

Program transformations

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High level features (stores, exceptions, I/O, ...) are essential:

- A program execution has a *raison d'être* only if it has I/O
- Processors have registers not functions
- Databases store persistent data
- Efficiency and clarity can be more easily achieved with exceptions

Question

Why some widespread languages, such as Haskell, insists on *purity*?

Advantages of a pure functional framework

- Much easier static analysis and correctness proofs
- Lazy evaluation
- Program optimizations

Static analysis: some examples

- Dependence analysis:

- control dependencies: the evaluation of a program's expressions depends on the result of a previous expression (eg, if_then_else)
- data dependencies: the result of a program's expressions depends on the result of a previous expression (eg, let-expression)

*Dependence analysis determines whether or not it is safe to **reorder** or **parallelize** the evaluation of expressions.*

- Data-flow analysis:

- reaching definitions: determines which definitions may reach a given point in the code (eg, registers allocation)
- live variable analysis: calculate for each program point the variables that may be potentially read (eg, use for dead-code elimination)

Data-flow analysis gathers information about the possible set of values calculated at various points.

- Type systems

Exercises

- 1 Try to imagine why the presence of impure expressions can make both dependence and data-flow analysis more difficult
- 2 Try to think about the problems of implementing a static type system to ensure that there won't be any uncaught exception.

Check also Appel's book

Lazy evaluation (1)

In lazy (as opposed to strict/eager) evaluation an expression passed as argument:

- is only evaluated if the result is required by the calling function (delayed evaluation)
- is only evaluated to the extent that is required by the calling function, called (short-circuit evaluation).
- is never evaluated more than once (as in applicative-order evaluation)

Example

$$\begin{aligned} & (\lambda x.(\text{fst } x, \text{fst } x))((\lambda y.(3 + y, e))5) \\ & \rightarrow (\text{fst } ((\lambda y.(3 + y, e))5), \text{fst } ((\lambda y.(3 + y, e))5)) \\ & \rightarrow (\text{fst } (3 + 5, e), \text{fst } (3 + 5, e)) \\ & \rightarrow (3 + 5, 3 + 5) \end{aligned}$$

(The last reduction is an optimization: common subexpressions elimination)

Lazy evaluation (2)

In OCaml lazy evaluation can be implemented by *memoization*:

```
# let rec boucle = function 0 -> () | n -> boucle (n-1);;
val boucle : int -> unit = <fun>
# let gros_calcul () = boucle 100000000; 4;;
val gros_calcul : unit -> int = <fun>
# let v = gros_calcul ();;      (* it is slow *)
val v : int = 4
# v + 1;;                      (* it is fast *)
- : int = 5
# let v () = gros_calcul ();;  (* it is fast *)
val v : unit -> int = <fun>
# v () + 1;;                   (* it is slow *)
- : int = 5
# v () + 1;;                   (* it is slow *)
- : int = 5
# let v =
let r = ref None in
  fun () -> match !r with
    | Some v -> v
    | None -> let v = (gros_calcul ()) in r := Some v; v;;
val v : unit -> int = <fun>
# v () + 1;;                   (* it is slow *)
- : int = 5
# v () + 1;;                   (* it is fast *)
- : int = 5
```

The Lazy module in OCaml

This is so frequent that OCaml provides this behavior natively via the special syntax `lazy` and the module `Lazy`:

```
# let v = lazy (gros_calcul ());;  
val v : int lazy_t = <lazy>  
  
# Lazy.force v;;                (* it is slow *)  
- : int = 4  
  
# Lazy.force v;;                (* it is fast *)  
- : int = 4
```

Advantages

- Lazy data structures: possibly infinite, efficient copy, low memory footprint
- Better performance due to avoiding unnecessary calculations (?),
- Maintains purity (!)

Rationale

Since also strict languages can be endowed with laziness (see Lazy library in OCaml) then the clear advantage of *pervasive* lazy evaluation is to keep purity and, thus, referential transparency (not the other way round).

Purity makes important optimizations possible

- 1 Obvious program transformations. In Haskell

$$\text{map } f \text{ (map } g \text{ lst)} = \text{map } (f.g) \text{ lst}$$

What if f and g had side effects?

This is called “deforestation” and works for non-strict languages (in strict languages it may transform a function that does not terminate into one that terminates).

- 2 Function inlining, partial evaluation
- 3 Memoization
- 4 Common subexpressions elimination
- 5 Parallelization
- 6 Speculative evaluation
- 7 Other optimizations (see CPS part later on)

Program transformations

Meaning:

In the broadest sense: all translations between programming languages that preserve the meaning of programs.

Usage:

Typically used as passes in a compiler. Progressively bridge the gap between high-level source languages and machine code.

In this course:

We focus on translations between different languages. Translations within the same language are for optimization and studied in compiler courses.

The interest is twofold:

- 1 Eliminate high-level features of a language and target a smaller or lower-level language.
- 2 To program in languages that lack a desired feature. E.g. use higher-order functions or objects in C; use imperative programming in Haskell or Coq.

Considered transformations

We will show how to get rid of higher level features:

- High-order functions
- “Impure” features: exceptions, state, call/cc

Note

In order to simulate higher level features we first have to formally define their semantics.

Let us take a refresher course on operational semantics and reduction strategies

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Syntax and small-step semantics

Syntax

<i>Terms</i>	$a, b ::= N$	Numeric constant
	x	Variable
	ab	Application
	$\lambda x.a$	Abstraction
<i>Values</i>	$v ::= \lambda x.a \mid N$	

Small step semantics for strict functional languages

Evaluation Contexts $E ::= [] \mid E a \mid v E$

BETA_v
 $(\lambda x.a) v \rightarrow a[x/v]$

CONTEXT
 $\frac{a \rightarrow b}{E[a] \rightarrow E[b]}$

Characteristics of the reduction strategy

Weak reduction: We cannot reduce under λ -abstractions;

Call-by-value: In an application $(\lambda x.a) b$, the argument b must be fully reduced to a value before β -reduction can take place.

Left-most reduction: In an application ab , we must reduce a to a value first before we can start reducing b .

Deterministic: For every term a , there is at most one b such that $a \rightarrow b$.

Big step semantics for strict functional languages

$$N \Rightarrow N \quad \lambda x.a \Rightarrow \lambda x.a \quad \frac{a \Rightarrow \lambda x.c \quad b \Rightarrow v_0 \quad c[x/v_0] \Rightarrow v}{ab \Rightarrow v}$$

The big step semantics induces an efficient implementation

```
type term =
  Const of int | Var of string | Lam of string * term | App of term * term

exception Error

let rec subst x v = function          (* assumes v is closed *)
  | Const n -> Const n
  | Var y -> if x = y then v else Var y
  | Lam(y, b) -> if x = y then Lam(y, b) else Lam(y, subst x v b)
  | App(b, c) -> App(subst x v b, subst x v c)

let rec eval = function
  | Const n -> Const n
  | Var x -> raise Error
  | Lam(x, a) -> Lam(x, a)
  | App(a, b) ->
    match eval a with
    | Lam(x, c) -> let v = eval b in eval (subst x v c)
    | _ -> raise Error
```

Exercises

- 1 Define the small-step and big-step semantics for the call-by-name
- 2 Deduce from the latter the interpreter
- 3 Use the technique introduced for the type 'a delayed earlier in the course to implement an interpreter with lazy evaluation.

Environments

- Implementing textual substitution $a[x/v]$ is inefficient. This is why compilers and interpreters *do not* implement it.
- Alternative: remember the binding $x \mapsto v$ in an *environment* e

$$\frac{e(x) = v}{e \vdash x \Rightarrow v} \qquad e \vdash N \Rightarrow N \qquad e \vdash \lambda x.a \Rightarrow \lambda x.a$$

$$\frac{e \vdash a \Rightarrow \lambda x.c \quad e \vdash b \Rightarrow v_0 \quad e; x \mapsto v_0 \vdash c \Rightarrow v}{e \vdash ab \Rightarrow v}$$

Giving up substitutions in favor of environments does not come for free

- Lexical scoping** requires careful handling of environments

```
let x = 1 in
let f = λy.(x+1) in
let x = "foo" in
f 2
```

In the environment used to evaluate `f 2` the variable `x` is bound to 1.

Function closures

To implement *lexical scoping in the presence of environments*, function abstractions $\lambda x.a$ must not evaluate to themselves, but to a function *closure*: a pair $(\lambda x.a)[e]$ (ie, the function and the *environment of its definition*)

Big step semantics with environments and closures

Values $v ::= N \mid (\lambda x.a)[e]$

Environments $e ::= x_1 \mapsto v_1; \dots; x_n \mapsto v_n$

$$\frac{e(x) = v}{e \vdash x \Rightarrow v} \qquad e \vdash N \Rightarrow N \qquad e \vdash \lambda x.a \Rightarrow (\lambda x.a)[e]$$

$$\frac{e \vdash a \Rightarrow (\lambda x.c)[e_0] \quad e \vdash b \Rightarrow v_0 \quad e_0; x \mapsto v_0 \vdash c \Rightarrow v}{e \vdash ab \Rightarrow v}$$

De Bruijn indexes

Identify variable not by names but by the number \underline{n} of λ 's that separate the variable from its binder in the syntax tree.

$$\lambda x. (\lambda y. y x) x \quad \text{is} \quad \lambda. (\lambda. \underline{0} \underline{1}) \underline{0}$$

\underline{n} is the variable bound by the n -th enclosing λ . Environments become sequences of values, the n -th value of the sequence being the value of variable $\underline{n-1}$.

$$\begin{array}{ll} \text{Terms} & a, b ::= N \mid \underline{n} \mid \lambda. a \mid ab \\ \text{Values} & v ::= N \mid (\lambda. a)[e] \\ \text{Environments} & e ::= v_0; v_1; \dots; v_n \end{array}$$

$$\frac{e = v_0; \dots; v_n; \dots; v_m}{e \vdash \underline{n} \Rightarrow v_n} \qquad e \vdash N \Rightarrow N \qquad e \vdash \lambda. a \Rightarrow (\lambda. a)[e]$$

$$\frac{e \vdash a \Rightarrow (\lambda. c)[e_0] \quad e \vdash b \Rightarrow v_0 \quad v_0; e_0 \vdash c \Rightarrow v}{e \vdash ab \Rightarrow v}$$

The canonical, efficient interpreter

```
# type term = Const of int | Var of int | Lam of term | App of term * term
  and value = Vint of int | Vclos of term * environment
  and environment = value list (* use Vec instead *)

# exception Error

# let rec eval e a =
  match a with
  | Const n -> Vint n
  | Var n -> List.nth e n (* will fail for open terms *)
  | Lam a -> Vclos(Lam a, e)
  | App(a, b) ->
    match eval e a with
    | Vclos(Lam c, e') ->
      let v = eval e b in
      eval (v :: e') c
    | _ -> raise Error

# eval [] (App (Lam (Var 0), Const (2))));; (* ( $\lambda x.x$ )2  $\rightarrow$  2 *)
- : value = Vint 2
```

Note: To obtain improved performance one should implement environments by persistent extensible arrays: for instance by the `Vec` library by Luca de Alfaro.

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Closure conversion

Goal: make explicit the construction of closures and the accesses to the environment part of closures.

Input: a fully-fledged functional programming language, with general functions (possibly having free variables) as first-class values.

Output: the same language where only closed functions (without free variables) are first-class values. Such closed functions can be represented at run-time as code pointers, just as in C for instance.

Idea: every function receives its own closure as an extra argument, from which it recovers values for its free variables. Such functions are closed. Function closures are explicitly represented as a tuple (closed function, values of free variables).

Uses: compilation; functional programming in C, Java, . . .

Definition of closure conversion

$$\llbracket x \rrbracket = x$$

$$\begin{aligned} \llbracket \lambda x. a \rrbracket &= \text{tuple}(\lambda(c, x). \text{let } x_1 = \text{field}_1(c) \text{ in} \\ &\quad \vdots \\ &\quad \text{let } x_n = \text{field}_n(c) \text{ in} \\ &\quad \llbracket a \rrbracket, \\ &\quad x_1, \dots, x_n) \end{aligned}$$

where x_1, \dots, x_n are the free variables of $\lambda x. a$

$$\llbracket a b \rrbracket = \text{let } c = \llbracket a \rrbracket \text{ in } \text{field}_0(c)(c, \llbracket b \rrbracket)$$

The translation extends isomorphically to other constructs, e.g.

$$\begin{aligned} \llbracket \text{let } x = a \text{ in } b \rrbracket &= \text{let } x = \llbracket a \rrbracket \text{ in } \llbracket b \rrbracket \\ \llbracket a + b \rrbracket &= \llbracket a \rrbracket + \llbracket b \rrbracket \end{aligned}$$

Example of closure conversion

Source program in Caml:

```
fun x lst ->
  let rec map f lst =
    match lst with
    [] -> []
    | hd :: tl -> f hd :: map f tl
  in
  map (fun y -> x + y) lst
```

Result of partial closure conversion for the f argument of map:

```
fun x lst ->
  let rec map f lst =
    match lst with
    [] -> []
    | hd :: tl -> field0(f)(f,hd) :: map f tl
  in
  map tuple( $\lambda(c,y). \text{let } x = \text{field}_1(c) \text{ in } x + y,$ 
           x)
         lst
```

Closure conversion for recursive functions

In a recursive function $\mu f.\lambda x.a$, the body a needs access to f , that is, the closure for itself. This closure can be found in the extra function parameter that closure conversion introduces.

$$\begin{aligned} \llbracket \mu f.\lambda x.a \rrbracket &= \text{tuple}(\lambda(f,x).\text{let } x_1 = \text{field}_1(f) \text{ in} \\ &\quad \vdots \\ &\quad \text{let } x_n = \text{field}_n(f) \text{ in} \\ &\quad \llbracket a \rrbracket, \\ &\quad x_1, \dots, x_n) \end{aligned}$$

where x_1, \dots, x_n are the free variables of $\mu f.\lambda x.a$

Notice that f is free in a and thus in $\llbracket a \rrbracket$, but bound in $\llbracket \mu f.\lambda x.a \rrbracket$.

In other terms, regular functions $\lambda x.a$ are converted exactly like pseudo-recursive functions $\mu c.\lambda x.a$ where c is a variable not free in a .

If the target of the conversion is an object-oriented language in the style of Java, C#, we can use the following variant of closure conversion:

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda x. a \rrbracket = \text{new } C_{\lambda x. a}(x_1, \dots, x_n)$$

where x_1, \dots, x_n are the free variables of $\lambda x. a$

$$\llbracket ab \rrbracket = \llbracket a \rrbracket.\text{apply}(\llbracket b \rrbracket)$$

Closure conversion in object-oriented style

The class $C_{\lambda x.a}$ (one for each λ -abstraction in the source) is defined (in C#) as follows:

```
public class C $\lambda x.a$  {
    protected internal Object x1, ..., xn;
    public C $\lambda x.a$ (Object x1 , ... , Object xn ) {
        this.x1 = x1 ; ...; this.xn = xn ;
    }
    public Object apply(Object x) {
        return  $\llbracket a \rrbracket$  ;
    }
}
```

Typing

In order to have a more precise typing the static types of the variables and of the function should be used instead of `Object`. In particular the method `apply` should be given the same input and return types as the encoded function.

In more general terms:

- Closure \approx Object with a single apply method
- Object \approx Closure with multiple entry points

Both function application and method invocation compile down to self application:

$$\begin{aligned} \llbracket fun\ arg \rrbracket &= \text{let } c = \llbracket fun \rrbracket \text{ in } \text{field}_0(c)(c, \llbracket arg \rrbracket) \\ \llbracket obj.meth(arg) \rrbracket &= \text{let } o = \llbracket obj \rrbracket \text{ in } o.meth(o, \llbracket arg \rrbracket) \end{aligned}$$

Where an object is interpreted as a record whose fields are methods which are parametrized by `self`.

First class closure

Modern OOL such as Scala and C# (and announced for Java in the JDK 7 but currently deferred to JDK 8 or later) provide syntax to define closures, without the need to encode them.

For instance C# provides a `delegate` modifier to define closures:

```
public delegate int DComparer (Object x, Object y)
```

Defines a new distinguished type `DComparer` whose instances are functions from two objects to `int` (i.e., $DComparer \equiv (Object * Object) \rightarrow int$)

Instances are created by passing to `new` a static or instance method (with compatible types):

```
DComparer mycomp = new DComparer(String.Comparer)
```

The closure `mycomp` can be passed around (wherever an argument of type `DComparer` is expected), or applied as in `mycomp("Scala", "Java")`

First class closure

Actually in C# it is possible to define “lambda expressions”:

Here how to write $(\lambda(x, y).x + y)$ in C#:

```
(x,y) => x + y
```

Lambda expressions can be used to instantiate closures:

```
DComparer myComp = (x,y) => x + y
```

Delegates (roughly, function types) can be polymorphic:

```
public delegate TResult Func<TArg0, TResult>(TArg0 arg0)
```

The delegate can be instantiated as `Func<int, bool> myFunc` where `int` is an input parameter and `bool` is the return value. The return value is always specified in the last type parameter. `Func<int, string, bool>` defines a delegate with two input parameters, `int` and `string`, and a return type of `bool`.

```
Func<int, bool> myFunc = x => x == 5;  
bool result = myFunc(4);           // returns false of course
```

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Goal: like closure conversion, make explicit the construction of closures and the accesses to the environment part of closures. Unlike closure conversion, do not use closed functions as first-class values.

Input: a fully-fledged functional programming language, with general functions (possibly having free variables) as first-class values.

Output: any first-order language (no functions as values). Idea: represent each function value $\lambda x.a$ as a data structure $C(v_1, \dots, v_n)$ where the constructor C uniquely identifies the function, and the constructor arguments v_1, \dots, v_n are the values of the variables x_1, \dots, x_n free in the body of the function.

Uses: functional programming in Pascal, Ada, Basic, . . .

Definition of defunctionalization

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda x. a \rrbracket = C_{\lambda x. a}(x_1, \dots, x_n)$$

where x_1, \dots, x_n are the free variables of $\lambda x. a$

$$\llbracket \mu f. \lambda x. a \rrbracket = C_{\mu f. \lambda x. a}(x_1, \dots, x_n)$$

where x_1, \dots, x_n are the free variables of $\mu f. \lambda x. a$

$$\llbracket ab \rrbracket = \text{apply}(\llbracket a \rrbracket, \llbracket b \rrbracket)$$

The difference between recursive and non-recursive functions is made in the definition of `apply`

(Other constructs: isomorphically.)

Definition of defunctionalization

The apply function collects the bodies of all functions and dispatches on its first argument. There is one case per function occurring in the source program.

```
let rec apply(fun, arg) =  
  match fun with  
  |  $C_{\lambda x.a}(x_1, \dots, x_n)$  -> let x = arg in  $\llbracket a \rrbracket$   
  |  $C_{\mu f.\lambda y.b}(x_1, \dots, x_m)$  -> let f = fun in let y = arg in  $\llbracket b \rrbracket$   
  | ...  
in  $\llbracket \text{program} \rrbracket$ 
```

Note

Unlike closure conversion, this is a whole-program transformation.

Defunctionalization of $(\lambda x.\lambda y.x) 1 2$:

```
let rec apply (fun, arg) =  
  match fun with  
  | C1()  -> let x = arg in C2(x)  
  | C2(x) -> let y = arg in x  
in  
  apply(apply(C1(), 1), 2)
```

We write C1 for $C_{\lambda x.\lambda y.x}$ and C2 for $C_{\lambda y.x}$.

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Small step semantics for exceptions

$$\begin{aligned}(\text{try } v \text{ with } x \rightarrow b) &\rightarrow v \\(\text{try raise } v \text{ with } x \rightarrow b) &\rightarrow b[x/v] \\P[\text{raise } v] &\rightarrow \text{raise } v \quad \text{if } P \neq [] \\ \frac{a \rightarrow b}{E[a] \rightarrow E[b]} &\end{aligned}$$

Exception propagation contexts P are like reduction contexts E but do not allow skipping past a try ... with

Reduction contexts:

$$E ::= [] \mid E a \mid v E \mid \text{raise } E \mid \text{try } E \text{ with } x \rightarrow a \mid \dots$$

Exception propagation contexts: (no try_with)

$$P ::= [] \mid P a \mid v P \mid \text{raise } P \mid \dots$$

Reduction semantics for exceptions

Assume the current program is $p = E[\text{raise } v]$, that is, we are about to raise an exception. If there is a `try ... with` that encloses the raise, the program will be decomposed as

$$p = E'[\text{try } P[\text{raise } v] \text{ with } x \rightarrow b]$$

where P contains no `try ... with` constructs. $P[\text{raise } v]$ head-reduces to `raise v`, and $E'[\text{try } [] \text{ with } x \rightarrow b]$ is an evaluation context. The reduction sequence is therefore:

$$\begin{aligned} p = E'[\text{try } P[\text{raise } v] \text{ with } x \rightarrow b] &\rightarrow E'[\text{try } \text{raise } v \text{ with } x \rightarrow b] \\ &\rightarrow E'[b[x/v]] \end{aligned}$$

If there are no `try ... with` around the raise, E is a propagation context and the reduction is therefore

$$p = E[\text{raise } v] \rightarrow \text{raise } v$$

When considering reduction sequences, a fourth possible outcome of evaluation appears: termination on an uncaught exception.

- Termination: $a \rightarrow^* v$
- **Uncaught exception**: $a \rightarrow \text{raise } v$
- Divergence: $a \rightarrow^* a' \rightarrow \dots$
- Error: $a \rightarrow a' \not\rightarrow$ where $a \neq v$ and $a \neq \text{raise } v$.

Big step semantics for exception

In big step semantics, the evaluation relation becomes $a \Rightarrow r$ where evaluation *results* are $r ::= v \mid \text{raise } v$. Add the following rules for try...with:

$$\frac{a \Rightarrow v}{\text{try } a \text{ with } x \rightarrow b \Rightarrow v}$$

$$\frac{a \Rightarrow \text{raise } v \quad b[x/v] \Rightarrow r}{\text{try } a \text{ with } x \rightarrow b \Rightarrow r}$$

as well as exception propagation rules such as:

$$\frac{a \Rightarrow \text{raise } v}{ab \Rightarrow \text{raise } v}$$

$$\frac{a \Rightarrow v' \quad b \Rightarrow \text{raise } v}{ab \Rightarrow \text{raise } v}$$

Conversion to exception-returning style

Goal: get rid of exceptions.

Input: a functional language featuring exceptions (`raise` and `try...with`).

Output: a functional language with pattern-matching but no exceptions.

Idea: every expression a evaluates to either $Val(v)$ if a evaluates normally, or to $Exn(v)$ if a terminates early by raising exception v . Val, Exn are datatype constructors.

Uses: giving semantics to exceptions; programming with exceptions in Haskell; reasoning about exceptions in theorem provers.

Definition of the transformation

$$\begin{aligned} \llbracket \text{raise } a \rrbracket &= \text{match } \llbracket a \rrbracket \text{ with} \\ &\quad | \text{Exn}(x) \rightarrow \text{Exn}(x) \\ &\quad | \text{Val}(x) \rightarrow \text{Exn}(x) \\ \llbracket \text{try } a \text{ with } x \rightarrow b \rrbracket &= \text{match } \llbracket a \rrbracket \text{ with} \\ &\quad | \text{Exn}(x) \rightarrow \llbracket b \rrbracket \\ &\quad | \text{Val}(x) \rightarrow \text{Val}(x) \end{aligned}$$

Definition of the transformation

$$\llbracket N \rrbracket = \text{Val}(N)$$

$$\llbracket x \rrbracket = \text{Val}(x)$$

$$\llbracket \lambda x. a \rrbracket = \text{Val}(\lambda x. \llbracket a \rrbracket)$$

$$\llbracket \text{let } x = a \text{ in } b \rrbracket = \text{match } \llbracket a \rrbracket \text{ with } \text{Exn}(x) \rightarrow \text{Exn}(x) \mid \text{Val}(x) \rightarrow \llbracket b \rrbracket$$

$$\begin{aligned} \llbracket ab \rrbracket &= \text{match } \llbracket a \rrbracket \text{ with} \\ &\quad \mid \text{Exn}(x) \rightarrow \text{Exn}(x) \\ &\quad \mid \text{Val}(x) \rightarrow \text{match } \llbracket b \rrbracket \text{ with} \\ &\quad \quad \mid \text{Exn}(y) \rightarrow \text{Exn}(y) \\ &\quad \quad \mid \text{Val}(y) \rightarrow xy \end{aligned}$$

Effect on types: if $a : \tau$ then $\llbracket a \rrbracket : \llbracket \tau \rrbracket$ where $\llbracket \tau_1 \rightarrow \tau_2 \rrbracket = (\tau_1 \rightarrow \llbracket \tau_2 \rrbracket)$ outcome and $\llbracket \tau \rrbracket = \tau$ outcome otherwise and where type 'a outcome = Val of 'a
| Exn of exn.

Example of conversion

Let fun and arg be two variables, then:

```
[[ try fun arg with w -> 0 ]] =  
  match  
    match Val(fun) with  
    | Exn(x) -> Exn(x)  
    | Val(x) ->  
      match Val(arg) with  
      | Exn(y) -> Exn(y)  
      | Val(y) -> x y  
  with  
  | Val(z) -> Val(z)  
  | Exn(w) -> Val(0)
```

Notice that the two inner `match` can be simplified yielding

```
[[ try fun arg with w -> 0 ]] =  
  match fun arg with  
  | Val(z) -> Val(z)  
  | Exn(w) -> Val(0)
```

This transformation can be generalized by defining *administrative reductions*.

Administrative reductions

The naive conversion generates many useless match constructs over arguments whose shape $Val(\dots)$ or $Exn(\dots)$ is known at compile-time.

These can be eliminated by performing administrative reductions \rightarrow at compile-time, just after the conversion:

Administrative reduction

$$\begin{aligned}(\text{match } Exn(v) \text{ with } Exn(x) \rightarrow b \mid Val(x) \rightarrow c) &\xrightarrow{\text{adm}} b[x/v] \\(\text{match } Val(v) \text{ with } Exn(x) \rightarrow b \mid Val(x) \rightarrow c) &\xrightarrow{\text{adm}} c[x/v]\end{aligned}$$

Correctness of the conversion

Define the conversion of a value $\mathcal{V}(v)$ as $\mathcal{V}(N) = N$ and $\mathcal{V}(\lambda x.a) = \lambda x. \llbracket a \rrbracket$.

Theorem

1. If $a \Rightarrow v$, then $\llbracket a \rrbracket \Rightarrow Val(\mathcal{V}(v))$.
2. If $a \Rightarrow \text{raise } v$, then $\llbracket a \rrbracket \Rightarrow Exn(\mathcal{V}(v))$.
3. If $a \uparrow$, then $\llbracket a \rrbracket \uparrow$.

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State (imperative programming)

The word *state* in programming language theory refers to the distinguishing feature of imperative programming: the ability to assign (change the value of) variables after their definition, and to modify data structures in place after their construction.

A simple yet adequate way to model state is to introduce references: indirection cells / one-element arrays that can be modified in place. The basic operations over references are:

ref a

Create a new reference containing initially the value of *a*.

deref a also written *!a*

Return the current contents of reference *a*.

assign a b also written *a := b*

Replace the contents of reference *a* with the value of *b*. Subsequent *deref a* operations will return this value.

Semantics of references

Semantics based on substitutions fail to account for sharing between references:

```
let r = ref 1 in r := 2; !r ↦ (ref 1) := 2; !(ref 1)
```

Left: the same reference r is shared between assignment and reading; result is 2.

Right: two distinct references are created, one is assigned, the other read; result is 1.

To account for sharing, we must use an additional level of indirection:

- *ref a* expressions evaluate to locations ℓ : a new kind of variable identifying references uniquely. (Locations ℓ are values.)
- A global environment called the store associates values to references.

Reduction semantics for references

The one-step reduction relation becomes $a \triangleleft s \rightarrow a' \triangleleft s'$
(read: in initial store s , a reduces to a' and updates the store to s')

$$(\lambda x. a) v \triangleleft s \rightarrow a[x/v] \triangleleft s$$

$$\text{ref } v \triangleleft s \rightarrow l \triangleleft (s + l \mapsto v) \quad \text{where } l \notin \text{Dom}(s)$$

$$\text{deref } l \triangleleft s \rightarrow s(l) \triangleleft s$$

$$\text{assign } l \ v \triangleleft s \rightarrow () \triangleleft (s + l \mapsto v)$$

CONTEXT

$$\frac{a \triangleleft s \rightarrow a' \triangleleft s'}{E(a) \triangleleft s \rightarrow E(a') \triangleleft s'}$$

Notice that we also added a new value, $()$, the result of a side-effect.

Example of reduction sequence

Let us reduce the following term

$$\text{let } r = \text{ref } 3 \text{ in } r := !r + 1; !r$$

that is

$$\text{let } r = \text{ref } 3 \text{ in let } x = r := !r + 1 \text{ in } !r$$

(recall that $e_1; e_2$ is syntactic sugar for $\text{let } _ = e_1 \text{ in } e_2$)

In red: the active redex at every step.

$$\begin{aligned} & \text{let } r = \text{ref } 3 \text{ in let } x = r := !r + 1 \text{ in } !r \triangleleft \emptyset \\ & \rightarrow \text{let } r = l \text{ in let } x = r := !r + 1 \text{ in } !r \triangleleft l \mapsto 3 \\ & \rightarrow \text{let } x = l := !l + 1 \text{ in } !l \triangleleft l \mapsto 3 \\ & \rightarrow \text{let } x = l := 3 + 1 \text{ in } !l \triangleleft l \mapsto 3 \\ & \rightarrow \text{let } x = l := 4 \text{ in } !l \triangleleft l \mapsto 3 \\ & \rightarrow \text{let } x = () \text{ in } !l \triangleleft l \mapsto 4 \\ & \rightarrow !l \triangleleft l \mapsto 4 \\ & \rightarrow 4 \end{aligned}$$

Conversion to state-passing style

Goal: get rid of state.

Input: a functional language featuring references.

Output: a pure functional language.

Idea: every expression a becomes a function that takes a run-time representation of the current store and returns a pair (result value, updated store).

Uses: give semantics to references; program imperatively in Haskell; reason over imperative code in theorem provers.

Definition of the conversion

Core constructs

$$\llbracket N \rrbracket = \lambda s.(N, s)$$

$$\llbracket x \rrbracket = \lambda s.(x, s)$$

$$\llbracket \lambda x.a \rrbracket = \lambda s.(\lambda x.\llbracket a \rrbracket, s)$$

$$\llbracket \text{let } x = a \text{ in } b \rrbracket = \lambda s.\text{match } \llbracket a \rrbracket s \text{ with } (x, s') \rightarrow \llbracket b \rrbracket s'$$

$$\begin{aligned} \llbracket ab \rrbracket &= \lambda s.\text{match } \llbracket a \rrbracket s \text{ with } (x_a, s') \rightarrow \\ &\quad \text{match } \llbracket b \rrbracket s' \text{ with } (x_b, s'') \rightarrow x_a x_b s'' \end{aligned}$$

Constructs specific to references

$$\begin{aligned} \llbracket \text{ref } a \rrbracket &= \lambda s. \text{match } \llbracket a \rrbracket s \text{ with } (x, s') \rightarrow \text{store_alloc } x \ s' \\ \llbracket !a \rrbracket &= \lambda s. \text{match } \llbracket a \rrbracket s \text{ with } (x, s') \rightarrow (\text{store_read } x \ s', s') \\ \llbracket a := b \rrbracket &= \lambda s. \text{match } \llbracket a \rrbracket s \text{ with } (x_a, s') \rightarrow \\ &\quad \text{match } \llbracket b \rrbracket s' \text{ with } (x_b, s'') \rightarrow (\epsilon, \text{store_write } x_a \ x_b \ s'') \end{aligned}$$

The operations `store_alloc`, `store_read` and `store_write` provide a concrete implementation of the store. Any implementation of the data structure known as persistent extensible arrays will do.

For instance we can use `Vec`, a library of extensible functional arrays by Luca de Alfaro. In that case we have that locations are natural numbers, a store is a vector `s` created by `Vec.empty`, a fresh location for the store `s` is returned by `Vec.length s`. Precisely, we have

```
store_alloc v s = (Vec.length s, Vec.append v s)
store_read  l s = Vec.get l s
store_write l v s = Vec.set l v s
```

Typing (assuming all values stored in references are of the same type `sval`):

```
store_alloc : sval → store → location × store
store_read  : location → store → sval
store_write : location → sval → store → store
where location is int and store is Vec.t.
```

Example of conversion

Administrative reductions: (where x, y, s , and s' are variables)

$$\begin{aligned}(\text{match } (a, s) \text{ with } (x, s') \rightarrow b) &\xrightarrow{\text{adm}} \text{let } x = a \text{ in } b[s'/s] \\ (\lambda s. b) s' &\xrightarrow{\text{adm}} b[s'/s] \\ \text{let } x = v \text{ in } b &\xrightarrow{\text{adm}} b[x/v] \\ \text{let } x = y \text{ in } b &\xrightarrow{\text{adm}} b[x/y]\end{aligned}$$

(the first reduction replaces only the store since replacing also a for x may change de evaluation order: a must be evaluated before the evaluation of b)

Example of translation after administrative reductions:

Consider again the term

$$\text{let } r = \text{ref } 3 \text{ in } r := !r + 1; !r$$

We have

$$\llbracket \text{let } r = \text{ref } 3 \text{ in let } x = r := !r + 1 \text{ in } !r \rrbracket =$$

```
λs. match store_alloc 3 s with (r, s1) ->
  let t = store_read r s1 in
  let u = t + 1 in
  match (ε, store_write r u s1) with (x, s2) -> (store_read r s2, s2)
```

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Notion of continuation

Given a program p and a subexpression a of p , the *continuation* of a is the computation that remains to be done once a is evaluated to obtain the result of p .

It can be viewed as a function: (value of a) \mapsto (value of p).

Example

Consider the program $p = (1 + 2) * (3 + 4)$.

The continuation of $a = (1 + 2)$ is $\lambda x. x * (3 + 4)$.

The continuation of $a' = (3 + 4)$ is $\lambda x. 3 * x$.

(Remember that $1 + 2$ has already been evaluated to 3.)

The continuation of the whole program p is of course $\lambda x. x$

Continuations and reduction contexts

Continuations closely correspond with reduction contexts in small-step operational semantics:

Nota Bene

If $E[a]$ is a reduct of ρ , then the continuation of a is $\lambda x.E[x]$.

Example

Consider again $\rho = (1 + 2) * (3 + 4)$.

$$\begin{aligned}(1 + 2) * (3 + 4) &= E_1[1 + 2] \text{ with } E_1 = [] * (3 + 4) \\ \rightarrow 3 * (3 + 4) &= E_2[3 + 4] \text{ with } E_2 = 3 * [] \\ \rightarrow \quad 3 * 7 &= E_3[3 * 7] \text{ with } E_3 = [] \\ \rightarrow \quad \quad 21 &\end{aligned}$$

The continuation of $1 + 2$ is $\lambda x.E_1[x] = \lambda x.x * (3 + 4)$.

The continuation of $3 + 4$ is $\lambda x.E_2[x] = \lambda x.3 * x$.

The continuation of $3 * 7$ is $\lambda x.E_3[x] = \lambda x.x$.

What continuations are for?

Historically continuations were introduced to define a denotational semantics for the `goto` statement in imperative programming

- Imagine we have a pure imperative programming language.
- As suggested by the state passing translation a program p of this language can be interpreted as a function that transforms states into states:

$$\llbracket p \rrbracket : \mathcal{S} \rightarrow \mathcal{S}$$

- This works as long as we do not have `GOTO`.

- Consider the following spaghetti code in BASIC

```
10 i = 0
20 i = i + 1
30 PRINT i; " squared = "; i * i
40 IF i >= 10 THEN GOTO 60
50 GOTO 20
60 PRINT "Program Completed."
70 END
```

- **Idea:** add to the interpretation of programs a further parameter: a continuation.
- In this framework a continuation is a function of type $\mathcal{S} \rightarrow \mathcal{S}$ since it takes the result of a statement (i.e. a state) and returns a new result (new state).

$$\llbracket p \rrbracket : \mathcal{S} \rightarrow (\mathcal{S} \rightarrow \mathcal{S}) \rightarrow \mathcal{S}$$

- Every (interpretation of a) statement will do their usual modifications on the state they received and then will pass the resulting state to the continuations they received
- Only the GOTO behaves differently: it throws away the received continuation and use instead the continuation of the statement to go to.
- For instance the statement in line 50 will receive a state and a continuation and will pass the received state to the continuation of the instruction 20.

Continuations for compiler optimizations

Explicit continuations are inserted by some compiler for optimization:

```
(*          defines the product of all prime numbers <= n          *)

let rec prodprime n =                                          (* bear with this          *)
  if n = 1                                                    (* horrible indentation *)
  then
    1
  else if
    isprime n                                                 (* receives k returns b *)
  then n * prodprime (n-1)                                     (* receives j returns p *)
  else prodprime (n-1);;                                       (* receives h returns q *)
```

The compiler adds points to control the flow of this function

- 1 `isprime` is given a return address `k` and returns a boolean `b`
- 2 The first `prodprime` call will return at point `j` an integer `p`
- 3 The second `prodprime` call will return at point `h` an integer `q`

Continuations for compiler optimizations

```
let rec prodprime(n,c) =
  if n = 1
  then
    c 1                                (* pass 1 to the current continuation c *)
  else
    let k b =                            (* continuation of isprime *)
      if b
      then
        let j p =                        (* continuation of prodprime *)
          let a = n * p in c a in
          let m = n - 1
          in prodprime(m,j) (*call prodprime(n-1) with its continuation*)
        else
          let h q =                      (* continuation of prodprime *)
            c q in
            let i = n - 1
            in prodprime(i,h) (*call prodprime(n-1) with its continuation*)
          in isprime(n,k) (* call isprime(n) with its continuation k *)
```

Notice that we added variables *m* and *i* to store intermediate results

Explicit continuations bring several advantages:

- **Tail recursion:** `prodprime` is now tail recursive. Also the call that was already call recursive has trivial continuation (`h` is equivalent to `c`) that can be simplified:

```
let h q =  
  c q in  
  let i = n - 1  
  in prodprime i h
```

 \Rightarrow

```
let i = n - 1  
in prodprime i c
```

- **Inlining:** In languages that are strict and/or have side effects inlining is very difficult to do directly. Explicit continuations overcome all the problems since *all actual parameters to functions are either variables or constants* (never a non-trivial sub-expression)
- **Dataflow analysis** describes static propagation of values. Continuation make this flow explicit and easy this analysis (for detection of dead-code or register allocation).

The Scheme language offers a primitive `callcc` (call with current continuation) that enables a subexpression a of the program to capture its continuation (as a function ‘value of a ’ \mapsto ‘value of the program’) and manipulate this continuation as a first-class value.

The expression `callcc($\lambda k.a$)` evaluates as follows:

- The continuation of this expression is passed as argument to $\lambda k.a$.
- Evaluation of a proceeds; its value is the value of `callcc($\lambda k.a$)`.
- If, during the evaluation of a *or later* (if we stored k somewhere or we passed it along), we evaluate `throw k v` , evaluation continues as if `callcc($\lambda k.a$)` returned v .

That is, the continuation of the `callcc` expression is reinstalled and restarted with v as the result provided by this expression.

Using first-class continuations

Libraries for lists, sets, and other collection data types often provide an imperative iterator `iter`, e.g.

```
(* list_iter: ('a -> unit) -> 'a list -> unit *)
```

```
let rec list_iter f l =  
  match l with  
  | [] -> ()  
  | head :: tail -> f head; list_iter f tail
```

Using first-class continuations

Using first-class continuations, an existing imperative iterator can be turned into a function that returns the first element of a collection satisfying a given predicate `pred` (of type `'a -> bool`).

```
let find pred lst =  
  callcc (λk.  
    list_iter  
      (λx. if pred x then throw k (Some x) else ())  
      lst;  
    None)
```

If an element `x` is found such that `pred x = true`, then the `throw` causes `Some x` to be returned immediately as the result of `find pred lst`. If no such element exists, `list_iter` terminates normally, and `None` is returned.

Using first-class continuations

The previous example can also be implemented with exceptions. However, `callcc` adds the ability to *backtrack* the search.

```
let find pred lst =
  callcc (λk.
    list_iter
      (λx. if pred x
           then callcc (λk'. throw k (Some(x, k')))
           else ())
    lst;
  None)
```

When `x` is found such that `pred x = true`, the function `find` returns not only `x` but also a continuation `k'` which, when thrown, will cause backtracking: the search in `lst` restarts at the element following `x`. This is used as shown in the next function.

Using first-class continuations

The following use of `find` will print all list elements satisfying the predicate:

```
let printall pred lst =  
  match find pred list with  
  | None -> ()  
  | Some(x, k) -> print_string x; throw k ()
```

The `throw k ()` restarts `find pred list` where it left the last time.

`callcc` and other control operators are difficult to use directly (“the `goto` of functional languages”), but in combination with references, can implement a variety of interesting control structures:

- Exceptions (seen)
- Backtracking (seen)
- Generators for imperative iterators such as Python’s and C# `yield` (next slides).
- Coroutines / cooperative multithreading (few slides ahead).
- Checkpoint/replay debugging (in order to save the intermediate state —*ie*, a checkpoint— of a process you can save the continuation).

Python's yield

yield inside a function makes the function a *generator* that when called returns an object of type *generator*. The object has a method `next` that executes the function till the expression `yield`, returns the value of the `yield`, and at the next call of `next`, starts again right after the `yield`.

```
>>> def gen_fibonacci():                # Generator of Fibonacci suite
...     a, b = 1, 2
...     while True:
...         yield a
...         a, b = b, a + b
...
>>> fib = gen_fibonacci()
>>> for i in range(4):
...     print fib.next()
...
1
2
3
5
>>> fib.next()
8
>>> fib.next()
13
```

Python's yield

Actually the argument of a `for` loop is a generator object.

At each loop the `for` calls the `next` method of the generator. When the generator do not find a next yield and exits, then it raises a exception that makes the `for` exit.

```
>>> for i in fib:
...     print i
...
21
34
55
89
144
233
377
610
987
:
:
:
```

Simulate yield by callcc

```
let return = ref (Obj.magic None);;
let resume = ref (Obj.magic None);;
let fib () = callcc (fun kk -> return := kk;
  let a,b = ref 1, ref 2 in
    while true do
      callcc (fun cc -> resume := cc; throw !return !a);
      b := !a + !b;          (* note: a,b ← b,a+b *)
      a := !b - !a;
    done; 0
)
val fib : unit -> int = <fun>
```

- 1 Use two references to store addresses to resume `fib` and return from it;
- 2 Save the return point in `return`
- 3 Save the resumption point in `resume`
- 4 Exit `fib()` by “going to” `return` and returning the value of `!a`
- 5 Adjust the types (the function must return an `int`)
- 6 Use `callcc(fun k -> return:=k; throw !resume ())` to resume

Example

```
# #load "callcc.cma";;
# open Callcc;;
# let return = ref (Obj.magic None);;
val return : '_a ref = contents = <poly>
# let resume = ref (Obj.magic None);;
val resume : '_a ref = contents = <poly>
# let fib() = callcc (fun kk -> return := kk;
  let a,b = ref 1, ref 2 in
    while true do
      callcc(fun cc -> (resume := cc; (throw !return !a)));
      b := !a + !b;
      a := !b - !a;
    done; 0)
;;
val fib : unit -> int = <fun>
# fib();;
- : int = 1
# callcc (fun k -> return:=k; throw !resume ());;
- : int = 2
# callcc (fun k -> return:=k; throw !resume ());;
- : int = 3
# callcc (fun k -> return:=k; throw !resume ());;
- : int = 5
# callcc (fun k -> return:=k; throw !resume ());;
- : int = 8
# callcc (fun k -> return:=k; throw !resume ());;
- : int = 13
```

Exercise

Rewrite the previous program without the `Object.magic` so that the references contain values of type `'a Callcc.cont option` (verbose)

```

# #load "callcc.cma";;
# open Callcc;;
# let return = ref None;;
val return : '_a option ref = contents = None
# let resume = ref None;;
val resume : '_a option ref = contents = None

# let fib() = callcc (fun kk -> return := (Some kk);
  let a,b = ref 1, ref 2 in
  while true do
    callcc(fun cc -> (
      resume := (Some cc);
      let Some k = !return in (throw k !a)));
    b := !a + !b;
    a := !b - !a;
  done; 0);;

Warning 8: this pattern-matching is not exhaustive.
Here is an example of a value that is not matched:
None
val fib : unit -> int = <fun>

# fib();;
- : int = 1
# callcc (fun k -> return:= Some k; let Some k = !resume in throw k ());;
Warning 8: this pattern-matching is not exhaustive.
Here is an example of a value that is not matched:
None
- : int = 2
# callcc (fun k -> return:= Some k; let Some k = !resume in throw k ());;

```

Loop and tail-recursion can be encoded by `callcc`

```
let fib () = callcc (fun kk ->
  return := kk;
  let a,b = ref 1, ref 2 in
  callcc(fun cc -> resume := cc);
  b := !a + !b;
  a := !b - !a;
  throw !return !a)
```

So for instance we can avoid to call multiple times the `throw ...` just do not modify the return address

```
# let x = fib () in
  if x < 100 then (
    print_int x; print_newline();
    throw !resume ())
  else ();;
```

```
1
2
3
5
8
13
21
```

Let us do it in a more functional way by using variables for a and b

```
# let resume = ref (Obj.magic None);;
val resume : '_a ref = contents = <poly>
# let fib () = callcc (fun kk ->
  let a,b = callcc(fun cc -> resume := cc ; (1,1) ) in
  throw kk (b,a+b) );;
val fib : unit -> int * int = <fun>
# let x,y = fib () in
  if x < 100 then (
    print_int x; print_newline();
    throw !resume (x,y))
  else ();;

1
2
3
5
8
13
21
34
55
89
- : unit = ()
```

Exercise

Modify `fib()` so as it does not need the reference `resume` for the continuation.

Coroutines

Coroutines are more generic than subroutines.

Subroutines can return only once; coroutines can return (yield) several times. Next time the coroutine is called, the execution just after the yield call.

An example in pseudo-code

```
var q := new queue

coroutine produce
  loop
    while q is not full
      create some new items
      add the items to q
      yield to consume

coroutine consume
  loop
    while q is not empty
      remove some items from q
      use the items
      yield to produce
```

Implementing coroutines with continuations

```
coroutine process1 n =
  loop
    print "1: received "; print_ln n
    yield n+1 to process2
coroutine process2 n =
  loop
    print "2: received "; print_ln n
    yield n+1 to process1
in process1 0
```

In OCaml with callcc

```
callcc (fun init_k ->
  let curr_k = ref init_k in
  let communicate x =
    callcc (fun k ->
      let old_k = !curr_k in curr_k := k; throw old_k x) in
  let rec process1 n =
    print_string "1: received "; print_int n; print_newline();
    process1(communicate(n+1))
  and process2 n =
    print_string "2: received "; print_int n; print_newline();
    process2(communicate(n+1)) in
  process1(callcc(fun start1 ->
    process2(callcc(fun start2 ->
      curr_k := start2; throw start1 0))))))
```

Coroutines and generators

Generators are also a generalization of subroutines to define iterators
They look less expressive since the `yield` statement in a generator does not specify a coroutine to jump to: this is not the case:

```
generator produce
  loop
    while q is not full
      create some new items
      add the items to q
    yield consume

generator consume
  loop
    while q is not empty
      remove some items from q
      use the items
    yield produce

subroutine dispatcher
  var d := new dictionary { generator → iterator }
  d[produce] := start produce
  d[consume] := start consume
  var current := produce
  loop current := d[current].next()
```

Rationale

It is possible to implement coroutines on top of a generator facility, with the aid of a top-level dispatcher routine that passes control explicitly to child generators identified by tokens passed back from the generators

Generators are a much more commonly found language feature

A number of implementations of coroutines for languages with generator support but no native coroutines use this or a similar model: e.g. Perl 6, C#, Ruby, Python (prior to 2.5),

In OCaml there is Jérôme Vouillon's lightweight thread library (`Lwt`) that provides cooperative multi-threading. This can be implemented by coroutines (see the concurrency part of the course).

Keep the same reductions “ \rightarrow ” and the same context rules as before, and add the following rules for `callcc` and `throw`:

$$\begin{aligned} E[\text{callcc } v] &\rightarrow E[v(\lambda x. E[x])] \\ E[\text{throw } k v] &\rightarrow kv \end{aligned}$$

(recall: the v argument of the `callcc` is a function that expects a continuation)

Same evaluation contexts E as before.

Example of reductions

$$\begin{aligned} E[\text{callcc}(\lambda k.1 + \text{throw } k \ 0)] \\ \rightarrow E[(\lambda k.1 + \text{throw } k \ 0)(\lambda x.E[x])] \\ \rightarrow E[1 + \text{throw } (\lambda x.E[x]) \ 0] \\ \rightarrow (\lambda x.E[x])0 \\ \rightarrow E[0] \end{aligned}$$

Note how `throw` discards the current context $E[1 + []]$ and reinstalls the saved context E instead.

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Conversion to continuation-passing style (CPS)

Goal: make explicit the handling of continuations.

Input: a call-by-value functional language with `callcc`.

Output: a call-by-value or call-by-name, pure functional language (no `callcc`).

Idea: every term a becomes a function $\lambda k \dots$ that receives its continuation k as an argument, computes the value v of a , and finishes by applying k to v .

Uses: compilation of `callcc`; semantics; programming with continuations in Caml, Haskell, ...

$$\begin{aligned}\llbracket N \rrbracket &= \lambda k. kN \\ \llbracket x \rrbracket &= \lambda k. kx \\ \llbracket \lambda x. a \rrbracket &= \lambda k. k(\lambda x. \llbracket a \rrbracket) \\ \llbracket \text{let } x = a \text{ in } b \rrbracket &= \lambda k. \llbracket a \rrbracket (\lambda x. \llbracket b \rrbracket k) \\ \llbracket a b \rrbracket &= \lambda k. \llbracket a \rrbracket (\lambda x. \llbracket b \rrbracket (\lambda y. x y k))\end{aligned}$$

A function $\lambda x. a$ becomes a function of two arguments, x and the continuation k that will receive the value of a .

In $\llbracket a b \rrbracket$, the variable x (which must not be free in b) will be bound to the value returned by a and y to the value of b .

Effect on types:

if $a : \tau$ then $\llbracket a \rrbracket : (\llbracket \tau \rrbracket \rightarrow \text{answer}) \rightarrow \text{answer}$ where

$$\begin{aligned}\llbracket b \rrbracket &= (b \rightarrow \text{answer}) \rightarrow \text{answer} && \text{for base types } b \\ \llbracket \tau_1 \rightarrow \tau_2 \rrbracket &= \llbracket \tau_1 \rrbracket \rightarrow (\llbracket \tau_2 \rrbracket \rightarrow \text{answer}) \rightarrow \text{answer}\end{aligned}$$

$$\begin{aligned} \llbracket \text{callcc } a \rrbracket &= \lambda k. \llbracket a \rrbracket k k \\ \llbracket \text{throw } a b \rrbracket &= \lambda k. \llbracket a \rrbracket (\lambda x. \llbracket b \rrbracket (\lambda y. x y)) \end{aligned}$$

In `callcc a`, the function value returned by $\llbracket a \rrbracket$ receives the current continuation k both as its argument and as its continuation.

In `throw a b`, we discard the current continuation k and apply directly the value of a (which is a continuation captured by `callcc`) to the value of b .

The CPS translation $\llbracket \dots \rrbracket$ produces terms that are more verbose than one would naturally write by hand. For instance, in the case of an application of a variable f to a variable x :

$$\llbracket f x \rrbracket = \lambda k. (\lambda k_1. k_1 f) (\lambda y_1. (\lambda k_2. k_2 x) (\lambda y_2. y_1 y_2 k))$$

instead of the more natural $\lambda k. f x k$. This clutter can be eliminated by performing β reductions at transformation time to eliminate the “administrative redexes” introduced by the translation. In particular, we have

$$(\lambda k. k v) (\lambda x. a) \xrightarrow{\text{adm}} (\lambda x. a) v \xrightarrow{\text{adm}} a[x/v]$$

whenever v is a value or variable.

$$\begin{aligned} \llbracket f(f x) \rrbracket \\ = \lambda k. f x (\lambda y. f y k) \end{aligned}$$

$$\begin{aligned} \llbracket \mu fact. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } fact(n-1) * n \rrbracket \\ = \lambda k_0. k_0 (\mu fact. \lambda n. \lambda k. \text{if } n = 0 \text{ then } k \ 1 \text{ else } fact(n-1) (\lambda v. k(v * n))) \end{aligned}$$

Notice that the factorial function has become tail-recursive

Execution of a program $prog$ is achieved by applying its CPS conversion to the initial continuation $\lambda x.x$:

$$\llbracket prog \rrbracket (\lambda x.x)$$

Theorem (Soundness)

If $a \rightarrow^* N$, then $\llbracket a \rrbracket (\lambda x.x) \rightarrow^* N$.

The λ -terms produced by the CPS transformation have a very specific shape, described by the following grammar:

$atom ::= x \mid N \mid \lambda x.body \mid \lambda x.\lambda k.body$	CPS atom
$body ::= atom \mid atom_1 atom_2 \mid atom_1 atom_2 atom_3$	CPS body

$\llbracket a \rrbracket$ is an atom, and $\llbracket a \rrbracket(\lambda x.x)$ is a body .

Reduction of CPS terms

$atom ::= x \mid N \mid \lambda v.body \mid \lambda x.\lambda k.body$ **CPS atom**
 $body ::= atom \mid atom_1 atom_2 \mid atom_1 atom_2 atom_3$ **CPS body**

Note that all applications (unary or binary) are in tail-position and at application-time, their arguments are closed atoms, that is, values.

The following reduction rules suffice to evaluate CPS-converted programs:

$$\begin{aligned}(\lambda x.\lambda k.body)atom_1 atom_2 &\rightarrow body[x/atom_1, k/atom_2] \\ (\lambda x.body)atom &\rightarrow body[x/atom]\end{aligned}$$

These reductions are always applied at the top of the program—there is no need for reduction under a context.

CPS terms can be executed by a stackless abstract machines with three registers, an environment and a code pointer.

Stacks are more efficient in terms of GC costs and memory locality, but need to be copied in full to implement `callcc`.

[*Compiling with continuations*, A. Appel, Cambridge University Press, 1992].

Theorem (Indifference (Plotkin 1975))

A closed CPS-converted program $\llbracket a \rrbracket (\lambda x. x)$ evaluates in the same way in call-by-name, in left-to-right call-by-value, and in right-to-left call-by-value.

CPS conversion encodes the reduction strategy in the structure of the converted terms. The one we gave corresponds to left-to-right call-by-value.

$$\llbracket a b \rrbracket = \lambda k. \llbracket a \rrbracket (\lambda x_a. \llbracket b \rrbracket (\lambda x_b. x_a x_b k))$$

Right-to-left call-by-value is obtained by taking

$$\llbracket a b \rrbracket = \lambda k. \llbracket b \rrbracket (\lambda x_b. \llbracket a \rrbracket (\lambda x_a. x_a x_b k))$$

while call-by-name is achieved by taking

$$\begin{aligned} \llbracket x \rrbracket &= \lambda k. x k \\ \llbracket a b \rrbracket &= \lambda k. \llbracket a \rrbracket (\lambda x_a. x_a \llbracket b \rrbracket k) \end{aligned}$$

Control operators and classical logic

Control operators such as `callcc` extend the Curry-Howard correspondence from *intuitionistic logic* to *classical logic*.

The *Pierce's law* $((P \rightarrow Q) \rightarrow P) \rightarrow P$ is not derivable in the intuitionistic logic while it is true in classical logic (in particular if we take $Q \equiv \perp$ then it becomes $((\neg P \rightarrow P) \rightarrow P$: if from $\neg P$ we can deduce P , then P must be true).

In terms of Curry-Howard it means that no term of the simply-typed λ -calculus has type $((P \rightarrow Q) \rightarrow P) \rightarrow P$.

But notice that

$$\text{callcc} : ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$$

`callcc` takes as argument a function f of type $((\alpha \rightarrow \beta) \rightarrow \alpha)$ which can either return a value of type α directly or apply an argument of type α to the continuation of type $(\alpha \rightarrow \beta)$. Since the existing context is deleted when the continuation is applied, the type β is never used and may be taken to be \perp .

`callcc` is a proof for Pierce's law. It extends the Curry-Howard correspondence from intuitionistic logic to classical logic

It is therefore possible to “prove” the excluded middle axiom $\forall P. P \vee \neg P$.

Modulo Curry-Howard, this axiom corresponds to the type $\forall P. P + (P \rightarrow False)$, where *False* is an empty type and $A + B$ is a datatype with two constructors

Left : $A \rightarrow A + B$ and *Right* : $B \rightarrow A + B$.

The following term “implements” (ie, it proves) excluded middle:

$$\text{callcc}(\lambda k. \text{Right}(\lambda p. \text{throw } k(\text{Left}(p))))$$

What about the CPS translation?

CPS and double negation

Let $\neg A = (A \rightarrow \perp)$ where \perp represent “false”. In intuitionistic logic

$$\vdash A \rightarrow \neg\neg A$$

whose proof is $\lambda x:A.\lambda f:\neg A.fx$. On the other hand:

$$\not\vdash \neg\neg A \rightarrow A$$

It is not possible to define a closed λ -term of the type above.

However

$$\vdash \neg\neg\neg A \rightarrow \neg A$$

whose proof is: $\lambda f : \neg\neg\neg A.\lambda x : A.f(\lambda g : \neg A.gx)$.

This suggests a *double negation* translation from classical to intuitionistic logic:

- $[\phi] = \neg\neg\phi$ if ϕ is *atomic* (ie, a basic type)
- $[A \rightarrow B] = [A] \rightarrow [B]$

Theorem (Glivenko 1929)

$$\vdash_{\text{classic}} A \quad \text{iff} \quad \vdash_{\text{intuitionistic}} [A]$$

In terms of the Curry Howard isomorphism

$$\vdash_{\text{classic}} M : A \quad \text{iff} \quad \vdash_{\text{intuitionistic}} [M] : [A]$$

where $[M]$ is (essentially) the CPS translation of M .

So the CPS translation extends the Curry-Howard isomorphism to the “double negation *encoding*” of the classical propositional logic

See A Formulæ-as-Types Notion of Control, T. Griffin, Symp. Principles of Programming Languages 1990.

- A. Appel. Programming with continuations.
- Slides of the course *Functional Programming Languages* by Xavier Leroy (from which the slides of this and the following part heavily borrowed) available on the web:

<http://cristal.inria.fr/~xleroy/mpri/progfunc>