Using structures to fix (or improve) several systems of nets derived from Linear Logic.

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PPS - Université Paris Diderot

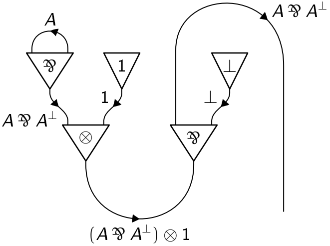
Collegium Logicum – 1 March 2011

Introduction / Interaction nets

Born from multiplicative linear logic [Laf90]:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\textit{ax}}{\vdash A \parr A^\perp}\,\textit{par} \qquad \dfrac{}{\vdash 1}\,\textit{one}}{\vdash (A \parr A^\perp) \otimes 1}\,\textit{ten} \qquad \dfrac{\dfrac{\dfrac{}{\vdash (A^\perp \otimes A), A \parr A^\perp}\,\textit{ax}}{\vdash (A^\perp \otimes A), \bot, A \parr A^\perp}\,\textit{bot}}{\vdash (A^\perp \otimes A) \parr \bot, A \parr A^\perp}\,\textit{par}}{\vdash A \parr A^\perp}\,\textit{cut}$$

$$\mapsto$$



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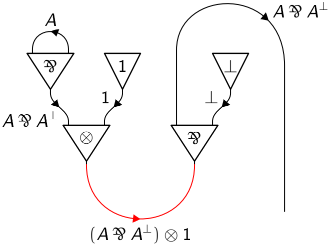

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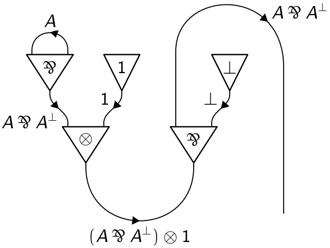
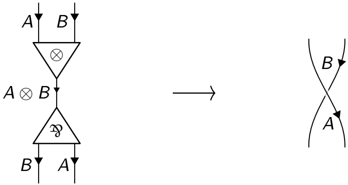
$$\longmapsto$$

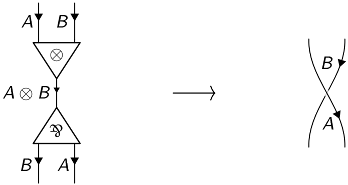


Introduction / Interaction nets

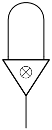


Flexible. . .

Introduction / Interaction nets

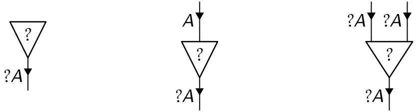


Flexible. . . too flexible.

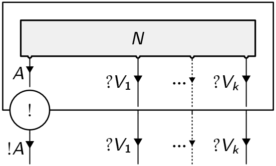


Introduction / Boxes

Exponentials:



Boxes are used to represent promotion: $\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}$ *prom*



Introduction / Problem

Nets are too flexible, but sequents are usually there to back them up.

New systems of nets appeared. What to do when sequents fail to back them up ? (Sequents are sometimes too rigid)

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New systems of nets appeared. What to do when sequents fail to back them up ? (Sequents are sometimes too rigid)

- Differential Interaction Nets [ER06]
  (sequent presentation is not fully satisfactory)
- Super-promotion
  (actually broken without backup)
- Localized Boxes
  (correctness criterion missing)

Systems to be fixed / Differential Interaction Nets

DIN come from differential λ-calculus:

$$\frac{\partial}{\partial x}\big(\lambda z.\,(x, x)\big)\cdot u \;=\; \lambda z.\,(u, x) \;+\; \lambda z.\,(x, u)$$

$$\frac{\partial}{\partial x}\big(\lambda z.\,z\big)\cdot u \;=\; 0$$

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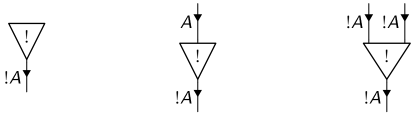
Differential cells (co-weakening, co-dereliction, co-corntraction):



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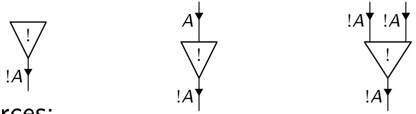
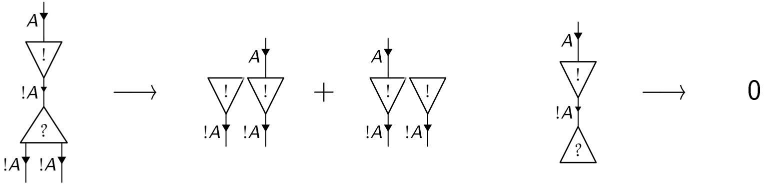
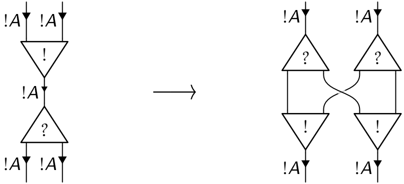


Routing of resources:



Systems to be fixed / Differential Interaction Nets

Having a look at this reduction:



Can it be written in sequent calculus ?

Systems to be fixed / Differential Interaction Nets

Multiplicative constructions:

$$\frac{\vdash \Gamma_1, A \quad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \otimes B}\ \textit{tensor} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \textit{par}$$

$$\frac{}{\vdash 1}\ \textit{unit} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \textit{bottom}$$

Differential constructions can be introduced in the same way:

$$\frac{\vdash \Gamma_1, !A \quad \vdash \Gamma_2, !A}{\vdash \Gamma_1, \Gamma_2, !A}\ \textit{co-contraction} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \textit{contraction}$$

$$\frac{}{\vdash !A}\ \textit{co-weakening} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \textit{weakening}$$

Systems to be fixed / Differential Interaction Nets

Cut elimination:

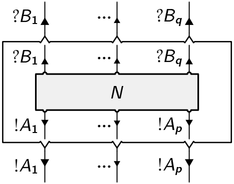
$$\dfrac{\dfrac{\dfrac{}{\vdash \Gamma_1, !A}\,\pi_1 \qquad \dfrac{}{\vdash \Gamma_2, !A}\,\pi_2}{\vdash \Gamma_1, \Gamma_2, !A}\,\textit{co-contraction} \qquad \dfrac{\dfrac{}{\vdash ?A^\perp, ?A^\perp, \Gamma_3}\,\pi_3}{\vdash ?A^\perp, \Gamma_3}\,\textit{contraction}}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}\,\textit{cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{}{\vdash \Gamma_1, !A}\,\pi_1 \quad \dfrac{}{\vdash \Gamma_2, !A}\,\pi_2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash ?A^\perp, !A}\,\textit{ax} \quad \dfrac{}{\vdash ?A^\perp, !A}\,\textit{ax}}{\vdash ?A^\perp, ?A^\perp, !A}\,\textit{co-contraction} \quad \dfrac{\dfrac{}{\vdash ?A^\perp, !A}\,\textit{ax} \quad \dfrac{}{\vdash ?A^\perp, !A}\,\textit{ax}}{\vdash ?A^\perp, ?A^\perp, !A}\,\textit{co-contraction} \quad \dfrac{}{\vdash ?A^\perp, ?A^\perp, \Gamma_3}\,\pi_3}{\vdash ?A^\perp, ?A^\perp, ?A^\perp, ?A^\perp, \Gamma_3}\,\textit{cut}^2}{\vdash ?A^\perp, ?A^\perp, ?A^\perp, ?A^\perp, \Gamma_3}\,\textit{commutation}}{\vdash ?A^\perp, ?A^\perp, \Gamma_3}\,\textit{contraction}^2}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}\,\textit{cut}^2$$

Systems to be fixed / Super-promotion

Super-promotion:



Systems to be fixed / Super-promotion

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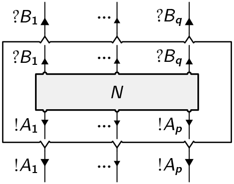
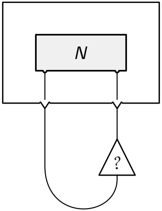
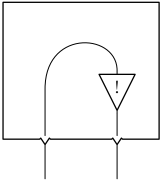


Correct:



Hidden cycle:



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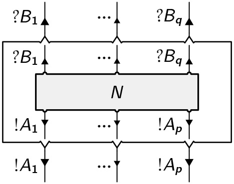

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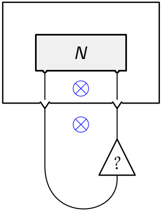

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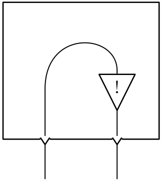
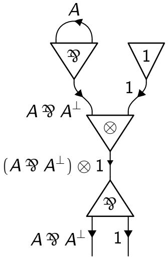


Structures / Typing nets

The problem: sequents can only type nets whose conclusions are in ⅋ relation.

Solution: we should remember connectives between wires.



Structures / Typing nets

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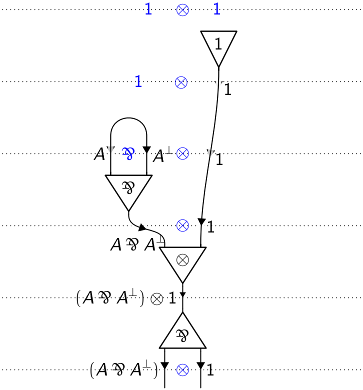

$$\frac{A \parr A^\perp}{A \parr A^\perp}$$

We can use deep inference up to blue connectives.

Structures / Typing nets

Structures over linear logic formulaes:

$$\sigma \quad ::= \quad \cdot \quad | \quad \sigma \,;\, \sigma \quad | \quad \circ \quad | \quad \sigma \,,\, \sigma \quad | \quad A$$

Derivations:

$$\cfrac{\sigma}{\cfrac{\vdots}{\cfrac{\omega_i}{\cfrac{\omega_{i+1}}{\cfrac{\vdots}{\tau}}}}}$$

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Structural rules [Str03]:

$$\frac{(\sigma \,,\, \tau) \,;\, \omega}{\sigma \,,\, (\tau \,;\, \omega)}\; \mathit{switch} \qquad \frac{\cdot}{\sigma \,,\, \sigma^{\perp}}\; i{\downarrow} \qquad \frac{\sigma^{\perp} \,;\, \sigma}{\circ}\; i{\uparrow}$$

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Logical rules:

$$\frac{A\,;B}{A\otimes B}\ \textit{tensor}{\downarrow} \qquad \frac{A\,,B}{A\parr B}\ \textit{par}{\downarrow} \qquad \frac{A\parr B}{A\,,B}\ \textit{tensor}{\uparrow} \qquad \frac{A\otimes B}{A\,;B}\ \textit{par}{\uparrow}$$

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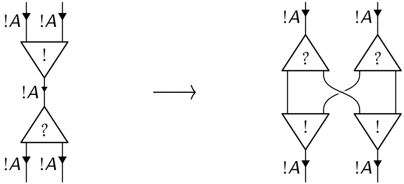
Symmetry, Deep inference!

Structures / Solving problems

Differential interaction nets:



Using structures:

$$\frac{?A\,,\,?A}{?A}\ \textit{contraction}\downarrow \qquad\qquad \frac{!A\,;\,!A}{!A}\ \textit{co-contraction}\downarrow$$

$$\begin{array}{c} !A\,;\,!A \\ \hline !A \\ \hline !A\,;\,!A \end{array}\ \begin{array}{l} \textit{co-contraction}\downarrow \\ \\ \textit{contraction}\uparrow \end{array} \quad\longrightarrow\quad \begin{array}{c} !A\,;\,!A \\ \hline !A\,;\,!A\,;\,!A\,;\,!A \\ \hline !A\,;\,!A\,;\,!A\,;\,!A \\ \hline !A\,;\,!A \end{array}\ \begin{array}{l} \textit{contraction}\uparrow\,;\,\textit{contraction}\uparrow \\ \\ \diamond \\ \\ \textit{co-contraction}\downarrow\,;\,\textit{co-contraction}\downarrow \end{array}$$

Structures / Solving problems

Promotion: In order to remain as close as possible to boxes used in nets, we can use promotion rules that are parametrized by a derivation:

$$
\dfrac{
  \begin{array}{c}
  !V_1 \;;\; \cdots \;;\, !V_k \\
  \boxed{\begin{array}{c} \dfrac{!V_1 \;;\; \cdots \;;\, !V_k}{\vdots} \\ \overline{A} \end{array}}
  \end{array}
}{
  !A
}\;\textit{promotion}\!\downarrow
\qquad\qquad
\dfrac{?A}{
  \begin{array}{c}
  \boxed{\begin{array}{c} \underline{A} \\ \dfrac{\vdots}{?V_1 \;,\; \cdots \;,\, ?V_k} \end{array}} \\
  ?V_1 \;,\; \cdots \;,\, ?V_k
  \end{array}
}\;\textit{promotion}\!\uparrow
$$

Structures / Solving problems

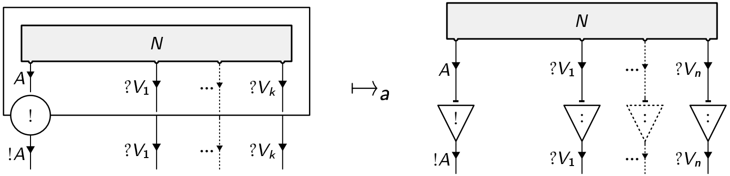
Super-promotion:

$$\dfrac{\begin{array}{c} !V_1 \, ; \, \cdots \, ; !V_q \\ \boxed{\begin{array}{c} \underline{!V_1 \, ; \, \cdots \, ; !V_q} \\ \vdots \\ \overline{!A_1 \, ; \, \cdots \, ; !A_p} \end{array}} \end{array}}{!A_1 \, ; \, \cdots \, ; !A_p} \; \textit{super-promotion}\!\downarrow
\qquad\qquad
\dfrac{?A_1 \, , \, \cdots \, , ?A_q}{\begin{array}{c} \boxed{\begin{array}{c} \underline{?A_1 \, , \, \cdots \, , ?A_q} \\ \vdots \\ \overline{?V_1 \, , \, \cdots \, , ?V_p} \end{array}} \\ ?V_1 \, , \, \cdots \, , ?V_p \end{array}} \; \textit{super-promotion}\!\uparrow$$

Not possible in sequent calculus.

Localized Exponential / Localized Boxes

Idea: materialize only interfaces of boxes.



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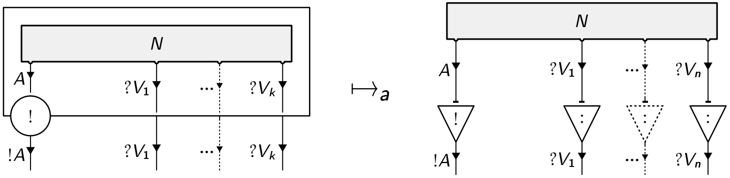
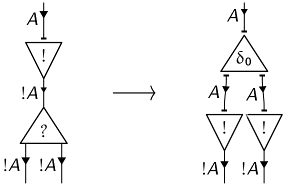


How it works:



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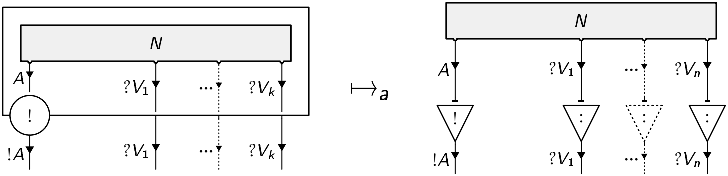

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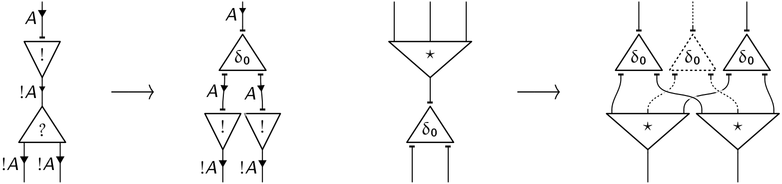

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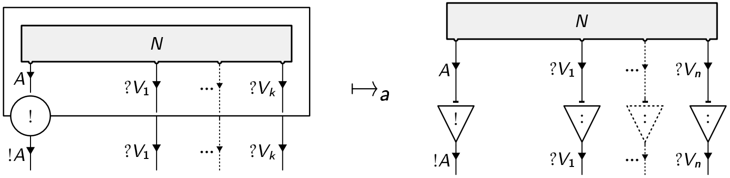

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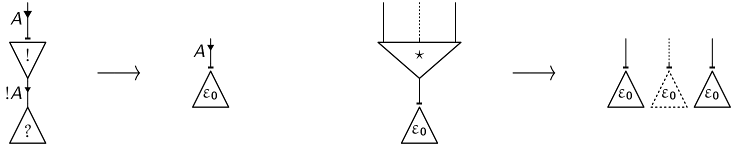

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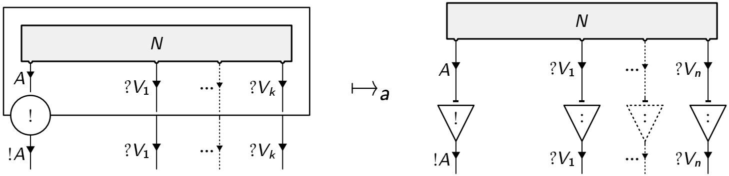

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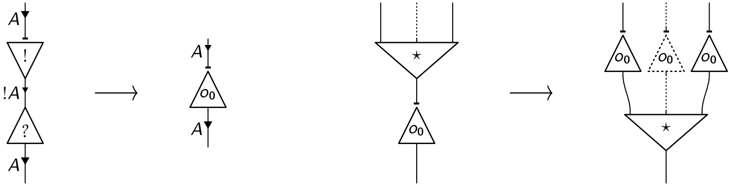

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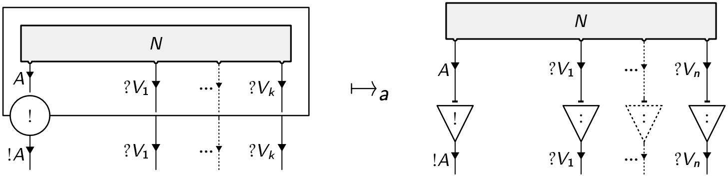

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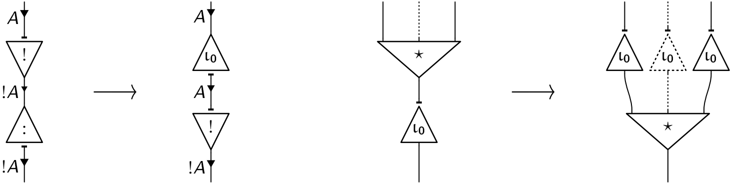
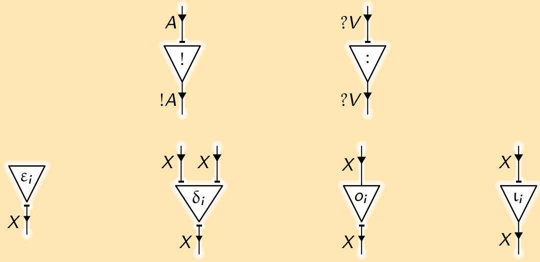


Localized Exponential / Localized Boxes

Boxes can be replaced by simple cells.

Localized exponential system (active version)



- Similar to "sharing graphs" [GAL92].
- Reduction in passive and controled versions preserve box structure. (Correctness of intermediate reduction steps?)

Localized Exponential / Benefits from/to Structures

$$\sigma \quad ::= \quad \cdot \;\mid\; \sigma \,;\, \sigma \;\mid\; \circ \;\mid\; \sigma \,,\, \sigma \;\mid\; [\sigma] \;\mid\; \{\sigma\} \;\mid\; A$$

Alternative structural rules for exponentials:

$$\frac{[\omega]}{\cdot}\ \textit{erase}\!\uparrow \qquad \frac{[\omega]}{[\omega]\,;\,[\omega]}\ \textit{dupl.}\!\uparrow \qquad \frac{[\omega]}{\omega}\ \textit{open}\!\uparrow \qquad \frac{[\omega]}{[[\omega]]}\ \textit{in}\!\uparrow$$

$$\frac{[\sigma]\,;\,[\tau]}{[\sigma\,;\,\tau]} \qquad \frac{\cdot}{[\cdot]}$$

Logical rules:

$$\frac{[A]}{!A}\ \textit{prom-out}\!\downarrow \qquad \frac{!A}{[!A]}\ \textit{prom-in}\!\uparrow$$

$$\frac{\circ}{?A}\ \textit{weak.}\!\downarrow \qquad \frac{?A\,,\,?A}{?A}\ \textit{contr.}\!\downarrow \qquad \frac{A}{?A}\ \textit{derel.}\!\downarrow \qquad \frac{??A}{?A}\ \textit{digg.}\!\downarrow$$

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$$\frac{[\sigma]\,;[\tau]}{[\sigma\,;\tau]} \qquad \frac{\cdot}{[\cdot]}$$

Logical rules:

$$\frac{[A]}{!A}\ \textit{prom-out}\!\downarrow \qquad \frac{!A}{[!A]}\ \textit{prom-in}\!\uparrow$$

$$\frac{\circ}{?A}\ \textit{weak.}\!\downarrow \qquad \frac{?A\,,?A}{?A}\ \textit{contr.}\!\downarrow \qquad \frac{A}{?A}\ \textit{derel.}\!\downarrow \qquad \frac{??A}{?A}\ \textit{digg.}\!\downarrow$$

- Promotion is just inference under [ ] contexts.

Conclusion / Work in progress

- Try structures when sequents are too rigid!
- Two levels of connectors fits nicely to Interaction Nets systems
- Boxes can be represented using:
  - a standard global construction
  - localized exponential structure
- A structured presentation of Nets might finally end up better than their graph presentation. (Simple induction principle)

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Thanks! Questions?

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