Towards A Homotopy Theory Of Higher Dimensional Transition Systems

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Modelling concurrency with geometry

- By topological models:
  - space of states
  - set of privileged continuous paths playing the role of execution paths (d-space, stream, local pospace)
- By categorical models:
  - set of states
  - space of execution paths modelled by topological spaces or ω-groupoids (flow, ω-category)
- By combinatorial models:
  - set of states
  - A n-transition is a labelled n-cube (labelled precubical set or any variant)

Cattani-Sassone formalism

- The concurrent execution of n actions from a state s to a state s' modelled by a multiset σ of n actions and a triple (s,σ,s') satisfying the following axioms:

CSA1 $(s,\{u\},s')$ and $(s,\{u'\},s')$ two transitions from a state s to a state s' and $\mu(u)=\mu(u')$ implies $u=u'$ (μ being the labelling map)

CSA2 $(s,\sigma_1\sqcup\sigma_2,s')$ a transition implies there exist two unique states ν_1 and ν_2 such that $(s,\sigma_1,\nu_1)\neq(s,\sigma_2,\nu_2)$ are two transitions and (ν_1,σ_2,s') and (ν_2,σ_1,s') are two transitions

CSA3 $(s,\sigma_1\sqcup\sigma_2\sqcup\sigma_3,s')$, (s,σ_1,ν_1), $(\nu_1,\sigma_2\sqcup\sigma_3,s')$, (ν_1,σ_2,ν_2), (ν_2,σ_3,s'), $(s,\sigma_1\sqcup\sigma_2,\nu_2')$, (ν_2',σ_3,s'), (s,σ_1,ν_1') and (ν_1',σ_2,ν_2') transitions imply $\nu_1=\nu_1'$ and $\nu_2=\nu_2'$

Combinatorial model category

- A locally presentable category
  - Axiomatized by a set of axioms in first-order logic of the form $(\forall x)\phi(x) \Rightarrow (\exists ! y\ \psi(x,y))$ with $\phi(x)$ and $\psi(x,y)$ conjunctions of atomic formulas
- A class of cofibrations Cof, a class of fibrations Fib, and a class of weak equivalences $\mathcal{W}$
  - $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ are two small weak factorization systems $(\mathbf{cof}(I), \mathbf{inj}(I))$ and $(\mathbf{cof}(J), \mathbf{inj}(J))$, I is the set of generating cofibrations, J is the set of generating trivial cofibrations
  - $\mathcal{W}$ contains all isomorphisms, is closed under retracts, and satisfies the two-out-of-three properties

Topological category

- Topological functor $F : \mathcal{C} \to \mathcal{D}$; the paradigm of such a functor is the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ from the category of general topological spaces to the category of sets
- A topological category $\mathcal{C}$ is a category which is topological over a product of the category of sets
- (Rosický) Such a category is axiomatized by a universal strict Horn theory without equality: $(\forall x)\phi(x) \Rightarrow \psi(x)$ where ϕ and ψ are conjunctions of atomic formulas without equalities in first-order logic (the class of axioms may be a proper class)
- Every topological category axiomatized by a set of axioms is locally presentable

Multisets are topological

- A non-empty set of labels Σ
- A higher dimensional transition presystem consists of a set of states S, a set of actions L, a labelling map $\mu : L \to \Sigma$, a subset of $\bigcup_{n \geqslant 1}(S \times L^n \times S)$, a tuple $(\alpha, u_1, \dots, u_n, \beta) \in S \times L^n \times S$ for $n \geqslant 1$ is called a n-transition satisfying the Multiset Axiom
  - For every permutation σ, if $(\alpha, u_1, \dots, u_n, \beta)$ is a n-transition, then $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$ is a n-transition
- The category of higher dimensional transition presystems is topological and locally finitely presentable

Cattani-Sassone transition system

- Higher dimensional transition presystem satisfying

CSA1 If (α, u, β) and (α, u', β) are two transitions such that $\mu(u) = \mu(u')$, then $u = u'$

CSA2 For $n \geqslant 2$ and $1 \leqslant p < n$, for every transition $(\alpha, u_1, \dots, u_n, \beta)$, there exists a unique state ν such that $(\alpha, u_1, \dots, u_p, \nu)$ and $(\nu, u_{p+1}, \dots, u_n, \beta)$ are transitions

CSA3 If the nine tuples $(\alpha, u_1, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$, $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2')$, $(\nu_2', u_{p+q+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1')$ and $(\nu_1', u_{p+1}, \dots, u_{p+q}, \nu_2')$ are transitions, then $\nu_1 = \nu_1'$ and $\nu_2 = \nu_2'$

The topological part of CSA3

- The full subcategory of Cattani-Sassone transition systems is not a reflective subcategory of the category of higher dimensional transition presystems
- The Coherence axiom
  - If $(\alpha, u_1, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$ and $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$ are transitions, then $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$ is a transition
- Higher dimensional transition presystems satisfying the Coherence axiom: a weak higher dimensional transition system (weak HDTS)

Weak HDTS

- Σ a nonempty set of labels
- A weak HDTS $X = (S, \mu : L \to \Sigma, T = \bigcup_{n\geqslant 1} T_n)$ with set of states S, set of actions L, labelling map μ, set of n-transitions $T_n \subset S \times L^n \times S$ with $n \geqslant 1$
  - **Multiset axiom** $(\alpha, u_1, \dots, u_n, \beta) \in T$ implies $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta) \in T$ for every permutation σ
  - **Coherence axiom** if $(\alpha, u_1, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$ and $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$ belong to T, then $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2) \in T$
- Note: the Coherence axiom is automatically satisfied in a cube

The category WHDTS

- $X = (S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n)$ and $X' = (S', \mu' : L' \to \Sigma, T' = \bigcup_{n \geqslant 1} T'_n)$
- A map of weak HDTS $f : X \to X'$ consists of
  - a set map $f_0 : S \to S'$
  - a map $\widetilde{f} \in (\mathbf{Set}{\downarrow}\Sigma)(\mu, \mu')$

  such that if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then $(f_0(\alpha), \widetilde{f}(u_1), \dots, \widetilde{f}(u_n), f_0(\beta))$ is a transition
- The forgetful functor $\omega : \mathbf{WHDTS} \to \mathbf{Set}^{\{s\} \cup \Sigma}$: $\omega(X) = (S, (\mu^{-1}(x))_{x \in \Sigma})$ is concrete topological
- $\mathbf{WHDTS}$ is locally finitely presentable

The category HDTS

- HDTS : weak HDTS satisfying CSA1 and CSA2
  - **HDTS** locally finitely presentable, but not topological
  - **HDTS** full reflective subcategory of **WHDTS**
- The pure n-transition $C_n[x_1, \dots, x_n]^{ext}$
  - Set of states $S = \{I, F\}$
  - Set of actions $L = \{(x_1, 1), \dots, (x_n, n)\}$
  - Labelling map $\mu(x_i, i) = x_i$
  - $T = \{(I, (x_{\sigma(1)}, \sigma(1)), \dots, (x_{\sigma(n)}, \sigma(n)), F)\}$
- The n-cube $C_n[x_1, \dots, x_n]$ is the reflection of $C_n[x_1, \dots, x_n]^{ext}$: 2^n states since $\{(x_1, 1), \dots, (x_n, n)\}$ contains n elements

HDTS as a small-orthogonality class

- Every HDTS is orthogonal to the set of maps $C_n[a_1,\dots,a_n]^{ext} \to C_n[a_1,\dots,a_n]$ for $n \geqslant 0$ and $a_1,\dots,a_n \in \Sigma$
- Every HDTS is orthogonal to the set of maps $C_1[x] \sqcup_{\partial C_1[x]} C_1[x] \to C_1[x]$ for $x \in \Sigma$
- Every weak HDTS orthogonal to the set of maps $C_n[a_1,\dots,a_n]^{ext} \to C_n[a_1,\dots,a_n]$ for $n \geqslant 0$ and $a_1,\dots,a_n \in \Sigma$ satisfies CSA2
- Every weak HDTS orthogonal to the set of maps $C_1[x] \sqcup_{\partial C_1[x]} C_1[x] \to C_1[x]$ for $x \in \Sigma$ satisfies CSA1
- HDTS is a small-orthogonality class

Homotopy theory of weak HDTS

- A cofibration of weak HDTS is a map inducing an injection on the set of actions
- There exists a left determined model category structure on $\mathbf{WHDTS}$ with respect to the class of cofibrations
  - i.e. the smallest localizer with respect to the cofibrations is the class of weak equivalences of a model category structure
  - A left proper combinatorial model category
  - All objects are cofibrant
- Note there exists a left determined model structure as well w.r.t. the class of monomorphisms but the class of weak equivalences is not convenient for concurrency

Construction of this model structure

- The proof uses Olschok's paper generalizing Cisinski's techniques from toposes to locally presentable categories
- The segment object $V = (\{0\}, \mathrm{pr}_1 : \Sigma \times \{0,1\} \to \Sigma, \{0\} \times (\bigcup_{n\geqslant 1}(\Sigma \times \{0,1\})^n) \times \{0\})$
  - Note: V does not satisfy CSA1
  - V is exponentiable
  - The map $\mathbf{1} \sqcup \mathbf{1} \longrightarrow \mathbf{1}$ factors as a composite $\mathbf{1} \sqcup \mathbf{1} \longrightarrow V \longrightarrow \mathbf{1}$ where the left-hand map is a cofibration and the right-hand map a trivial fibration
  - $(X' \sqcup X') \sqcup_{X \sqcup X} \mathrm{Cyl}(X) \to \mathrm{Cyl}(X')$ and $X' \sqcup_X \mathrm{Cyl}(X) \to \mathrm{Cyl}(X')$ are cofibrations for every cofibration $X \to X'$ with $\mathrm{Cyl}(-) := V \times -$

Cubical transition system (I)

- A cubical transition system is a weak HDTS which is union of its subcubes
- A cubical transition system is not necessarily a colimit of cubes
- $\begin{array}{c}\xrightarrow{x}\\ \xrightarrow{x}\end{array}$, denoted by $\uparrow x \uparrow$, is not a colimit of cubes:

$\uparrow x \uparrow = \varinjlim(C_1[x] \leftarrow \underline{x} \rightarrow C_1[x])$ with $\underline{x} = (\varnothing, \{x\}, \varnothing)$

- $\begin{array}{c}\xrightarrow{x_1}\\ \xrightarrow{x_2}\end{array}$ with $\mu(x_1) = \mu(x_2) = x$ is a colimit of cubes

Cubical transition system (II)

- The full subcategory $\mathbf{CTS}$ is locally finitely presentable, coreflective, not concretely coreflective and not topological
- It is also a small-injectivity class w.r.t.
  - the maps $\underline{x} \subset C_1[x]$ for $x \in \Sigma$; all actions are used in a transition
  - $C_n[x_1, \dots, x_n]^{ext} \subset C_n[x_1, \dots, x_n]$ for $n \geqslant 0$ and $x_1, \dots, x_n \in \Sigma$; all higher dimensional transitions have faces (note: in $\mathbf{CTS}$, a square may have more than 4 edges...)

Homotopy theory in $\mathbf{CTS}$

- A cofibration of cubical transition systems is a map inducing an injection on the set of actions
- There exists a left determined model category structure on $\mathbf{CTS}$ with respect to the class of cofibrations
    - i.e. the smallest localizer with respect to the cofibrations is the class of weak equivalences of a model category structure
    - A left proper combinatorial model category
- The adjunction $\mathbf{CTS} \leftrightarrows \mathbf{WHDTS}$ is a Quillen adjunction
- A monomorphism (resp. cofibration) of $\mathbf{CTS}$ is not necessarily a cofibration (monomorphism)

Descent theorem for Olschok’s structure

- Let $\mathcal{B}$ be a combinatorial model category constructed using a segment object V and Olschok’s techniques
- Let $\mathcal{A} \subset \mathcal{B}$ be a full subcategory which is
  - coreflective
  - a small cone-injectivity class such that each map of each cone is a cofibration of $\mathcal{B}$
  - $V \in \mathcal{A}$
  - the inclusion $\mathcal{A} \subset \mathcal{B}$ preserves binary products
- THEN Olschok’s techniques apply and provide a left proper left determined model category on $\mathcal{B}$ w.r.t. the class of cofibrations between objects of $\mathcal{A}$

The homotopy category of CTS

- The canonical map $X \to \mathbf{1}$ functorially factors as a composite
$$X \longrightarrow \mathrm{CSA}_1(X) \longrightarrow \mathbf{1}$$
with $X \to \mathrm{CSA}_1(X)$ transfinite composition of pushouts of $C_1[x] \sqcup_{\partial C_1[x]} C_1[x] \to C_1[x]$ for $x \in \Sigma$ and $\mathrm{CSA}_1(X)$ safisfying CSA1
- The homotopy category of $\mathbf{CTS}$ is equivalent to $\mathbf{CTS}[\mathrm{CSA}_1^{-1}]$: in particular, two weakly equivalent cubical transition systems satisfying CSA1 (e.g. HDTS) are isomorphic
- $C_1[x] \sqcup C_1[x] \to \varinjlim(C_1[x] \leftarrow \underline{x} \to C_1[x])$ is not a weak equivalence

Bousfield localization w.r.t. $\underline{\mathrm{Cub}}$

- $\underline{\mathrm{Cub}}(X)$ the colimits (and not the union) of the cubes of X:

$$\boxed{\underline{\mathrm{Cub}}(X) = \varinjlim_{C_n[a_1,\dots,a_n]\to X} C_n[a_1,\dots,a_n]}$$

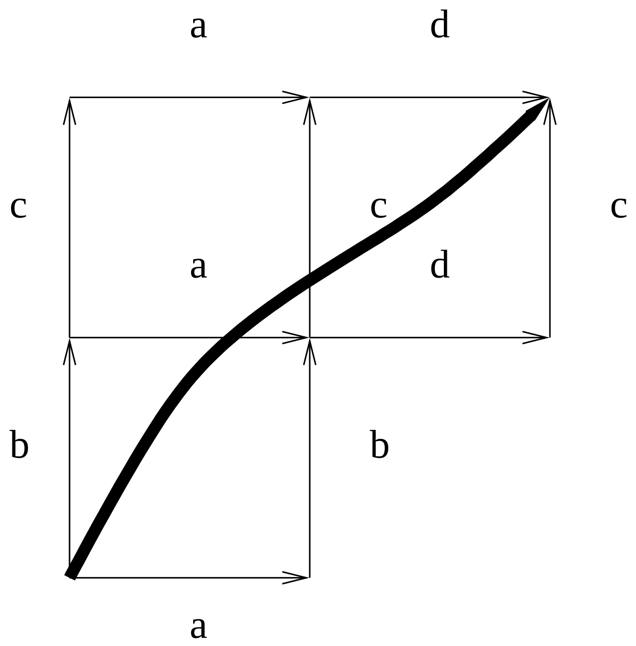
- $\underline{\mathrm{Cub}}(\varnothing) \cong \underline{\mathrm{Cub}}(\underline{x}) \cong \varnothing$
- $\underline{\mathrm{Cub}}(C_1[x] \sqcup C_1[x]) \cong \underline{\mathrm{Cub}}\,\varinjlim(C_1[x] \leftarrow \underline{x} \rightarrow C_1[x])$
- The Bousfield localization w.r.t. the class of maps f such that $\underline{\mathrm{Cub}}(f)$ is a weak equivalence exists
- It is equal to the Bousfield localization $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ w.r.t. the set of maps

$$\mathcal{S} = \{C_1[x] \sqcup C_1[x] \rightarrow \varinjlim(C_1[x] \leftarrow \underline{x} \rightarrow C_1[x]) \mid x \in \Sigma\}$$

The homotopy category of $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$

- For a cubical transition system X, let $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ be the cubical transition system obtained by replacing each action u by its label $\mu(u) \in \Sigma$
  - There is a canonical map $X \longrightarrow \underline{\mathbf{L}}_{\mathcal{S}}(X)$
  - $\underline{\mathbf{L}}_{\mathcal{S}}(C_1[x] \sqcup C_1[x]) = \uparrow x \uparrow$
  - In fact, $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is always fibrant in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$
- The cubical transition system $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ satisfies CSA1
  - Two transitions (α, u, β) and (α, v, β) with $\mu(u) = \mu(v)$ are taken to $(\alpha, \mu(u), \beta) = (\alpha, \mu(v), \beta)$
- A map f is a weak equivalence of $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ if and only if $\underline{\mathbf{L}}_{\mathcal{S}}(f)$ is an isomorphism

Paths in a cubical transition system



A path is a cubical transition system of the form $C_{m_1} \sqcup_{C_{n_1}} \cdots \sqcup_{C_{n_{p-1}}} C_{m_p}$ where the C_k are cubes and C_{n_i} is a sub-cube of the target of C_{m_i} and of the source of $C_{m_{i+1}}$

- A map $f : X \longrightarrow Y$ is open if it is injective w.r.t. the inclusions $\{0_n\} \subset P$ where P is a path
- Two cubical transition systems X and Y are bisimilar if there is a zig-zag of open maps $X \longleftarrow T \longrightarrow Y$ (Joyal-Nielsen-Winskel)

Bousfield localizing w.r.t. open maps

- The class of open maps is accessible and (finitely) accessibly-embedded
- The Bousfield localization w.r.t. the class of open maps exists (standard argument):
  - there exists a regular cardinal λ such that λ-filtered colimits are homotopy colimits; so any cubical transition system local w.r.t. this set of generators is local w.r.t. to the whole class of open maps
  - It suffices to localize w.r.t. a set of λ-presentable generators
- In $\underline{\mathbf{L}}_{open}\,\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$, all open maps are weak equivalences and two bisimilar cubical transition systems are weakly equivalent

The homotopy category of $\underline{\mathbf{L}}_{open}\,\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$

- There exists a fibrant replacement functor $(-)^{fib}$ such that f is a weak equivalence of $\underline{\mathbf{L}}_{open}\,\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ if and only if f^{fib} is an isomorphism
  - Take the image $\mathcal{O}$ by $\underline{\mathbf{L}}_{\mathcal{S}}$ of the previous set of generators of open maps
  - Every map $X \longrightarrow \mathbf{1}$ factors as a composite $X \longrightarrow \underline{\mathbf{L}}_{\mathcal{O}}(X) \longrightarrow \mathbf{1}$ with the left-hand map in $\mathbf{cell}(\mathcal{O})$ and the right-hand map in $\mathbf{inj}(\mathcal{O})$
  - $(-)^{fib} = \underline{\mathbf{L}}_{\mathcal{O}}\,\underline{\mathbf{L}}_{\mathcal{S}}$
- Conjecture: for two cubical transitions systems with one-to-one labelling maps and satisfying CSA2, weakly equivalent $\Leftrightarrow$ bisimilar

Reachable cubical transition systems

- A reachable cubical transition system is an object of the coslice category $(\{i\}{\downarrow}\mathbf{CTS})$ such that every state α is reachable from i by a finite sequence of 1-transitions $(\alpha_i, u_i, \alpha_{i+1})$ with $0 \leqslant i \leqslant n$, $\alpha_0 = i$ and $\alpha_{n+1} = \alpha$
- The category $\mathbf{CTS}_\bullet$ of reachable cubical transition systems is
  - a small cone-injectivity class of the coslice category $(\{i\}{\downarrow}\mathbf{CTS})$: the cone is the set of maps

$$\{i, f\} \subset i \xrightarrow{x_1} \cdots \xrightarrow{x_n} f$$

  - a coreflective full subcategory of $(\{i\}{\downarrow}\mathbf{CTS})$
  - it is locally finitely presentable

Homotopy theory of CTS.

- A cofibration of reachable cubical transition systems is a map inducing a one-to-one map between the set of actions
- There exists a left determined model structure w.r.t. $\mathbf{CTS}$.
- The Bousfield localization w.r.t. the maps of $\mathcal{S}$ and the open maps exists
- Conjecture: for two cubical transitions systems with one-to-one labelling maps and satisfying CSA2, weakly equivalent ⇔ bisimilar