T-homotopy and Refinement of Observation

Application : Reconstructing Whitehead's theorem for the full dihomotopy relation

Philippe Gaucher
gaucher@pps.jussieu.fr
http://www.pps.jussieu.fr/˜gaucher/

Preuves, Programmes et Systèmes, CNRS UMR 7126 et Paris 7

Bibliography

- P. Gaucher, *Introduction*, math.AT/0505152
- P. Gaucher, *Adding new T-homotopy equivalences*, math.AT/0505328
- P. Gaucher, *Invariance of the branching and merging homologies*, math.AT/0505329
- P. Gaucher, *Invariance of the underlying homotopy type*, math.AT/0505331
- P. Gaucher, *Strom model structure for branching and merging homologies*, math.AT/0401033
- P. Gaucher, *Inverting weak dihomotopy equivalence using homotopy continuous flow*, math.AT/0506548

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- $\mathbb{P}X = \bigsqcup_{\alpha,\beta} \mathbb{P}_{\alpha,\beta}X$ is called the path space

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The corresponding category is denoted by $\mathbf{Flow}$

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A flow is exactly a small categories enriched over the category of (compactly generated) topological spaces without identity morphisms

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The branching space functor $\mathbb{P}^- : \mathbf{Flow} \longrightarrow \mathbf{Top}$ defined by

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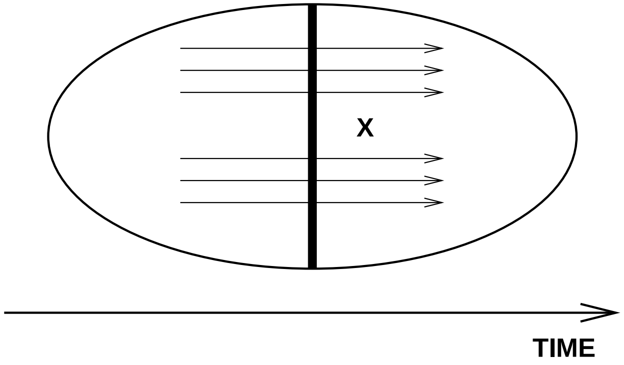
The branching space $\mathbb{P}^- X$ is discrete, and therefore not very interesting...

Fundamental examples of flows



- The globe $\mathrm{Glob}(Z)$ of the topological space Z
  - $\mathrm{Glob}(Z) = \{\widehat{0}, \widehat{1}\}$
  - $\mathbb{P}\mathrm{Glob}(Z) = Z$
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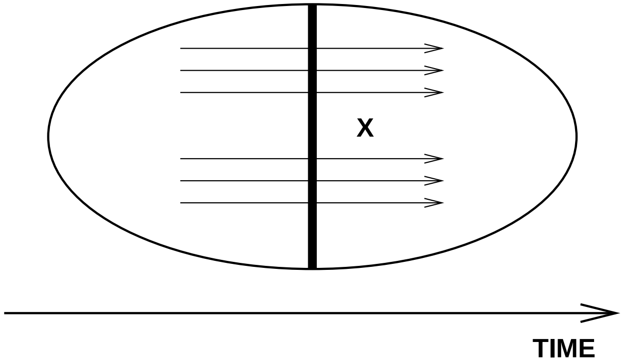

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- If $Z = \{*\}$, $\mathrm{Glob}(Z) = \overrightarrow{I}$ is called the directed segment

Deforming the time flow of a HDA

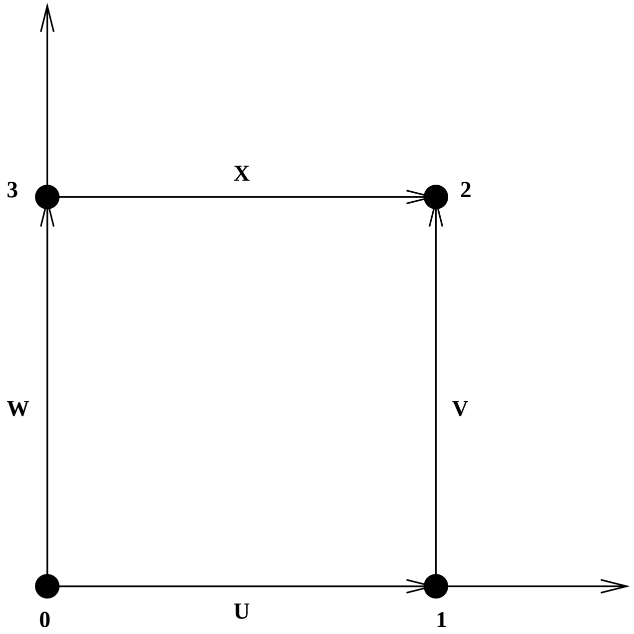

Figure 1: Two non-concurrent processes

- The flow $\partial\overrightarrow{C}_2$ defined by $(\partial\overrightarrow{C}_2)^0 = \{0,1,2,3\}$, $\mathbb{P}_{0,1}\partial\overrightarrow{C}_2 = \{U\}$, $\mathbb{P}_{1,2}\partial\overrightarrow{C}_2 = \{V\}$, $\mathbb{P}_{0,3}\partial\overrightarrow{C}_2 = \{W\}$, $\mathbb{P}_{3,2}\partial\overrightarrow{C}_2 = \{X\}$

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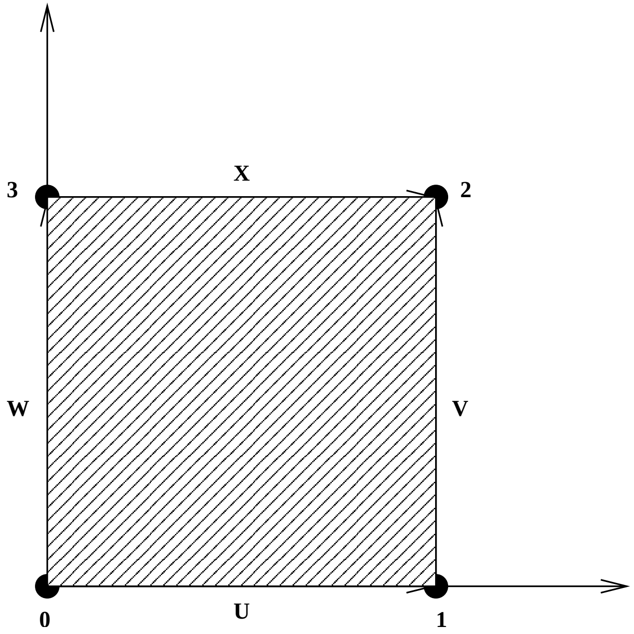

Figure 2: Two concurrent processes

- The flow $\color{red}{\partial\overrightarrow{C}_2}$ defined by $(\partial\overrightarrow{C}_2)^0 = \{0, 1, 2, 3\}$, $\mathbb{P}_{0,1}\partial\overrightarrow{C}_2 = \{U\}$, $\mathbb{P}_{1,2}\partial\overrightarrow{C}_2 = \{V\}$, $\mathbb{P}_{0,3}\partial\overrightarrow{C}_2 = \{W\}$, $\mathbb{P}_{3,2}\partial\overrightarrow{C}_2 = \{X\}$
- $$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^0) \arrow[r, "q"] \arrow[d] & \partial\overrightarrow{C}_2 \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^1) \arrow[r] & \color{red}{\overrightarrow{C}_2} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$ and $q(\mathbf{S}^0) = \{U * V, W * X\}$

Weak S-homotopy equivalence

- It does not matter for $\mathbb{P}_{0,2}\overrightarrow{C}_2$ to be homeomorphic to $\mathbf{D}^1$ or only homotopy equivalent to $\mathbf{D}^1$, or even only weakly homotopy equivalent to $\mathbf{D}^1$. The only thing that matters is that the topological space $\mathbb{P}_{0,2}\overrightarrow{C}_2$ be weakly contractible.

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- A morphism of flows $f : X \longrightarrow Y$ is a weak S-homotopy equivalence if
  - $f^0 : X^0 \longrightarrow Y^0$ bijection of sets
  - $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ weak homotopy equivalence

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This model structure is cofibrantly generated

Generating cofibrations

- The set of generating cofibrations is

$$I_+^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\} \cup \{C, R\}$$

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- The set of generating trivial cofibrations is

$$J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}$$

The cofibrant replacement functor Q

- Factoring the canonical morphism $\varnothing \longrightarrow X$ as a composite

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- Q is a functor
- The cofibrant replacement $Q(X)$ of a flow X is the closest computer scientific interpretation of X

Full n-dimensional cube

- n processes running concurrently

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- Let X and Y be two flows. There exists a unique flow $X \otimes Y$ such that
  - $(X \otimes Y)^0 = X^0 \times Y^0$
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  - $s(x,y) = (s(x), s(y))$, $t(x,y) = (t(x), t(y))$, $(x,y) * (z,t) = (x * z, y * t)$

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- Note: $\overrightarrow{I}^{\otimes n}$ never cofibrant if $n \geqslant 2$

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- $\overrightarrow{C}_m$ and $\overrightarrow{C}_n$ not weakly S-homotopy equivalent if $m \neq n$

Full directed ball

- Some hints given by $\overrightarrow{C}_m$ ($m \geqslant 1$):
  - $(\overrightarrow{C}_m)^0 = \{\widehat{0} < \widehat{1}\}^m$ is a finite bounded poset
  - $\overrightarrow{C}_m$ loopless
  - $\alpha, \beta \in (\overrightarrow{C}_m)^0$ and $\alpha < \beta$ implies $\mathbb{P}_{\alpha,\beta}\overrightarrow{C}_m$ contractible

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- $\overrightarrow{C}_m$ is a full directed ball

From finite bounded posets to flows

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- Each finite bounded poset P is a flow with
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- This defines a functor $\mathcal{P} \longrightarrow \mathbf{Flow}$

Generating T-homotopy equivalences

- Consider the set of $f : P_1 \longrightarrow P_2$ such that
  - $f(\widehat{0}) = \widehat{0}$
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  - f one-to-one

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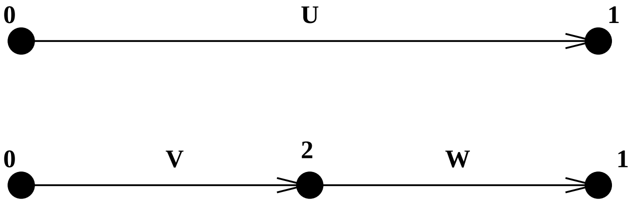
- Is it a reasonable definition of T-homotopy ?

The simplest T-homotopy equivalence



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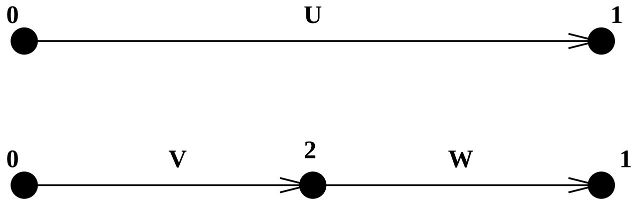

- $U \mapsto V * W$
- $\{\widehat{0} < \widehat{1}\} \longrightarrow \{\widehat{0} < 2 < \widehat{1}\}$
- T-homotopy equivalences model invariance by refinement of observation

Known dihomotopy invariants

Only three so far...

- The branching homology $H_*^-(X)$ of a flow X
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Very poor invariant, but very useful anyway, for instance to try new definitions of T-homotopy equivalences...

Branching homology

- Let $n \geqslant -1$

$$\mathbb{P}^- X := \mathbb{P}X/(x = x * y)$$

$$\boxed{H_{n+1}^-(X) := H_n\left(\operatorname{Sing}\mathbb{P}^- Q(X) \xrightarrow{s} X^0\right)}$$

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- $H^-_0(X)$ is the free group generated by the final states of X

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- $H_n^{-}(X)$ contains the non-deterministic branchings of dimension n for $n \geqslant 1$
- $H_0^{-}(X)$ is the free group generated by the final states of X

Theorem. *H_n^{-} sends weak S-homotopy equivalences and T-homotopy equivalences into isomorphisms of groups*

About the branching homology

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- There exists a weak S-homotopy equivalence of flows $f : X \longrightarrow Y$ such that $\mathbb{P}^{-}X$ is homotopy equivalent to $\mathbf{S}^2 \sqcup \{0\}$ and such that $\mathbb{P}^{-}Y$ is homotopy equivalent to $\{0, 1\}$

About the branching homology

- The cofibrant replacement functor necessary for obtaining a dihomotopy invariant
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- The augmentation is necessary for obtaining a dihomotopy invariant

About the branching homology

- The cofibrant replacement functor necessary for obtaining a dihomotopy invariant
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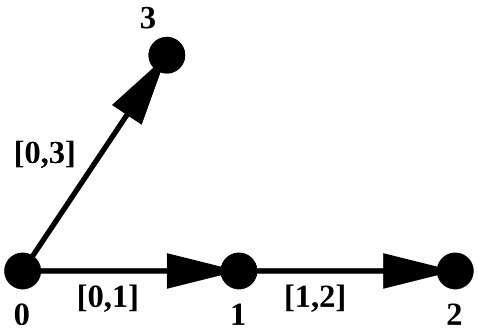
- The augmentation is necessary for obtaining a dihomotopy invariant
- The functor $X \mapsto H_0(\mathbb{P}^{-}Q(X))$ is invariant with respect to weak S-homotopy, but not with respect to T-homotopy equivalences

1-dimensional branching



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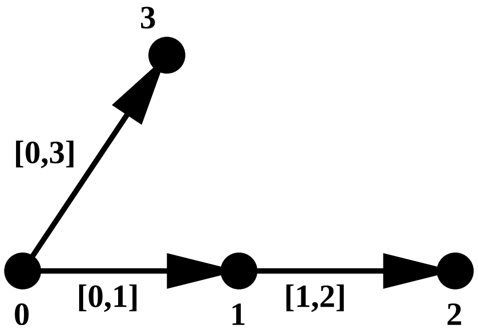

$X^0 = \{0,1,2,3\}$, $\mathbb{P}_{0,1}X = \{[0,1]\}$, $\mathbb{P}_{1,2}X = \{[1,2]\}$,
$\mathbb{P}_{0,3}X = \{[0,3]\}$, $\mathbb{P}_{0,2}X = \{[0,2]\}$ and $\mathbb{P}_{\alpha\beta}X = \varnothing$ otherwise

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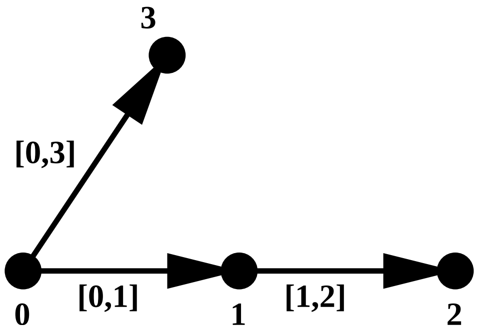

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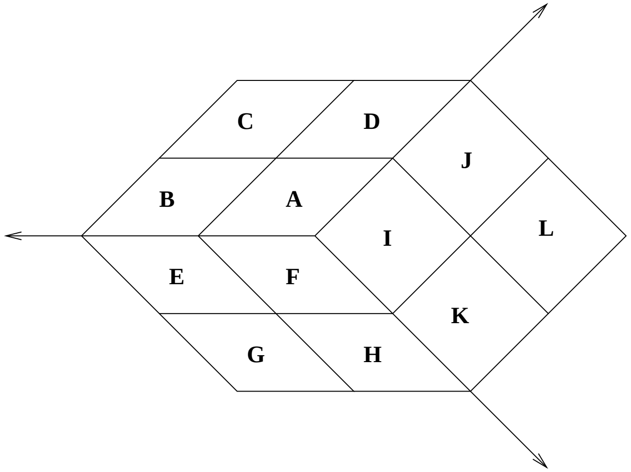
$H^-_n(X) = 0$ for $n \geqslant 2$, $H^-_1(X) = \mathbb{Z}$ (generated by $[0,3] - [0,1]$), and $H^-_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ (generated by the final states 2 and 3)

2-dimensional branching



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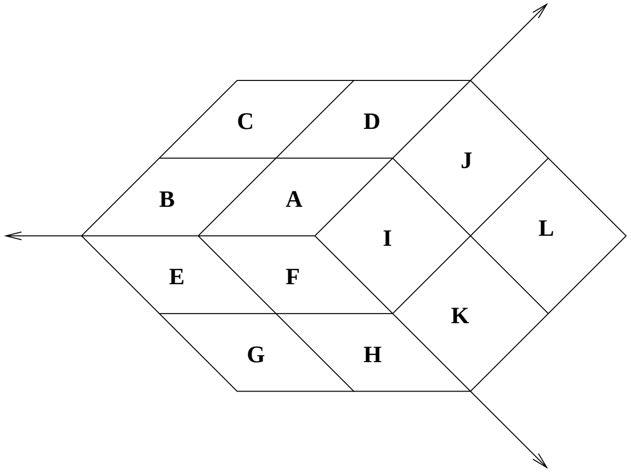

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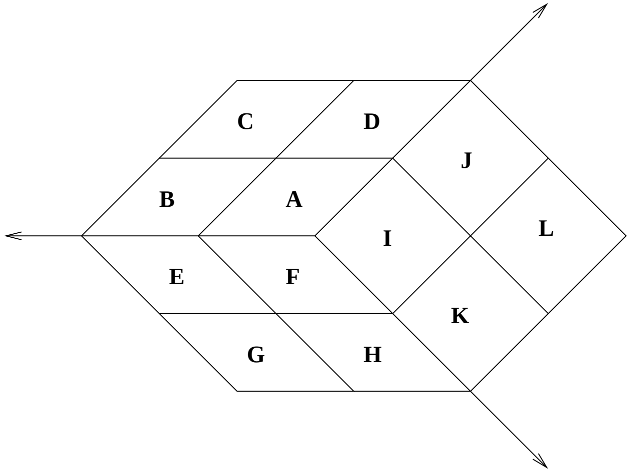

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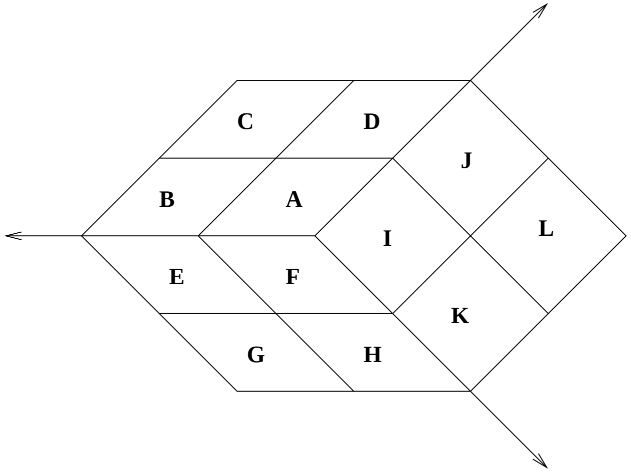

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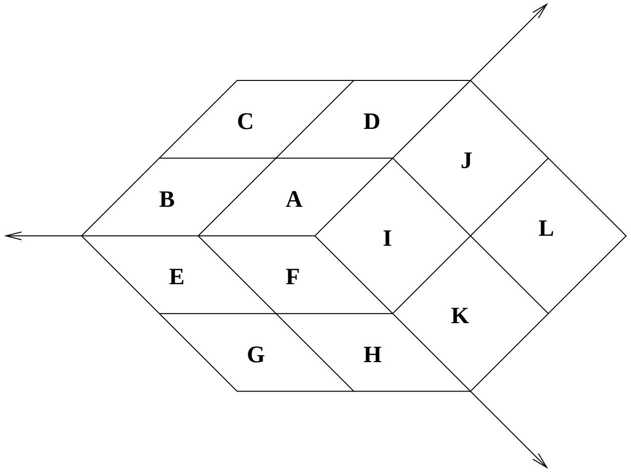

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Invariance of branching homology

The key point : proving that for every finite bounded poset P, the (cofibrant) topological space $\mathbb{P}^{-}_{\widehat{0}}Q(P)$ contractible

$$\begin{tikzcd}
& A \arrow[dr, "\leqslant"] & \\
\widehat{0} \arrow[ur, "\leqslant"] \arrow[dr, "\leqslant"] & & \widehat{1} \\
& B \arrow[ur, "\leqslant"] &
\end{tikzcd}$$

Figure 3: The simplest example of finite bounded posets for which P not cofibrant

Some calculations (I)

$$\begin{tikzcd}
\mathbb{P}_{\widehat{0},A} \times \mathbb{P}_{A,\widehat{1}} \arrow[d, dashed, "pr_1"'] \arrow[dr, hook, "*"] & & \mathbb{P}_{\widehat{0},B} \times \mathbb{P}_{B,\widehat{1}} \arrow[d, dashed, "pr_1"] \arrow[dl, hook', "*"'] \\
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\{0\} \times \{0\} \arrow[d, "pr_1"'] \arrow[dr, "*"] & & \{0\} \times \{0\} \arrow[d, "pr_1"] \arrow[dl, "*"'] \\
\{0\} & \{0\} & \{0\}
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Not Reedy cofibrant: $\mathbb{P}_{\widehat{0},A} \times \mathbb{P}_{A,\widehat{1}} \sqcup \mathbb{P}_{\widehat{0},B} \times \mathbb{P}_{B,\widehat{1}} \longrightarrow \mathbb{P}_{\widehat{0},\widehat{1}}$ not one-to-one.

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Reedy cofibrant:

$\mathbb{P}_{\widehat{0},A} \times \mathbb{P}_{A,\widehat{1}} \sqcup_{\mathbb{P}_{\widehat{0},A} \times \mathbb{P}_{A,B} \times \mathbb{P}_{B,\widehat{1}}} \mathbb{P}_{\widehat{0},B} \times \mathbb{P}_{B,\widehat{1}} \longrightarrow \mathbb{P}_{\widehat{0},\widehat{1}}$ one-to-one.

Merging homology

- Let $n \geqslant -1$

$$\mathbb{P}^+ X := \mathbb{P}X/(y = x * y)$$

$$\boxed{H^+_{n+1}(X) := H_n\left(\operatorname{Sing}\mathbb{P}^+Q(X) \xrightarrow{t} X^0\right)}$$

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Theorem. *H_n^+ sends weak S-homotopy equivalences and T-homotopy equivalences into isomorphisms of groups*

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The underlying homotopy type $|X|$ of a flow X

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Theorem. $X \mapsto |X|$ *from* $\mathbf{Flow}$ *to* $\mathbf{Ho}(\mathbf{Top})$ *sends weak S-homotopy equivalences and T-homotopy equivalences into isomorphisms*

Some calculations (III)

Calculating the underlying homotopy type of the flow X (supposed cofibrant, and U, V and W nonempty)

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\widehat{0} \arrow[r, "U"] \arrow[rr, bend left, "W"] & 2 \arrow[r, "V"] & \widehat{1}
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Homotopy continuous flow

Theorem. *For any model structure (cofibrantly generated or not) on* $\mathbf{Flow}$ *such that* $Q(\{\widehat{0} < \widehat{1}\}) \longrightarrow Q(\{\widehat{0} < 2 < \widehat{1}\})$ *is a weak equivalence, there exists a pushout of* $R : \{0, 1\} \longrightarrow \{0\}$ *which is a weak equivalence*

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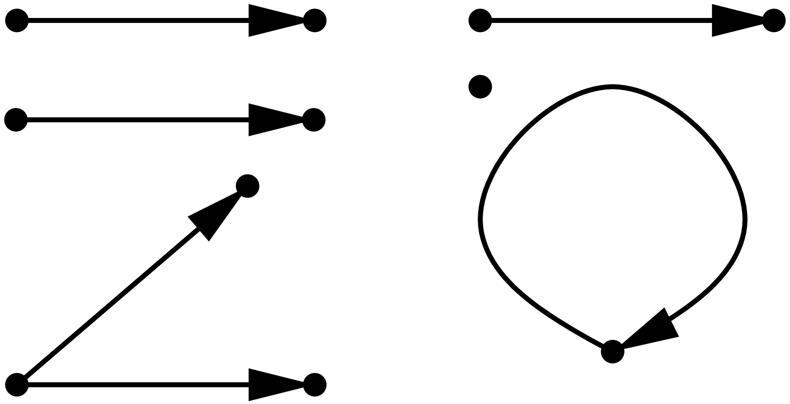

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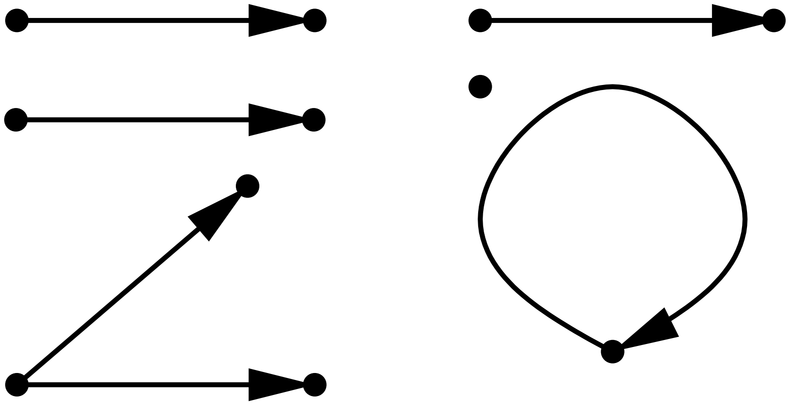

There is anyway a good notion of fibrant objects for the full dihomotopy relation

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X homotopy continuous if and only if $X \longrightarrow \mathbf{1}$ satisfies the right lifting property with respect to the set of generating T-homotopy equivalences

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- *The inclusion functor $\mathbf{Flow}^{\textit{homotopy continuous}}_{\textit{cofibrant}} \subset \mathbf{Flow}$ induces the equivalence of categories*

$$\boxed{\mathbf{Flow}^{\textit{homotopy continuous}}_{\textit{cofibrant}} / \sim_{T} \simeq \mathbf{Flow}[(\mathcal{S}_{\mathcal{T}})^{-1}]}$$

Path functor for dihomotopy

Theorem. *There exists a functor* $\mathrm{Path}_{\mathcal{T}} : \mathbf{Flow} \longrightarrow \mathbf{Flow}$

$$\begin{tikzcd}
X \arrow[r, "\in \mathbf{cof}(J^{gl} \cup \mathcal{T})"'] \arrow[rr, bend left, "(\mathrm{Id}_X{,}\mathrm{Id}_X)"] & \mathrm{Path}_{\mathcal{T}}\, X \arrow[r, two heads, "\in \mathbf{inj}(J^{gl} \cup \mathcal{T})"'] & X \times X
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Congruence $\sim_{\mathcal{T}}$ = transitive closure of right dihomotopy