such that*

- $\mathrm{Fib} = \{f : X \to Y$ *s.t.* $\mathbb{P}f$ *Serre fibration*$\}$
- $\mathcal{W} = \{$*weak S-homotopy*$\}$ *(*$\Rightarrow R : \{0,1\} \to \{0\} \in \mathrm{Cof}$*)*

It is cofibrantly generated, proper, simplicial, not cellular, not topological. Every flow is fibrant.

$$\boxed{I^{gl}_+ = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}}$$ with $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$ (generating cofibrations)

$$\boxed{J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \to \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}}$$

(generating trivial cofibrations)

Realizing posets as flows

- $(P, \leqslant)$ poset
- let
  - $F(P)^0 = P$
  - $\mathbb{P}_{\alpha,\beta}F(P) = \{u_{\alpha,\beta}\}$ iff $\alpha < \beta$
  - $\mathbb{P}_{\alpha,\beta}F(P) = \varnothing$ iff $\alpha \geqslant \beta$
- $F(P)$ loopless
- F : {poset+strictly increasing map} $\rightarrow$ $\mathbf{Flow}$
- $F(P)$ also denoted by P
- $\leqslant$ represents observable time ordering

Localizing w.r.t. generating T-homotopy

- $\{\widehat{0} < \widehat{1}\} \to \{\widehat{0} < 2 < \widehat{1}\}$, $U \mapsto V * W$

$$\widehat{0} \xrightarrow{U} \widehat{1}$$

$$\widehat{0} \xrightarrow{V} 2 \xrightarrow{W} \widehat{1}$$

- A representative set of finite bounded posets (P bounded implying $\widehat{0} = \min P \neq \max P = \widehat{1}$)
- $\boxed{\mathcal{T} = \{Q(P_1 \subset P_2) \text{ s.t. } \uparrow\uparrow \text{ and } \widehat{0} \mapsto \widehat{0} \text{ and } \widehat{1} \mapsto \widehat{1}\}}$ with bottom element $\widehat{0}$ and top element $\widehat{1}$
- How to prove the existence of the Bousfield localization of the weak S-homotopy model category structure with respect to $\mathcal{T}$ (using Bousfield-Friedlander localization or a notion of quasi-cellular model category ?)

Dihomotopy and quasidihomotopy

- f weak S-homotopy equivalence iff $Q(f)$ isomorphism of $\mathbf{Flow}_{cof}[\mathbf{cell}(J^{gl})^{-1}]$
- f weak dihomotopy equivalence iff $Q(f)$ isomorphism of $\mathbf{Flow}_{cof}[\mathbf{cell}(\mathcal{T} \cup J^{gl})^{-1}]$
- f weak quasidihomotopy equivalence iff f $\mathcal{T}$-local equivalence, i.e. iff $Q(f)$ isomorphism of $\mathbf{Flow}_{cof}[\mathbf{cell}(\Lambda(\mathcal{T}) \cup J^{gl})^{-1}]$ where

  $$\Lambda(\mathcal{T}) = \{(A \otimes \Delta[n]) \sqcup_{(A \otimes \partial\Delta[n])} (B \otimes \partial\Delta[n]) \to (B \otimes \Delta[n])\}$$

  with $A \to B \in \mathcal{T}$ and $n \geqslant 0$
- $\{$weak S-homotopy$\} \subset \{$weak dihomotopy$\} \subset \{$weak quasidihomotopy$\}$

Branching and merging spaces

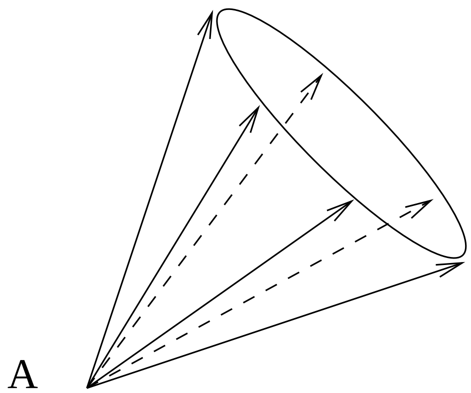
- $\mathbb{P}^- X := \mathbb{P}X/(x = x * y)$ (branching space functor) : space of germs of non-constant execution paths beginning in the same way
- $\mathbb{P}^+ X := \mathbb{P}X/(y = x * y) \cong \mathbb{P}^- X^{op}$ (merging space functor) where X^{op} is the opposite flow

Theorem. *There exists a weak S-homotopy equivalence $f : X \to Y$ such that $\mathbb{P}^- X \cong \mathbf{S}^2 \sqcup \{0\}$ and $\mathbb{P}^- Y \cong \{0\} \sqcup \mathbb{R}$.*

Theorem. *For all weak S-homotopy equivalences $f : X \to Y$, the maps $\mathbb{P}^- Q(f) : \mathbb{P}^- Q(X) \to \mathbb{P}^- Q(Y)$ and $\mathbb{P}^+ Q(f) : \mathbb{P}^+ Q(X) \to \mathbb{P}^+ Q(Y)$ are weak homotopy equivalences of topological spaces.*

Branching homology

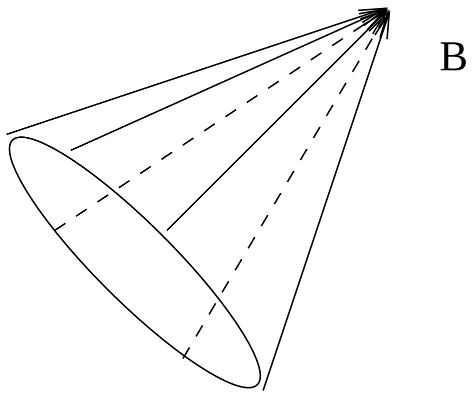
- Let $n \geqslant -1$ $\boxed{H^-_{n+1}(X) := H_n\left(\operatorname{Sing}\mathbb{P}^-Q(X) \xrightarrow{s} X^0\right)}$ ($s(x) = s(x * y)$)
- $H^-_n(X)$ contains the non-deterministic branchings of dimension n for $n \geqslant 1$
- $H^-_0(X)$ is the free abelian group generated by the final states of X



$$\mathbb{P}^-_A \cong \mathbf{S}^1$$

Merging homology

- Let $n \geqslant -1$ $\boxed{H^+_{n+1}(X) := H_n\left(\operatorname{Sing}\mathbb{P}^+Q(X)\xrightarrow{t} X^0\right)}$ ($t(y) = t(x * y)$)
- $H^+_n(X)$ contains the non-deterministic mergings of dimension n for $n \geqslant 1$
- $H^+_0(X)$ is the free abelian group generated by the initial states of X



$$\mathbb{P}^+_B \cong \mathbf{S}^1$$

Underlying homotopy type of a flow

The underlying homotopy type $|X|$ of a flow X

- Write $\varnothing \to Q(X)$ as a transfinite composition of pushouts of $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$, and of $\mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{Glob}(\mathbf{D}^n)$ with $n \geqslant 0$
- Replace this transfinite composition by a transfinite composition of pushouts of $C : \varnothing \to \{0\}$, $R : \{0,1\} \to \{0\}$ and of $|\mathrm{Glob}(\mathbf{S}^{n-1})| \to |\mathrm{Glob}(\mathbf{D}^n)|$ with $n \geqslant 0$ (with $|\mathrm{Glob}(Z)| = Z \times [0,1]/((z,0) = \widehat{0}, (z,1) = \widehat{1})$ if $Z \neq \varnothing$)
- One obtains $\varnothing \to |X|$
- $|-| : \mathbf{Flow} \to \mathbf{Ho}(\mathbf{Top})$

About dihomotopy

Theorem. *Dihomotopy preserves the branching and merging homologies, and the underlying homotopy type.*

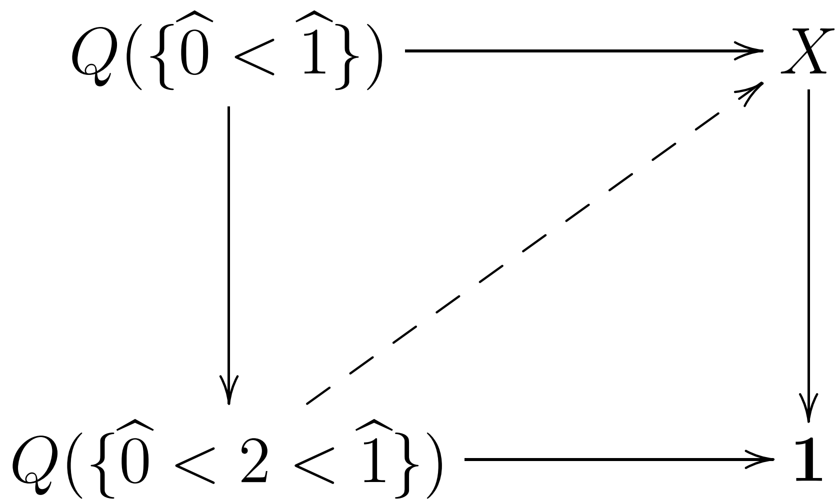
Theorem. *There does not exist any model structure on* $\mathbf{Flow}$ *(cofibrantly generated or not) such that the class of weak equivalences is exactly the class of weak dihomotopy equivalences.*

Theorem. *For any model structure (cofibrantly generated or not) on* $\mathbf{Flow}$ *such that* $Q(\{\widehat{0} < \widehat{1}\}) \to Q(\{\widehat{0} < 2 < \widehat{1}\})$ *is a weak equivalence, there exists a pushout of* $R : \{0, 1\} \to \{0\}$ *which is a weak equivalence.*

There is a good notion of fibrant objects and a cocylinder functor for the full dihomotopy relation

Homotopy continuous flow

Homotopy continuous flow = Indefinitely divisible up to S-homotopy



X homotopy continuous if and only if $X \longrightarrow \mathbf{1}$ satisfies the right lifting property with respect to the set of maps $J^{gl} \cup \mathcal{T}$

Whitehead's theorem for dihomotopy

Theorem. *There exists a congruence* $\sim_{\mathcal{T}}$ *on the morphisms of flows such that the inclusion functor* $\mathbf{Flow}^{\textit{homotopy continuous}}_{\textit{cofibrant}} \subset \mathbf{Flow}$ *induces the equivalence of categories*

$$\boxed{\mathbf{Flow}^{\textit{homotopy continuous}}_{\textit{cofibrant}} / \sim_{\mathcal{T}} \simeq \mathbf{Flow}[(\mathcal{S}_{\mathcal{T}})^{-1}]}$$

where $\mathcal{S}_{\mathcal{T}}$ *is the class of weak dihomotopy equivalences.*

Path functor for dihomotopy

Theorem. *There exists a functor* $\mathrm{Path}_{\mathcal{T}} : \mathbf{Flow} \longrightarrow \mathbf{Flow}$

$$X \xrightarrow[\in\, \mathbf{cof}(J^{gl} \cup \mathcal{T})]{} \mathrm{Path}_{\mathcal{T}}\, X \xrightarrow[\in\, \mathbf{inj}(J^{gl} \cup \mathcal{T})]{} X \times X, \qquad (\mathrm{Id}_X, \mathrm{Id}_X) : X \longrightarrow X \times X$$

$$X \xrightarrow[H]{} \mathrm{Path}_{\mathcal{T}}\, Y \longrightarrow Y \times Y, \qquad (f,g) : X \longrightarrow Y \times Y$$

H right dihomotopy from f to g : reflexive and symmetric

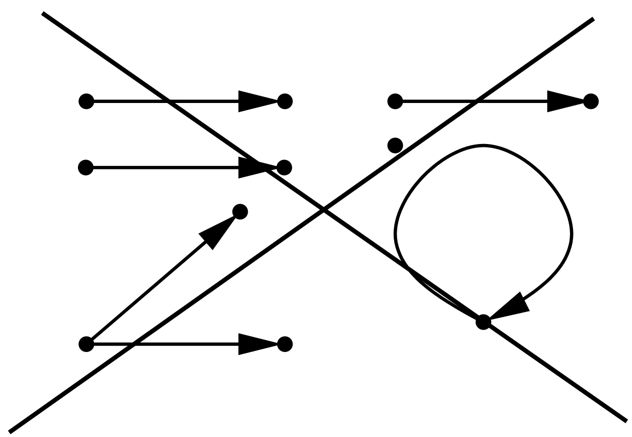
Congruence $\sim_{\mathcal{T}}$ = transitive closure of right dihomotopy

About quasidihomotopy

Theorem. *There exist pushouts of* $R : \{0,1\} \to \{0\}$ *which are weak quasidihomotopy equivalences.*

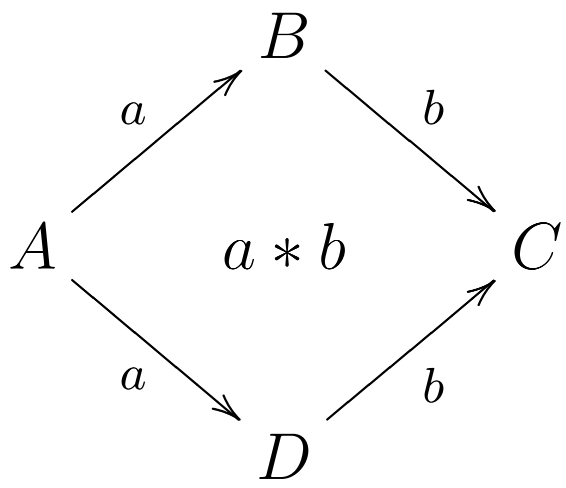
Theorem. *Quasidihomotopy preserves the initial and final states.*

Open question. *Characterizing the weak dihomotopy equivalences as the subclass of that of weak quasidihomotopy equivalences preserving branching and merging homologies and underlying homotopy type.*



Quasidihomotopy and pushout of R

- Identifying $B = D$ in

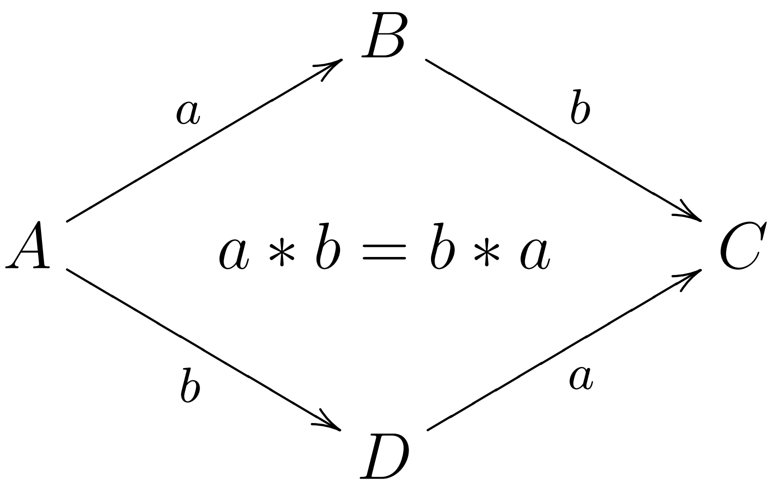


corresponding to the sequential execution of a and b

- Underlying homotopy type not preserved: $\mathbf{S}^1$ not contractible
- $\mathbb{P}_A^-$ and $\mathbb{P}_C^+$ contractible: the identification $B = D$ not observable from outside

Not a quasidihomotopy

- Identifying $B = D$ in



  corresponding to the concurrent execution of a and b

- After the identification $B = D$, two possible actions between A and $B = D$ if $a \neq b$

Some remarks

- Quasidihomotopy behaves like dihomotopy except in non-observable areas of the time flow : the same phenomena happen with the higher dimensional maps of $\Lambda(\mathcal{T})$
- Quasidihomotopy is an observational equivalence
- The Bousfield localization functor (which does exist) erases the non-observable areas
- An observational equivalence does not necessarily preserve the underlying homotopy type

Part II

- Geometry of time flows
  - Time flow, weak S-homotopy, weak dihomotopy, weak quasidihomotopy, branching homology, merging homology, underlying homotopy type
- CCS (Calculus of the Communicating Systems)
  - Very Short Course about CCS (VSC)
  - Semantics of process algebra using CCS precubical sets (modification of Fahrenberg's Goubault's and Worytkiewicz's constructions)
  - Semantics of process algebra using CCS flows
- Bisimulation, weak flow
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  - Segal flow and Rezk model category

[VSC] A CCS-like language

(channel, port) names: $a, b, c, \ldots$

co-names: $\overline{a}, \overline{b}, \overline{c}, \ldots$ ($\overline{\overline{a}} = a$)

silent action: τ (synchronized action of a and $\overline{a}$, b and $\overline{b}$, etc...)

actions, prefixes: $\mu ::= a, \overline{a}, \tau$

$P, Q ::= nil$	idle action,
$\mu.P$	prefix,
$P + Q$	non-deterministic choice,
$P \vert Q$	concurrent execution with synchronization,
$(\nu a)P$	restriction to a local use of a,
$\mathrm{rec}(x)P(x)$	recursion, x guarded variable in $P(x)$
$P \xrightarrow{\mu} Q$	P behaves like Q after executing μ

[VSC] Operational semantics of CCS

Act $$\frac{}{a.P \xrightarrow{a} P}$$

Res $$\frac{P \xrightarrow{\mu} P' \quad \mu \neq a, \overline{a}}{(\nu a)P \xrightarrow{\mu} (\nu a)P'}$$

Sum1 $$\frac{P \xrightarrow{\mu} P'}{P + Q \xrightarrow{\mu} P'}$$

Sum2 $$\frac{Q \xrightarrow{\mu} Q'}{P + Q \xrightarrow{\mu} Q'}$$

Par1 $$\frac{P \xrightarrow{\mu} P'}{P|Q \xrightarrow{\mu} P'|Q}$$

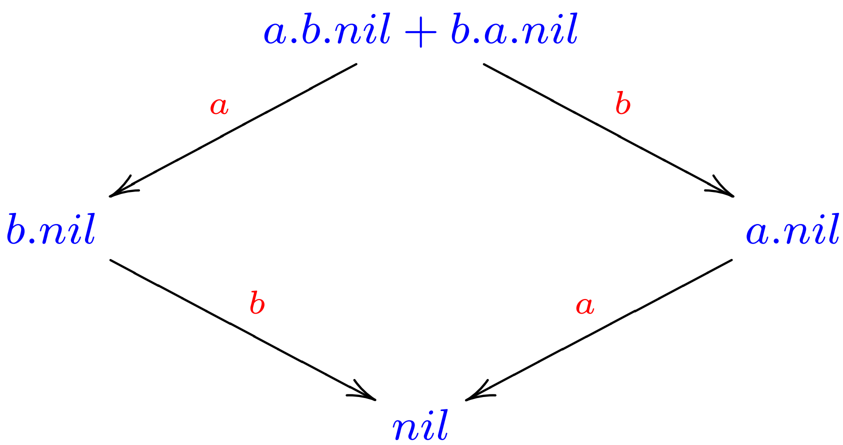
Par2 $$\frac{Q \xrightarrow{\mu} Q'}{P|Q \xrightarrow{\mu} P|Q'}$$

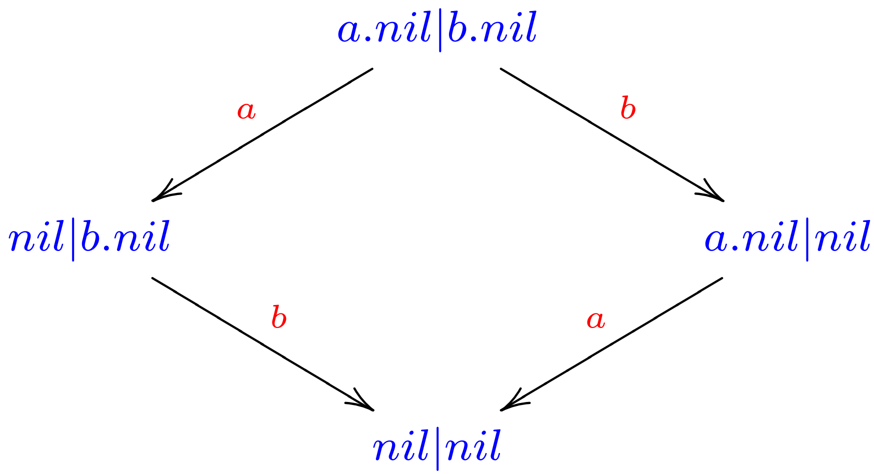
Com $$\frac{P \xrightarrow{a} P', Q \xrightarrow{\overline{a}} Q'}{P|Q \xrightarrow{\tau} P'|Q'}$$

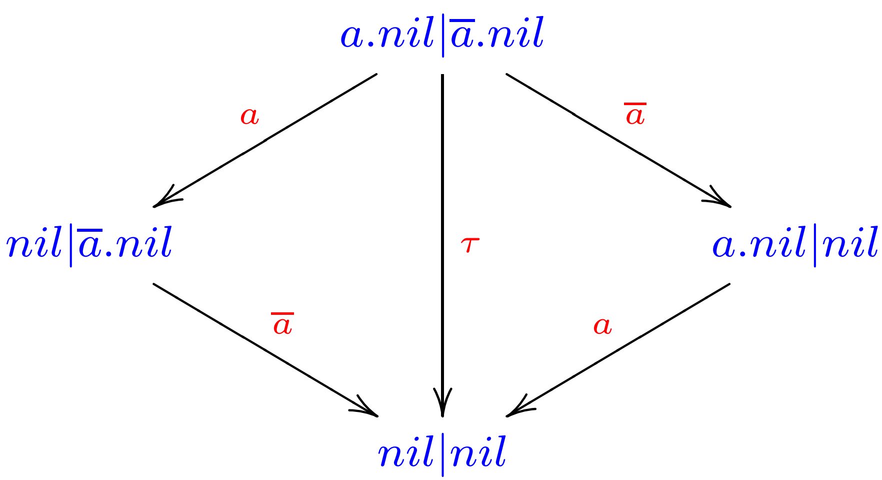
Rec $$\frac{P(\mathrm{rec}(x)P(x)) \xrightarrow{a} P'}{\mathrm{rec}(x)P(x) \xrightarrow{a} P'}$$

[VSC] Examples of CCS processes

Labelled precubical sets decorated by CCS terms: the decoration does not belong to the structure of labelled precubical set







With $P(x) = \mu.x$,

$$rec(x)P(x) \xrightarrow{\mu} rec(x)P(x) \xrightarrow{\mu} \ldots$$

$$(\nu a)a.nil|\overline{a}.nil \xrightarrow{\tau} nil|nil$$

CCS precubical set

- $\Sigma = \{a, b, c, \dots\} \cup \{\overline{a}, \overline{b}, \overline{c}, \dots\} \cup \{\tau\}$ with $\overline{\overline{a}} = a$ (set of labels)
- Put an arbitrary total ordering $(\Sigma, \leqslant)$: label for the concurrent execution of a and b = label for the concurrent execution of b and a
- Precubical set of labels $!\Sigma$ defined by
  - $(!\Sigma)_0 = \{()\}$ (the empty word)
  - for $n \geqslant 1$, $(!\Sigma)_n = \{(a_1, \dots, a_n) \in \Sigma \times \dots \times \Sigma, a_1 \leqslant \dots \leqslant a_n$ and $i \neq j \Rightarrow a_i \neq \overline{a_j}\}$
  - $\partial_i^0(a_1, \dots, a_n) = \partial_i^1(a_1, \dots, a_n) = (a_1, \dots, \widehat{a_i}, \dots, a_n)$
- CCS precubical set: object of $\square^{op}\mathbf{Set} \downarrow (!\Sigma)$
- Two opposite faces are labelled in the same way

Example of CCS 2-cubes

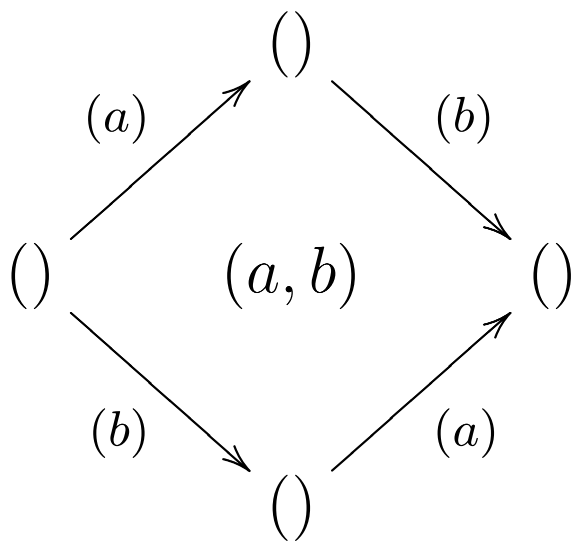


Figure 1: Concurrent execution of a and b with $a \neq \overline{b}$ and $a \leqslant b$

- If $b = \overline{a}$, then the 2-cube cannot be filled out

CCS n-coskeleton

- Let $\ell : K \to !\Sigma$ be a CCS precubical set. A CCS n-shell of K is a $2(n+1)$-uple $(x_1^0, \ldots, x_{n+1}^0, x_1^1, \ldots, x_{n+1}^1)$ of CCS n-cubes of K such that for any $1 \leqslant i < j \leqslant n+1$, one has $\partial_i^\alpha x_j^\beta = \partial_{j-1}^\beta x_i^\alpha$ for any $\alpha, \beta \in \{0,1\}$ and such that $\ell(x_i^0) = \ell(x_i^1)$ for any $1 \leqslant i \leqslant n+1$.
- The CCS n-truncature functor $\mathrm{tr}_n^{CCS} : (\square^{op}\mathbf{Set}{\downarrow}!\Sigma) \to (\square_n^{op}\mathbf{Set}{\downarrow}!\Sigma)$ has a right adjoint $\mathrm{cosk}_n^{CCS} : (\square_n^{op}\mathbf{Set}{\downarrow}!\Sigma) \to (\square^{op}\mathbf{Set}{\downarrow}!\Sigma)$ which freely generates CCS p-cubes for all $p > n$
- Using the CCS labelling, and the fact that a precubical set contains no degenerate cubes, the right adjoint is well-behaved !

Synchronized tensor product of cubes

- Take two CCS cubes $\ell : \square[m] \to !\Sigma$ and $\ell : \square[n] \to !\Sigma$
- Consider the CCS 1-dimensional precubical set Z defined by
  - $Z_0 = \square[m]_0 \times \square[n]_0$
  - $Z_1 = (\square[m]_0 \times \square[n]_1) \sqcup (\square[m]_1 \times \square[n]_0) \sqcup \{(x,y) \in \square[m]_1 \times \square[n]_1, \ell(x) = \overline{\ell(y)}\}$
  - face maps defined in an obvious way
- $(\square[m] \to !\Sigma) \otimes_s (\square[n] \to !\Sigma) := \operatorname{cosk}_1^{CCS}(Z)$
- $K \otimes_s L := \varinjlim_{\square[m]\to K, \square[n]\to L} (\square[m] \to !\Sigma) \otimes_s (\square[n] \to !\Sigma)$
- $\otimes$ (non-synchronized tensor product) closed monoidal structure of $\square^{op}\mathbf{Set}$

Denotational semantics of CCS (I)

- for every P, every state of $\Box[\![P]\!]$ decorated, and the unique initial one decorated by P
- $\Box[\![nil]\!] = \Box[0]$ and $\Box[\![\mu.nil]\!] = () \xrightarrow{(\mu)} ()$
- $$\begin{array}{ccc} \{0\} & \xrightarrow{0\mapsto nil} & \Box[\![\mu.nil]\!] \\ {\scriptstyle 0\mapsto P}\downarrow & & \downarrow \\ \Box[\![P]\!] & \longrightarrow & \Box[\![\mu.P]\!] \end{array} \qquad \begin{array}{ccc} \Box[\![(\nu a)P]\!] & \longrightarrow & \Box[\![P]\!] \\ \downarrow & & \downarrow \\ !(\Sigma\backslash\{a,\overline{a}\}) & \longrightarrow & !\Sigma \end{array}$$
- $\Box[\![P+Q]\!] = \Box[\![P]\!] \oplus \Box[\![Q]\!]$ where $\oplus$ binary coproduct in the comma category $\{i\}\downarrow\mathbf{Flow}\downarrow ?\Sigma$
- $\Box[\![P|Q]\!] = \Box[\![P]\!] \otimes_s \Box[\![Q]\!]$
- $\Box[\![\mathrm{rec}(x)P(x)]\!] = \varinjlim_n \Box[\![P^n(nil)]\!]$ (using $nil \rightarrow P(nil) \rightarrow P(P(nil)) \rightarrow \dots$)

Realizing precubical sets as flows

- K precubical set
- $\square[n]$ n-cube, $n \geqslant 0$
- Q cofibrant replacement functor
- Let
  - $|\square[0]| = \{0\}$
  - $|\square[n]| = Q(\{\widehat{0} < \widehat{1}\}^n)$, $n \geqslant 1$
  - $|K| = \int^{[n]} K_n.|\square[n]|$
- functor $|-| : \{$precubical set$\} \to \mathbf{Flow}$
- it has a right adjoint

Note: Q necessary. Otherwise $|\partial\square[n]| \cong |\square[n]|$ for $n \geqslant 3$.

CCS flow

- $\Sigma = \{a, b, c, \dots\} \cup \{\overline{a}, \overline{b}, \overline{c}, \dots\} \cup \{\tau\}$ with $\overline{\overline{a}} = a$ (set of labels)
- Flow of labels $?\Sigma$ defined by
  - $(?\Sigma)^0 = \{0\}$
  - $\mathbb{P}(?\Sigma)$ free associative discrete monoid without unit generated by Σ and the relations $a * b = b * a$ if $a \neq \overline{b}$ and $a * \tau = \tau * a$ and $\overline{a} * \tau = \tau * \overline{a}$
- CCS flow: object of $\mathbf{Flow} \downarrow (?\Sigma)$
- Two executions paths of the same connected component of $\mathbb{P}_{\alpha,\beta}X$ have same label
- The realization of a CCS precubical set gives rise to a CCS flow (there is a canonical map $|!\Sigma| \to ?\Sigma$)

Examples of CCS 2-cubes

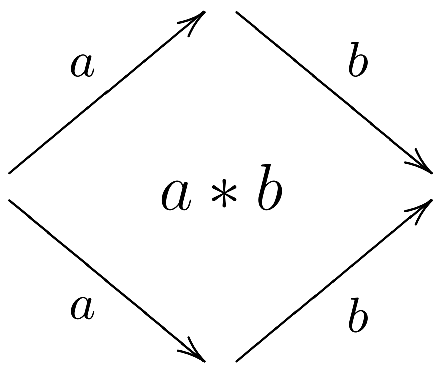


Figure 2: Sequential execution of a and b

Not the realization of a CCS precubical set for $a \neq b$

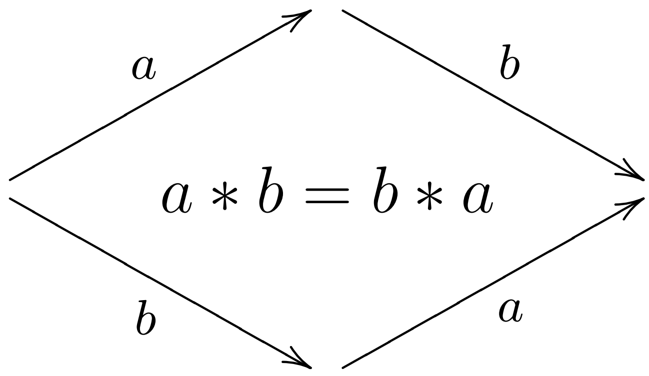


Figure 3: Concurrent execution of a and b with $a \neq \overline{b}$

If $b = \overline{a}$, then the 2-cube cannot be filled out

Synchronized tensor product of flows

- The flow $X \otimes Y$ defined by
  - $(X \otimes Y)^0 = X^0 \times Y^0$
  - $\mathbb{P}(X \otimes Y) = (X^0 \times \mathbb{P}Y) \sqcup (\mathbb{P}X \times Y^0) \sqcup (\mathbb{P}X \times \mathbb{P}Y)$
  - $s(x,y) = (s(x), s(y))$, $t(x,y) = (t(x), t(y))$,
    $(x,y) * (x',y') = (x * x', y * y')$
- $\otimes$ closed monoidal structure of $\mathbf{Flow}$

$$
\begin{array}{ccc}
X \otimes_s Y & \longrightarrow & X \otimes Y \\
\downarrow & h & \downarrow \\
?(\Sigma \cup \{\tau_a, a \text{ name}\}) & \xrightarrow{\tau_a \mapsto a * \overline{a}} & (?\Sigma)^{ab} \\
{\scriptstyle \tau_a \mapsto \tau} \downarrow & & \\
?\Sigma & &
\end{array}
$$

- $\tau_a * \tau_b = \tau_b * \tau_a$
- $\tau_a * b = b * \tau_a$
- $\tau_a * \overline{b} = \overline{b} * \tau_a$
- $\tau_a * \tau = \tau * \tau_a$

Denotational semantics of CCS (II)

- $[\![nil]\!] = \{0\}$ and $[\![\mu.nil]\!] = \{0\} \xrightarrow{\mu} \{0\}$
- $$\begin{array}{ccc} \{0\} & \xrightarrow{0\mapsto nil} & [\![\mu.nil]\!] \\ {\scriptstyle 0\mapsto P}\downarrow & {\scriptstyle h} & \downarrow \\ [\![P]\!] & \longrightarrow & [\![\mu.P]\!] \end{array} \qquad \begin{array}{ccc} [\![(\nu a)P]\!] & \longrightarrow & [\![P]\!] \\ \downarrow & {\scriptstyle h} & \downarrow \\ !(\Sigma\backslash\{a,\overline{a}\}) & \longrightarrow & !\Sigma \end{array}$$
- $[\![P+Q]\!] \cong [\![P]\!] \oplus [\![Q]\!]$ where $\oplus$ binary coproduct in the comma category $\{i\}{\downarrow}\square^{op}\mathbf{Set}{\downarrow}!\Sigma$
- $[\![P|Q]\!] \cong [\![P]\!] \otimes_s [\![Q]\!]$
- $[\![\mathrm{rec}(x)P(x)]\!] = \varinjlim_n^{\mathrm{holim}} [\![P^n(nil)]\!]$

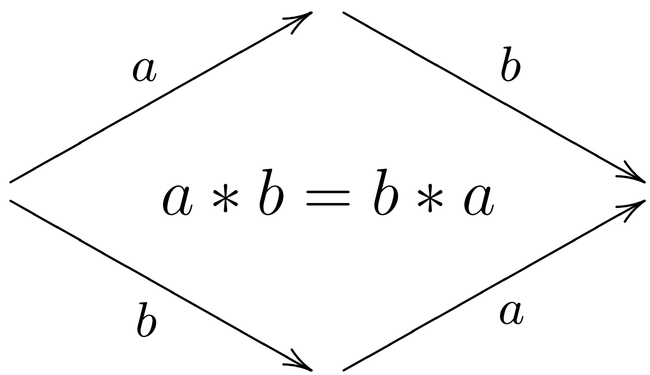
Theorem. *(not yet !) For every closed term P of CCS, the flows $|\square[\![P]\!]|$ and $[\![P]\!]$ are weakly S-homotopy equivalent.*

Part III

- Geometry of time flows
  - Time flow, weak S-homotopy, weak dihomotopy, weak quasidihomotopy, branching homology, merging homology, underlying homotopy type
- CCS (Calculus of the Communicating Systems)
  - Very Short Course about CCS (VSC)
  - Semantics of process algebra using CCS precubical sets (modification of Fahrenberg's Goubault's and Worytkiewicz's constructions)
  - Semantics of process algebra using CCS flows
- Bisimulation, weak flow
  - Cubical bisimulation and Bousfield bisimulation
  - Segal flow and Rezk model category

Introducing a notion of bisimulation

- Let $a, b \in \Sigma$ with $a \neq \bar{b}$ and $a \neq b$; $[\![a.b.nil]\!]$ and $[\![b.a.nil]\!]$ are not weakly S-homotopy equivalent
- But they are weakly dihomotopy equivalent, since there exists a zig-zag of cofibrations which are also weak dihomotopy equivalences between them
- Quasidihomotopy preserves the geometrical part of the causal structure, not the syntactical part



[VSC] CCS bisimulation

- P and Q two closed terms of CCS
- Q simulates P if there exists a binary relation $\mathcal{R}$ on the set of closed terms of CCS called a simulation relation such that
  - $(P, Q) \in \mathcal{R}$
  - $\forall (R, S) \in \mathcal{R}\ \forall \mu \in \Sigma\ \forall R \xrightarrow{\mu} R'\ \exists S \xrightarrow{\mu} S'$ and $(R', S') \in \mathcal{R}$
  - intuitively, every 1-dimensional path of P is a 1-dimensional path of Q
- P and Q bisimilar if there exists a binary relation called a bisimulation relation such that P simulates Q and Q simulates P
- $a.b.nil$ and $b.a.nil$ not bisimilar if $a \neq b$

Bisimulation of flows

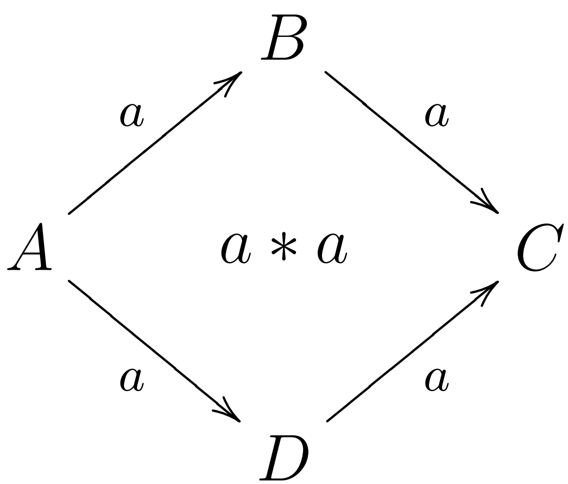
- Let $\mathcal{P}$ be a nonempty set of cofibrant flows which a unique initial state $\widehat{0}$; let $\ell_X : X \to ?\Sigma$ and $\ell_Y : Y \to ?\Sigma$ be two CCS flows ; assume $\{\overrightarrow{I}\} \subset \mathcal{P}$
- Y $\mathcal{P}$-simulates X (with $\mathcal{R}$) if there exists $\mathcal{R} \subset X^0 \times Y^0$ (the simulation relation) such that:
  - $\alpha \in X^0 \Longrightarrow \exists \beta \in Y^0, (\alpha, \beta) \in \mathcal{R}$
  - $\forall (\alpha, \beta) \in \mathcal{R}$, $\forall C \in \mathcal{P}$, $\forall f : C \to X$ with $f(\widehat{0}) = \alpha$ implies $\exists g : C \to Y$ with $g(\widehat{0}) = \beta$ and $\forall x \in C^0, (f(x), g(x)) \in \mathcal{R}$ and $\ell_X \circ f = \ell_Y \circ g$
- X and Y are $\mathcal{P}$-bisimilar if there exists $\mathcal{R} \subset X^0 \times Y^0$ (the bisimulation relation) such that X $\mathcal{P}$-simulates Y with $\mathcal{R}^{op}$ and Y $\mathcal{P}$-simulates X with $\mathcal{R}$

Partial results about bisimulation

- $\mathcal{P}$-bisimulation is an equivalence relation ; two weakly S-homotopy equivalent flows are $\mathcal{P}$-bisimilar ; two $\mathcal{P}$-bisimilar flows have same initial states
- $\{\overrightarrow{I}\}$-bisimulation coincides with CCS bisimulation ; $[\![a.b.nil]\!]$ and $[\![b.a.nil]\!]$ are not $\mathcal{P}$-bisimilar
- $\mathcal{P} = \{Q(\{\widehat{0} < \widehat{1}\}^n), n \geqslant 1\}$: cubical bisimulation
- $\mathcal{P}$ the set of domains and codomains of the maps of $J^{gl} \cup \Lambda(\mathcal{T})$: Bousfield bisimulation
- Two precubical flows are cubical bisimilar iff they are Bousfield bisimilar ??
- $[\![a.b.nil + a.b.nil]\!]$ and $[\![a.b.nil]\!]$ are Bousfield bisimilar and not weakly quasidihomotopy equivalent

Quasidihomotopy and precubical flows

- Describing the quasidihomotopy equivalences between precubical flows in terms of dihomotopy and some elementary transformations.
- Identifying $B = D$ in



is a weak quasidihomotopy equivalence between precubical flows which is not a weak dihomotopy equivalence.

Segal flow

- The prenerve functor $\mathcal{N}^{pre} : \mathbf{Flow} \to \Delta^{op}_{<}\mathbf{Top}$ associating a flow with a presimplicial topological space

Theorem. *There exists a model category structure (the Rezk model structure) on $\Delta^{op}_{<}\mathbf{Top}$ such that the fibrant objects are exactly the Reedy fibrant presimplicial spaces s.t. the Segal maps are weak homotopy equivalences. This model category is cellular. X and Y weakly S-homotopy equivalent iff $\mathcal{N}^{pre}(X)$ and $\mathcal{N}^{pre}(Y)$ weakly equivalent.*

- A Segal flow M is a fibrant object of this model category ; its set of states is $\pi_0(M_0)$
- Bousfield-localization w.r.t $\mathcal{N}^{pre}(P_1) \subset \mathcal{N}^{pre}(P_2)$

Work in progress about Segal flow

- Constructing the branching and merging homology theories
- Constructing the underlying homotopy type functor
- How to prove that $H_*^{\pm}(\mathcal{N}^{pre}(X)) \cong H_*^{\pm}(X)$, $|\mathcal{N}^{pre}(X)| \simeq |X|$ for any strict flow X ?
- These questions are very important for (at least) two reasons:
  - To get the “good” notion of Segal flow
  - To understand what happens in the Bousfield localization of the Rezk model category structure by the $\mathcal{N}^{pre}(P_1) \subset \mathcal{N}^{pre}(P_2)$
- Bisimulation and weak flow