Realizability Proof for Normalization of Full Differential Linear Logic

Stéphane Gimenez

PPS - Université Paris Diderot

30 Juin 2011 (based on Novi Sad's talk)

Reducibility
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Plan

Introduction

- Differentials and λ-terms
- Interaction nets, Boxes, and Reduction
- Differential constructions
- Full-DiLL

Normalization

- Previous results
- A new proof
- WN of LL (proof by Girard)
- WN of Full-DiLL

Conclusion

Introduction / Differentials and λ-terms

Analytic functions.

$$\begin{aligned} h \quad : \quad & \mathbb{R} \to \mathbb{R} \\ & x \mapsto 2x\, g(x) \end{aligned}$$

$$\begin{aligned} Dh \quad : \quad & \mathbb{R} \to \mathcal{L}(\mathbb{R}, \mathbb{R}) \\ & x \mapsto u \mapsto Dh(x)(u) \end{aligned}$$

$$Dh(x)(u) = \tfrac{\partial}{\partial x}\ (2x\, g(x))\ \cdot u$$

$$= 2\, g(x) \cdot u + 2x\, \tfrac{\partial}{\partial x}\, (g(x)) \cdot u$$

$$= 2\, g(x) \cdot u + 2x\, Dg(x)(u)$$

Introduction / Differentials and λ-terms

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λ-terms.

$$\begin{aligned} t &: A \to B \\ &= \lambda x.\, x(g\, x) \end{aligned}$$

$$\begin{aligned} Dt &: A \to A \multimap B \\ &= \lambda x.\lambda u.\, Dt\, x\, u \end{aligned}$$

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Introduction / Differentials and λ-terms

Differential λ-calculus [Ehrhard, Regnier, Vaux]:

$$t \quad ::= \quad \begin{array}{c|c|c|c} x & \lambda x\, t & t\, t' & Dt \cdot t' \\ & 0 & t + t' & \end{array}$$

β-reduction:

$$(\lambda z.t)u \longrightarrow t[{}^{u}/_{z}]$$

differential reduction:

$$D(\lambda z.t) \cdot u \longrightarrow \lambda z.\, \frac{\partial}{\partial z}\, t \cdot u$$

where $\frac{\partial}{\partial z}\, t \cdot u$ accounts for "linear substitution of z by u in t"

- Resource aware calculus.

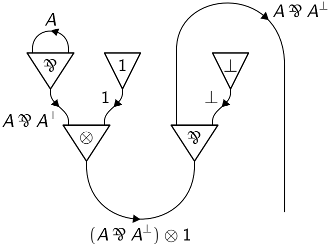
on the other side of the Curry-Howard mirror...

Introduction / Interaction nets, Boxes, and Reduction

LL in Sequent Calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\mathit{ax}}{\vdash A \parr A^\perp}\,\mathit{par} \qquad \dfrac{}{\vdash 1}\,\mathit{one}}{\vdash (A \parr A^\perp) \otimes 1}\,\mathit{ten} \qquad \dfrac{\dfrac{\dfrac{}{\vdash (A^\perp \otimes A), A \parr A^\perp}\,\mathit{ax}}{\vdash (A^\perp \otimes A), \perp, A \parr A^\perp}\,\mathit{bot}}{\vdash (A^\perp \otimes A) \parr \perp, A \parr A^\perp}\,\mathit{par}}{\vdash A \parr A^\perp}\,\mathit{cut}$$

LL using Interaction Nets [Lafont]:



Introduction / Interaction nets, Boxes, and Reduction

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,ax}{\vdash A \parr A^\perp}\,par \qquad \dfrac{}{\vdash 1}\,one}{\vdash \color{red}{(A \parr A^\perp) \otimes 1}}\,ten \qquad \dfrac{\dfrac{\dfrac{}{\vdash (A^\perp \otimes A), A \parr A^\perp}\,ax}{\vdash (A^\perp \otimes A), \perp, A \parr A^\perp}\,bot}{\vdash \color{red}{(A^\perp \otimes A) \parr \perp}, A \parr A^\perp}\,par}{\vdash A \parr A^\perp}\,\color{red}{cut}$$

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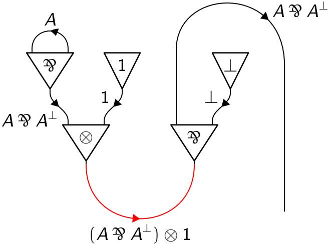
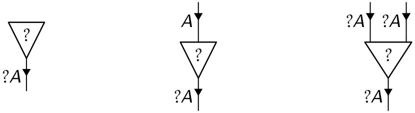
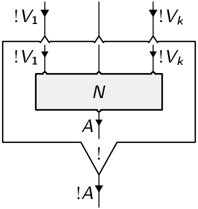


Introduction / Interaction nets, Boxes, and Reduction

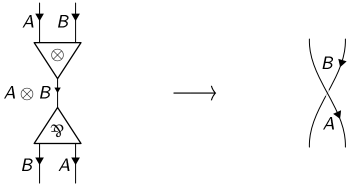
Exponentials:

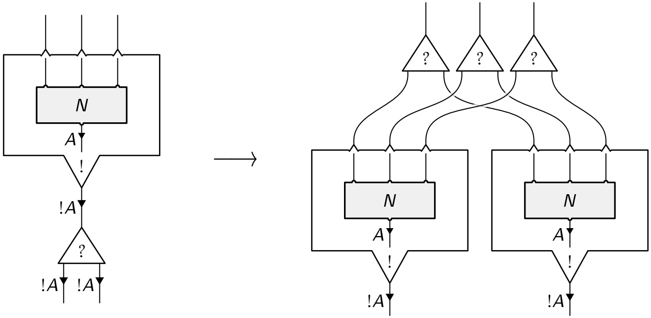




$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$ *promotion*

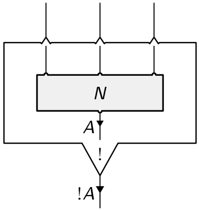
Introduction / Interaction nets, Boxes, and Reduction

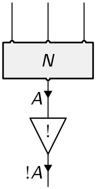




Introduction / Differential constructions

In a differential setting, we want both replicable resources and single-use resources (co-dereliction):





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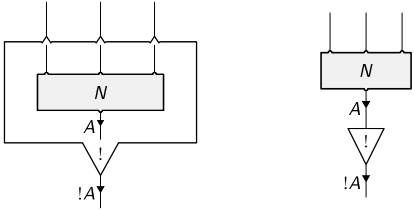
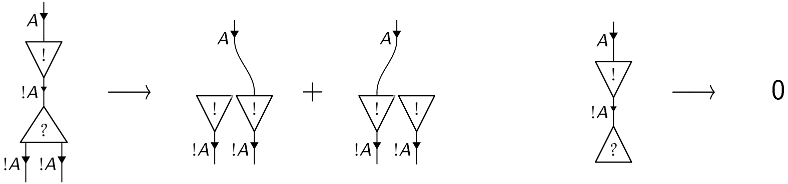
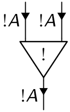


Routing of single-use resources:

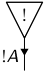


Introduction / Differential constructions

We need a way to merge bags of resources (co-contraction):



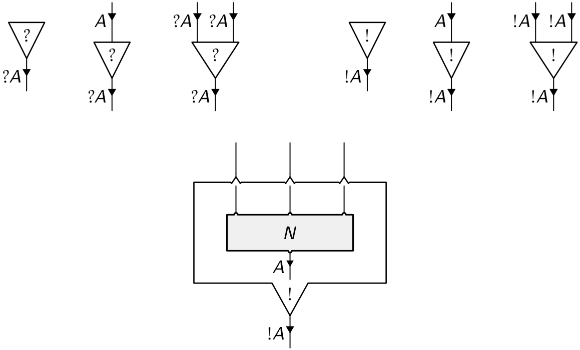
and to build empty bags of resources (co-weakening):



(Illustration with some concrete examples on the blackboard!)

Introduction / Full-DiLL

Full-DiLL constructions:

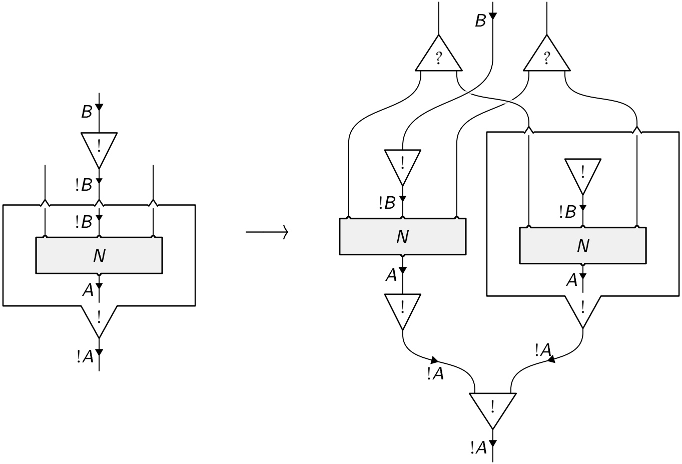


Reduction rules:

- standard *Linear Logic* reductions (4).
- *Differential Interaction Nets* reductions (9).
- 3 more rules.

Introduction / Full-DiLL

Chain rule:



Introduction / Full-DiLL

Motivations for Full-DiLL:

- Embeds Differential λ-calculus.
- Duality in consuming and producing resources.
- Links with concurrency (finite π-calculus):

  $a(x).P$ and $\bar{a}(z).P$ processes can be seen as resource producers and resource consumers.
- Promotion brings guarded replication into this picture:

  $!a(x).P$ is a replicable resource.

Does this logic enjoy the WN property? (cut-elimination theorem)
the SN property?

Normalization / Previous results

Previous normalisation results:

- Differential Interaction Nets
  [Ehrhard, Regnier, 2006]
  Combinatorial proof for SN of Promotion free *DiLL*.
- The cut-elimination theorem for Differential Linear Logic
  [Pagani, 2009]
  Combinatorial proof for WN of Full-DiLL.
- The Conservation Theorem for Differential Nets
  [Pagani, Tranquilli, to appear]
  "This turns the quest for strong normalization into one for non-erasing weak normalization"
  Upgrades the previous proof to SN of Full-DiLL.

Normalization / A new proof

Other approach:

- extend Girard's proof for WN of LL.

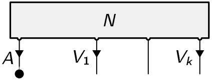
Using a reducibility technique, we obtain a modular proof. Further extensions comes for free:

- second order fragment
- additive fragment...

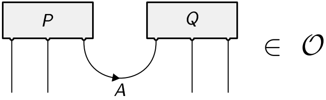
The reducibility technique (Girard).

Normalization / WN of LL (proof by Girard)

A *pointed net* is a net with a distinguished port:



Let $\mathcal{O}$ be a chosen set of nets called *observable*, $P \star Q$ holds when:



If $\mathcal{P}$ is a set of pointed nets of type A, its *dual* is defined as:

$$\mathcal{P}^\star \;:=\; \{\, Q \mid \forall P \in \mathcal{P},\ P \star Q \,\}$$

Normalization / WN of LL (proof by Girard)

For any set of pointed nets $\mathcal{P} \subseteq \mathcal{N}_A$ and $\mathcal{Q} \subseteq \mathcal{N}_A$, the following properties hold:

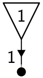
$$\mathcal{P} \subseteq \mathcal{Q} \implies \mathcal{Q}^\star \subseteq \mathcal{P}^\star \qquad \mathcal{P} \subseteq \mathcal{P}^{\star\star} \qquad \mathcal{P}^{\star\star\star} = \mathcal{P}^\star$$

A *behavior* at type A is any set $\mathcal{P}$ of pointed nets of type A which is equal to its double dual:

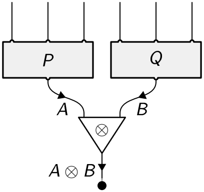
$$\mathcal{P} \;=\; \mathcal{P}^{\star\star}$$

Normalization / WN of LL (proof by Girard)

We can define a behaviour $[\![1]\!] \subseteq \mathcal{N}_1$ as follows:

$[\![1]\!] \;:=\; \Big\{$  $\Big\}^{\star\star}$

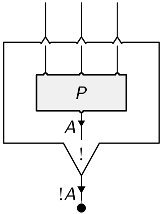
Behavior $[\![A \otimes B]\!] \subseteq \mathcal{N}_{A \otimes B}$ is defined from behaviors $[\![A]\!] \subseteq \mathcal{N}_A$ and $[\![B]\!] \subseteq \mathcal{N}_B$:

$[\![A \otimes B]\!] \;:=\; \Bigg\{$  $\Bigg|\; P \in [\![A]\!], Q \in [\![B]\!] \Bigg\}^{\star\star}$

Normalization / WN of LL (proof by Girard)

The behavior $[\![!A]\!] \subseteq \mathcal{N}_{!A}$ is defined from $[\![A]\!] \subseteq \mathcal{N}_A$ as follows:

$$[\![!A]\!] := \left\{ \quad \right.$$



$$\left. \middle| \begin{array}{c} P \in [\![A]\!] \\ \text{with adequate} \\ \text{interface} \end{array} \right\}^{\star\star}$$

And $[\![?A]\!] \subseteq \mathcal{N}_{?A}$ is defined by:

$$[\![?A]\!] := [\![!A^{\perp}]\!]^{\star}$$

Normalization / WN of LL (proof by Girard)

We now chose for $\mathcal{O}$ the set of all weakly normalizing nets. So, $P \star Q$ means that the interaction between P and Q reduces to a normal form.

Property

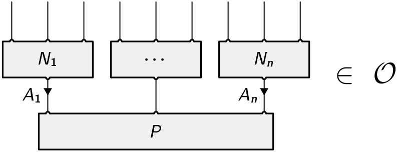
Behaviors $[\![A]\!]$ previously defined are reducibility candidates:

$$Axiom_A \in [\![A]\!] \qquad\qquad [\![A]\!] \subseteq \mathcal{O}_A$$

Normalization / WN of LL (proof by Girard)

Definition (Net reducibility)

A net P is *reducible* when for all $N_i \in [\![A_i]\!]$, we have:



Main lemma:

Lemma

Every valid net is reducible.

Normalization / WN of LL (proof by Girard)

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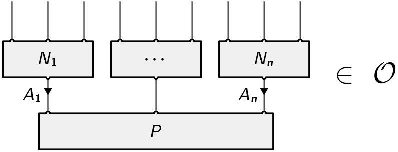


Main lemma:

Lemma

Every valid net is reducible.

(Proof by induction on net sequentializations).

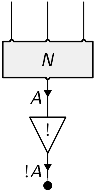
From LL to Full-DiLL...

reducibiblity candidates for exponential types must be built differently

Normalization / WN of Full-DiLL

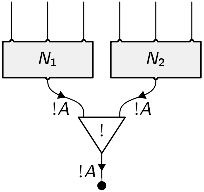
Given a set $\mathcal{S}_A$ of pointed nets, one can build $T(\mathcal{S}_A)$ such that:

- if $N \in \mathcal{S}_A$, then



$\in T(S_A)$.

- if $N_1, N_2 \in T(\mathcal{S}_A)$, then

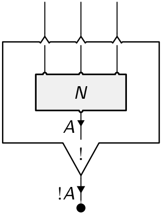


$\in$ $T(S_A)$.

- 

$\in T(S_A)$.

- if $N \in \mathcal{S}_A$ has adequate interface, then



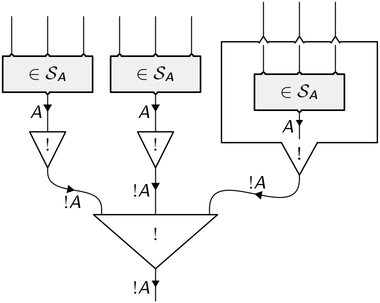
$\in T(S_A)$.

- the axiom

$\in T(S_A)$.

Normalization / WN of Full-DiLL

Going from $\mathcal{S}_A$ to $T(\mathcal{S}_A)$ consists in applying an exponential macro-construction, restricted to a single exponential layer.



Now, we can define Full-DiLL exponential candidates as follows:

$$[\![!A]\!] := (\, T(\, [\![A]\!] \,)\,)^{\star\star}$$

Normalization / WN of Full-DiLL

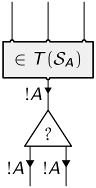
The rest of the proof is an induction and relies on:

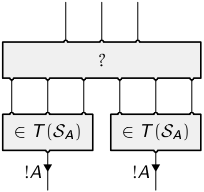
- A property about reduction of exponential macro-constructions (weakening, dereliction, contraction cases)
- A technical lemma to deal with the chain rule (promotion case).

And we need to reuse a little trick to deal with the co-contraction case.

Normalization / WN of Full-DiLL

Property

- 

reduces to a sum of nets of the form


Normalization / WN of Full-DiLL

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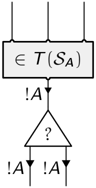
- 

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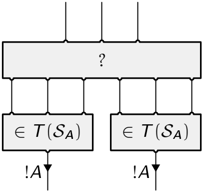
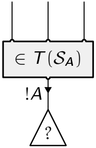


- 

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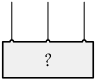
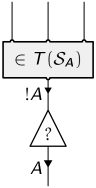


- 

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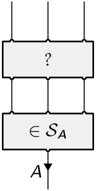
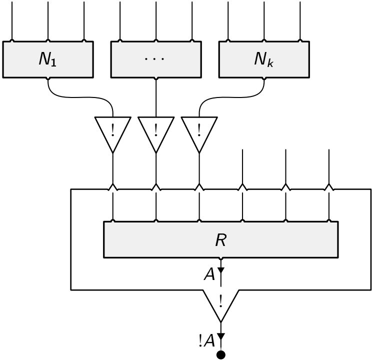


Normalization / WN of Full-DiLL

Lemma

Take any reducible pointed net R with [adequate interface], and let $N_1 \in [\![V_1]\!], \dots, N_k \in [\![V_k]\!]$, we have:



$$\in \ [\![!A]\!]$$

Conclusion

Result:

- A modular normalization proof for WN of Full-DiLL.
- Due to modularity this proof extends trivially to WN of "Second Order Full-DiLL with Additives".

Conclusion

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- A modular normalization proof for WN of Full-DiLL.
- Due to modularity this proof extends trivially to WN of "Second Order Full-DiLL with Additives".

Related work:

- Strong normalization property for second order linear logic [Pagani, Tortora de Falco, 2010]
- The Conservation Theorem for Differential Nets [Pagani, Tranquilli, to appear]

Pending work:

- Strong normalization of "Second Order Full-DiLL".

Conclusion

Result:

- A modular normalization proof for WN of Full-DiLL.
- Due to modularity this proof extends trivially to WN of "Second Order Full-DiLL with Additives".

Related work:

- Strong normalization property for second order linear logic [Pagani, Tortora de Falco, 2010]
- The Conservation Theorem for Differential Nets [Pagani, Tranquilli, to appear]

Pending work:

- Strong normalization of "Second Order Full-DiLL".

Thank you!