T-homotopy and Quillen model category

How to construct a model category whose homotopy category is exactly the category of flows up to weak S-homotopy and up to T-homotopy ?

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Definition of a flow

- A topological space $\mathbb{P}X$ called the path space, and whose elements are called non-constant execution paths
- A discrete space X^0 called the 0-skeleton, and whose elements are called states
- Two continuous maps s (the source map) and t (the target map) from $\mathbb{P}X$ to X^0
- A continuous and associative map (the composition law)

$$* : \{(x,y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$$

such that $s(x*y) = s(x)$ and $t(x*y) = t(y)$

Morphism of flows $f : X \longrightarrow Y$

- A set map $f^0 : X^0 \longrightarrow Y^0$
- A continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ such that
  - $f(s(x)) = s(f(x))$
  - $f(t(x)) = t(f(x))$
  - $f(x * y) = f(x) * f(y)$

The corresponding category is denoted by **Flow**

Why no identity morphisms ?

A flow is exactly a small categories enriched over the category of (compactly generated) topological spaces without identity morphisms

The branching space functor $\mathbb{P}^- : \mathbf{Flow} \longrightarrow \mathbf{Top}$ defined by

$$\mathbb{P}^- X := \mathbb{P}X/(x = x * y)$$

$$i : X^0 \longrightarrow \mathbb{P}X$$

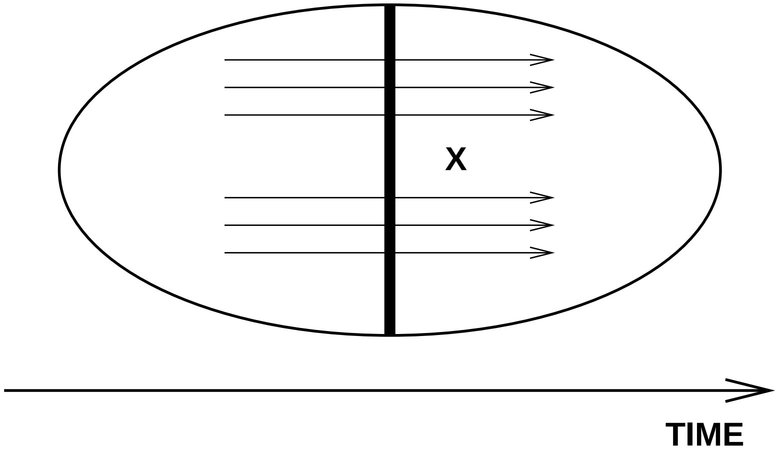
$$x = i(s(x)) * x \sim i(s(x))$$

The branching space $\mathbb{P}^- X$ is discrete, and therefore not very interesting...

Fundamental examples of flows

- The globe $\mathrm{Glob}(Z)$ of the topological space Z
  - $\mathrm{Glob}(Z) = \{0,1\}$
  - $\mathbb{P}\mathrm{Glob}(Z) = Z$
  - no composition law
- If $Z = \{*\}$, $\mathrm{Glob}(Z) = \overrightarrow{I}$ is called the directed segment

Symbolic representation of $\mathrm{Glob}(X)$ for some topological space X



Weak S-homotopy equivalence

A morphism of flows $f : X \longrightarrow Y$ is a weak S-homotopy equivalence if

- $f^0 : X^0 \longrightarrow Y^0$ bijection of sets
- $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ weak homotopy equivalence

Concurrent execution paths cannot be distinguished by observation

The weak S-homotopy model structure

Theorem 1. *There exists exactly one model structure on* $\mathbf{Flow}$ *such that*

- *the weak equivalences are the weak S-homotopy equivalences*
- *the fibration* $f : X \longrightarrow Y$ *are the morphisms of flows such that* $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ *is a fibration of topological spaces*

This model structure is cofibrantly generated.

Generating cofibrations

The set of generating cofibrations is

$$\{\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}$$

with $C : \varnothing \longrightarrow \{0\}$, $R : \{0,1\} \longrightarrow \{0\}$
The set of generating trivial cofibrations is

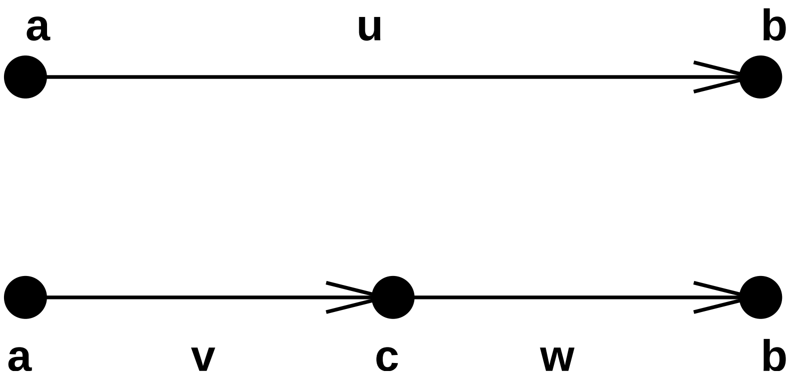
$$\{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}$$

This model structure is not cellular

The cofibrant replacement $Q(X)$ of a flow X can be interpreted as the closest computer scientific interpretation of X

T-homotopy equivalence

T-homotopy equivalences model invariance by refinement of observation

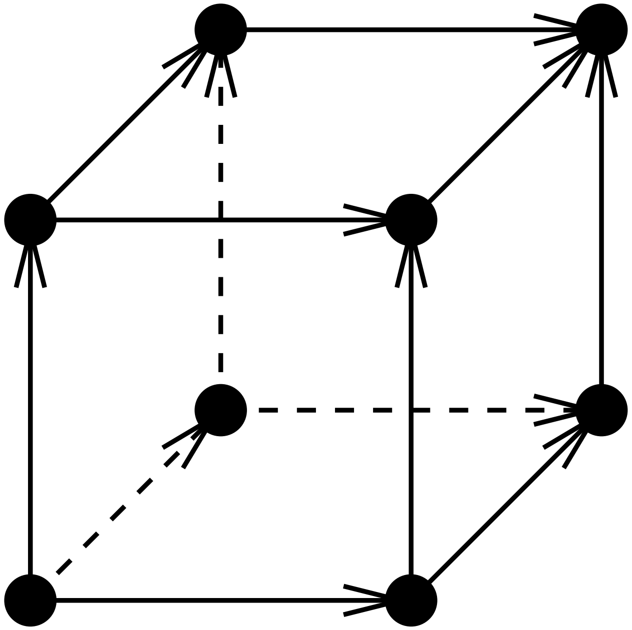


Let $\mathcal{T}$ be the class of transfinite compositions of pushouts of morphisms of the form $\overrightarrow{I} \longrightarrow Q(\overrightarrow{I} * \overrightarrow{I})$ (Q being the cofibrant replacement of the weak S-homotopy model structure)

A T-homotopy equivalence is in this talk an isomorphism of the (categorical) localization $\mathbf{Flow}[\mathcal{T}^{-1}]$

Not enough T-homotopy

The preceding class of T-homotopy equivalences is not big enough



The 3-cube cannot be identified to the directed segment because any point is connected to 3 edges

Known dihomotopy invariants

Only three so far...

- The branching homology $H_*^-(X)$ of a flow X
- The merging homology $H_*^+(X)$ of a flow X
- The underlying homotopy type $|X|$ of a flow X

Very poor invariant, but very useful anyway, for instance to try new definitions of T-homotopy equivalences...

Branching homology

Let $n \geqslant -1$

$$\mathbb{P}^{-}X := \mathbb{P}X/(x = x * y)$$

$$\boxed{H^{-}_{n+1}(X) := H_n\left(\operatorname{Sing}\mathbb{P}^{-}Q(X) \xrightarrow{s} X^0\right)}$$

$H^{-}_{n}(X)$ contains the non-deterministic branchings of dimension n for $n \geqslant 1$

$H^{-}_{0}(X)$ is the free group generated by the final states of X

Theorem 2. *H^{-}_{n} sends weak S-homotopy equivalences and T-homotopy equivalences into isomorphisms of groups.*

About the branching homology

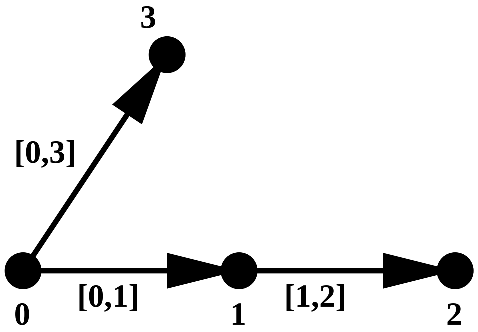
The cofibrant replacement functor necessary for obtaining a dihomotopy invariant

Theorem 3. *There exists a weak S-homotopy equivalence of flows* $f : X \longrightarrow Y$ *such that* $\mathbb{P}^- X$ *is homotopy equivalent to* $\mathbf{S}^2 \sqcup \{0\}$ *and such that* $\mathbb{P}^- Y$ *is homotopy equivalent to* $\{0, 1\}$.

The augmentation is necessary for obtaining a dihomotopy invariant

Theorem 4. *The functor* $X \mapsto H_0(\mathrm{ho}\mathbb{P}^- X)$ *is invariant with respect to weak S-homotopy, but not with respect to T-homotopy equivalences.*

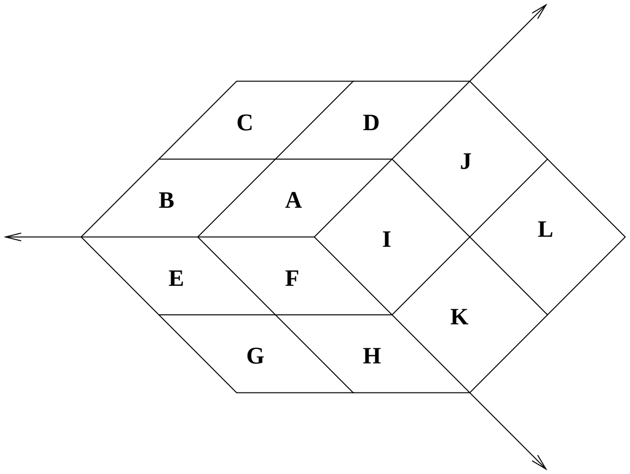
1-dimensional branching



$X^0 = \{0,1,2,3\}$, $\mathbb{P}_{0,1}X = \{[0,1]\}$, $\mathbb{P}_{1,2}X = \{[1,2]\}$, $\mathbb{P}_{0,3}X = \{[0,3]\}$, $\mathbb{P}_{0,2}X = \{[0,2]\}$ and $\mathbb{P}_{\alpha\beta}X = \varnothing$ otherwise.

$\mathbb{P}^-_0 X = \{[0,1],[0,3]\}$, $\mathbb{P}^-_1 X = \{[1,2]\}$, $\mathbb{P}^-_2 X = \mathbb{P}^-_3 X = \varnothing$. $H^-_n(X) = 0$ for $n \geqslant 2$, $H^-_1(X) = \mathbb{Z}$ (generated by $[0,3] - [0,1]$), and $H^-_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ (generated by the final states 2 and 3).

2-dimensional branching



$H_1^- = 0$ and $H_n^- = 0$ for $n \geqslant 2$.
$H_1^- = \mathbb{Z}$, the generating branching being the one corresponding to the alternate sum $(A) - (F) + (I)$.
$H_0^- = \mathbb{Z} \oplus \mathbb{Z} \oplus Z$, the generators being the final states of the three squares (C), (G) and (L).
If α is the common initial state of (A), (F) and (I), then $\mathbb{P}_\alpha^- = \mathbf{S}^1$.

Merging homology

Let $n \geqslant -1$

$$\mathbb{P}^+X := \mathbb{P}X/(y = x * y)$$

$$\boxed{H^+_{n+1}(X) := H_n\left(\operatorname{Sing}\mathbb{P}^+Q(X) \xrightarrow{t} X^0\right)}$$

$H^+_n(X)$ contains the non-deterministic mergings of dimension n for $n \geqslant 1$

$H^+_0(X)$ is the free group generated by the initial states of X

Theorem 5. *H^+_n sends weak S-homotopy equivalences and T-homotopy equivalences into isomorphisms of groups.*

Underlying homotopy type of a flow

The underlying homotopy type $|X|$ of a flow X

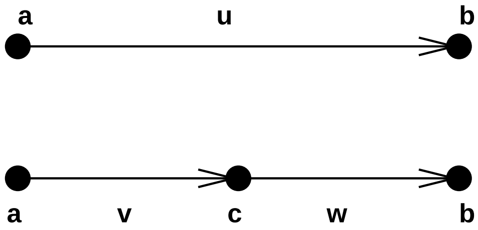
- Take the cofibrant replacement $Q(X)$
- Write $\varnothing \longrightarrow Q(X)$ as a transfinite composition of pushouts of $C : \varnothing \longrightarrow \{0\}$, $R : \{0,1\} \longrightarrow \{0\}$, and of $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ with $n \geqslant 0$
- Replace this transfinite composition by a transfinite composition of pushouts of $C : \varnothing \longrightarrow \{0\}$, $R : \{0,1\} \longrightarrow \{0\}$ and of $\Sigma\mathbf{S}^{n-1} \longrightarrow \Sigma\mathbf{D}^n$ with $n \geqslant 0$
- One obtains $\varnothing \longrightarrow |X|$

Theorem 6. $X \mapsto |X|$ *from* $\mathbf{Flow}$ *to* $\mathbf{Ho}(\mathbf{Top})$ *sends weak S-homotopy equivalences and T-homotopy equivalences into isomorphisms.*

Model categories for computer science

There exists a model structure on $\mathbf{Flow}$ such that the weak equivalences are exactly the weak S-homotopy equivalences

A model structure on $\mathbf{Flow}$ is relevant for computer science if any of its weak equivalences is sent into an isomorphism by H_*^-, H_*^+ and $|-|$



$$\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I} \text{ such that } \phi(u) = v * w$$

What is the behaviour of the Bousfield localization of the weak S-homotopy model structure of $\mathbf{Flow}$ with respect to ϕ ? And does it exist ?

First negative results

It is unknown whether the Bousfield localization of the weak S-homotopy model structure of $\mathbf{Flow}$ with respect to ϕ exists but

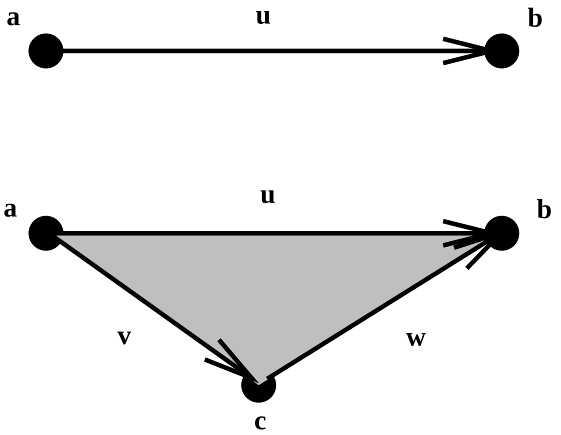
Theorem 7. *Even if this Bousfield localization exists, there exists for any* $n \geqslant 1$ *a weak equivalence* f *such that* $H_n^-(f)$ *is not an isomorphism.*

Theorem 8. *Even if this Bousfield localization exists, there exists for any* $n \geqslant 1$ *a weak equivalence* f *such that* $H_n^+(f)$ *is not an isomorphism.*

Theorem 9. *Even if this Bousfield localization exists, there exists a weak equivalence* f *such that* $|f|$ *is not an isomorphism.*

The main example

$$Q(\phi) : \overrightarrow{I} \longrightarrow W$$



Theorem 10. *The weak S-homotopy model structure of* **Flow** *is simplicial.*

Theorem 11. *The functor* $\mathbb{P}^{-} : \mathbf{Flow} \longrightarrow \mathbf{Top}$ *is a left Quillen functor compatible with the simplicial enrichments of* **Flow** *and* **Top**.

Explanation of the problem

For $n \geqslant 2$

$$\begin{tikzcd}
\overrightarrow{I} \otimes \mathbf{S}^{n-1} \arrow[r] \arrow[d] & \overrightarrow{I} \otimes \mathbf{D}^{n} \arrow[d] \arrow[ddr, bend left] & \\
W \otimes \mathbf{S}^{n-1} \arrow[r] \arrow[drr, bend right] & X \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, dashed, "g"] & \\
 & & W \otimes \mathbf{D}^{n}
\end{tikzcd}$$

$$\mathbb{P}^{-}_{c} X \cong \mathbf{S}^{n-1} \longrightarrow \mathbb{P}^{-}_{c}(W \otimes \mathbf{D}^{n}) \cong \mathbf{D}^{n}$$

This is not a weak homotopy equivalence

Flows over simplicial sets

Consider the category $\mathbf{Flow}(\Delta^{op}\mathbf{Set})$ of flows over the simplicial sets

The Quillen equivalence

$$\mathbf{Flow}(\Delta^{op}\mathbf{Set}) \leftrightarrows \mathbf{Flow}$$

induced by the well-known Quillen equivalence

$$\Delta^{op}\mathbf{Set} \leftrightarrows \mathbf{Top}$$

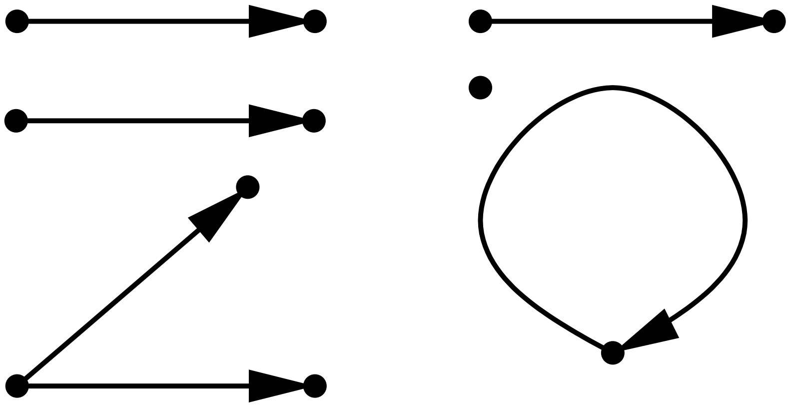
Theorem 12. *Under Vopěnka's principle, the Bousfield localization of* $\mathbf{Flow}(\Delta^{op}\mathbf{Set})$ *with respect to* ϕ *exists.*

The same pathological phenomenon as $\mathbf{Flow}$

Other pathological phenomena

Theorem 13. *For any model structure (cofibrantly generated or not) on* **Flow** *such that* $\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I}$ *is a weak equivalence, there exists a pushout of* $R : \{0,1\} \longrightarrow \{0\}$ *which is a weak equivalence.*

Theorem 14. *For any model structure (cofibrantly generated or not) on* **Flow** *such that* $R : \{0,1\} \longrightarrow \{0\}$ *is not a cofibration, there exists a pushout of* R *which is a weak equivalence.*



A presheaf category

Consider the full small subcategory $\mathcal{B}$ of countable objects of $\mathbf{Flow}(\Delta^{op}\mathbf{Set})$

Theorem 15. *There exists a Bousfield localization of the Bousfield-Kan model structure of* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *which is Quillen equivalent to the weak S-homotopy model structure of* $\mathbf{Flow}$.

The latter model category is cellular

$$Re : \Delta^{op}\mathrm{Presh}(\mathcal{B}) \leftrightarrows \mathbf{Flow}(\Delta^{op}\mathbf{Set}) : \mathrm{Sing}$$

$$Re(F) = \int^{b \in \mathcal{B}} \int^{[n] \in \Delta} F_n(b).(b \otimes \Delta[n])$$

$$\mathrm{Sing}(X)(b) = \mathbf{Flow}(\Delta^{op}\mathbf{Set})(b \otimes \Delta[*], X)$$

Again negative results

Theorem 16. *In the Bousfield localization of the Heller model structure of* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *with respect to* ϕ, *there exists for any* $n \geqslant 1$ *a weak equivalence* f *such that* $H_n^-(f)$ *(resp.* $H_n^+(f)$*) is not an isomorphism.*

Theorem 17. *In the Bousfield localization of the Bousfield-Kan model structure of* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *with respect to* ϕ, *there exists for any* $n \geqslant 1$ *a weak equivalence* f *such that* $H_n^-(f)$ *(resp.* $H_n^+(f)$*) is not an isomorphism.*

More negative results with topos theory

Theorem 18. *In any model structure (cofibrantly generated or not) of the category of simplicial presheaves* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *such that the objectwise weak homotopy equivalences are weak equivalences, such that the branching homology and the merging homology are homotopy invariant (w.r.t. this model structure), and such that* ϕ *is a weak equivalence, there exists a monomorphism of* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *which is not a cofibration*.

... and with Vopěnka's principle

Theorem 19. *Assume Vopěnka's principle. In any model structure (cofibrantly generated or not) of the category of simplicial presheaves* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *such that the objectwise weak homotopy equivalences are weak equivalences, such that the branching homology and the merging homology are homotopy invariant (w.r.t. this model structure), and such that* ϕ *is a weak equivalence, there exists a Bousfield-Kan cofibration of* $\Delta^{op}\mathrm{Presh}(\mathcal{B})$ *which is not a cofibration.*

Morally speaking...

$\mathcal{L}$ denotes the Bousfield localization functor

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{L}"'] & Y \arrow[d, "\mathcal{L}"'] \\
\mathcal{L}X \arrow[r, "\mathcal{L}f"] & \mathcal{L}Y
\end{tikzcd}$$

f is a dihomotopy equivalence if and only if $\mathcal{L}f$ is a weak S-homotopy equivalence

Impossible to subdivide $\mathcal{L}X$, $\mathcal{L}Y$ by T-homotopy, i.e. $\mathcal{L}X$, $\mathcal{L}Y$ are continuous objects

fibrant = continuous
fibration = more subdivided

The terminal object of **Flow** is continuous

Perspectives

The difficulty of constructing a model structure relevant for the study of T-homotopy seems to have two causes:

- the rigidity of the category of flows and the lack of continuous objects
- the behaviour of the simplicial enrichment

There exists other frameworks for doing abstract homotopy theory (i.e. Bousfield localization, homotopy limits and colimits, construction of invariants), for instance Grothendieck derivators and Heller homotopy theory, Rezk's approach, etc...

The use of symmetric simplicial sets could be explored

Appendix: generalized T-homotopy

A full directed ball D is a flow D such that:

- D^0 is finite
- D has exactly one initial state I and one final state F
- any other state of D^0 is between I and F
- D does not contain any loop
- for any $(\alpha, \beta) \in \mathbb{P}_{\alpha,\beta}D$, $\mathbb{P}_{\alpha,\beta}D$ is empty or weakly contractible

Appendix: generalized T-homotopy

Attempt of definition of a T-homotopy equivalence

Let $\mathcal{T}$ be the class of transfinite compositions of pushouts of morphisms of the form $\overrightarrow{I} \longrightarrow Q(D)$ (Q being the cofibrant replacement of the weak S-homotopy model structure), where D a full directed ball

A T-homotopy equivalence is an isomorphism of the (categorical) localization $\mathbf{Flow}[\mathcal{T}^{-1}]$

Appendix: generalized T-homotopy

Conjecture 20. *The underlying homotopy type of a full directed ball is contractible.*

Conjecture 21. *Let* D *be a full directed ball. Then the homotopy branching space of* D *is contractible on any point* $\alpha \in D^0$ *not equal to the final state.*

Conjecture 22. *Let* D *be a full directed ball. Then the homotopy merging space of* D *is contractible on any point* $\alpha \in D^0$ *not equal to the initial state.*

Proof in progress...

Appendix: long exact sequence

Let $f : X \longrightarrow Y$ be a morphism of flows. Let Cf be the cone of f for the weak S-homotopy model structure of $\mathbf{Flow}$.

$$\begin{aligned}
&\dots \to H_n^-(X) \to H_n^-(Y) \to H_n^-(Cf) \to \dots \\
&\dots \to H_3^-(X) \to H_3^-(Y) \to H_3^-(Cf) \to \\
&H_2^-(X) \to H_2^-(Y) \to H_2^-(Cf) \to \\
&H_0(\mathbb{P}^-Q(X)) \to H_0(\mathbb{P}^-Q(Y)) \to H_0(\mathbb{P}^-Q(Cf)) \to 0.
\end{aligned}$$

$$\begin{aligned}
&\dots \to H_n^+(X) \to H_n^+(Y) \to H_n^+(Cf) \to \dots \\
&\dots \to H_3^+(X) \to H_3^+(Y) \to H_3^+(Cf) \to \\
&H_2^+(X) \to H_2^+(Y) \to H_2^+(Cf) \to \\
&H_0(\mathbb{P}^+Q(X)) \to H_0(\mathbb{P}^+Q(Y)) \to H_0(\mathbb{P}^+Q(Cf)) \to 0.
\end{aligned}$$