

# Cofibrant complexes are free

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## Abstract

We define a notion of cofibration among  $\infty$ -categories and show that the cofibrant objects are exactly the free ones, that is those generated by polygraphs.

## 1 Introduction

Polygraphs [Bur93], also known as computads [Str76, Pow91] are structured systems of generators for  $\infty$ -categories, extending the familiar notion of presentation by generators and relations beyond monoids or groups, and have recently proved extremely well-adapted to higher-dimensional rewriting [Gui06a, Gui06b].

They also lead to a simple definition of a homology for  $\infty$ -categories [Mét03, LM06], based on the following construction: a *polygraphic resolution* of an  $\infty$ -category  $C$  is a pair  $(S, p)$  where

- $S$  is a polygraph, generating a free  $\infty$ -category  $S^*$ ;
- the morphism  $p : S^* \rightarrow C$  is a trivial fibration (see 3.1).

$S$  gives rise to an abelian complex  $\mathbb{Z}S$ , whose homology only depends on  $C$ , so that we may define a polygraphic homology by

$$H_*^{\text{pol}}(C) =_{\text{def}} H_*(\mathbb{Z}S).$$

Here the main property of free  $\infty$ -categories is that they are *cofibrant*. In other words, given a polygraph  $S$  and a trivial fibration  $p : D \rightarrow C$ , any morphism  $f : S^* \rightarrow C$  lifts to a morphism  $g : S^* \rightarrow D$  (figure 1).

$$\begin{array}{ccc} & & D \\ & \nearrow g & \downarrow p \\ S^* & \xrightarrow{f} & C \end{array}$$

Figure 1: lifting

The purpose of the present work is to prove the converse, namely that all cofibrant  $\infty$ -categories are freely generated by polygraphs, thus establishing a simple, abstract characterization of the free objects, otherwise defined by a rather complex inductive construction.

We first give a brief review of the basic categories in play (section 2): **Glob**, **Compl** and **Pol** stand respectively for the category of globular sets,  $\infty$ -categories (or “complexes”) and polygraphs. Then we investigate trivial fibrations and cofibrations (section 3). In section 4, we reduce our theorem to the fact that the full subcategory of **Compl** whose objects are

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free is Cauchy-complete, in other words that all its idempotents split. This is proved in appendix B.

Let us sketch the Cauchy-completeness argument in the simpler case of monoids: thus, let  $\mathbf{Mon}$  denote the category of monoids, and  $\mathbf{Fmon}$  the full subcategory of  $\mathbf{Mon}$  whose objects are the free monoids. It is well-known that a submonoid of a free monoid is not necessarily free itself. However, if  $M = S^*$  is the free monoid on the alphabet  $S$  and  $h : M \rightarrow M$  is an *idempotent* endomorphism of  $M$ , then the submonoid  $\text{Fix}(h) = \{m \in M \mid h(m) = m\}$  of fixpoints of  $h$  is free, which easily leads to a splitting of  $h$  in  $\mathbf{Fmon}$ , hence to the fact that  $\mathbf{Fmon}$  is Cauchy-complete. Here the keypoint is to find a set of generators of  $\text{Fix}(h)$  without non-trivial relations in  $M$ . A simple way to build such a set is by considering the subset  $S_1 \subset S$  of those  $s \in S$  such that  $h(s) = usv$  where  $h(u) = h(v) = 1$ . Then we define  $T = \{h(s) \mid s \in S_1\}$ . It turns out that the obvious inclusion  $T^* \rightarrow M$  sends  $T^*$  isomorphically to  $\text{Fix}(h)$ , as shown by the existence of a retraction  $M \rightarrow T^*$ .

Now the same ideas carry into higher dimensions, with  $\infty$ -categories instead of monoids and polygraphs instead of generating sets. The general case involves additional technicalities, due to the presence of higher dimensional compositions. Of particular importance is the notion of *context*, defined and explored in appendix A.

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## 2 Basic categories

### 2.1 Globular sets

Let  $\mathbf{O}$  be the small category defined as follows:

- the objects of  $\mathbf{O}$  are integers  $0, 1, \dots$ ;
- the arrows are generated by composition of  $s_n, t_n : n \rightarrow n + 1$ ,  $n \in \mathbb{N}$  subject to the following equations

$$\begin{aligned} s_{n+1} \circ s_n &= t_{n+1} \circ s_n, \\ s_{n+1} \circ t_n &= t_{n+1} \circ t_n. \end{aligned}$$

As a consequence,  $\mathbf{O}(m, n)$  has exactly two elements if  $m < n$ , namely  $s_{m,n} = s_{n-1} \circ \dots \circ s_m$  and  $t_{m,n} = t_{n-1} \circ \dots \circ t_m$ .  $\mathbf{O}(m, n) = \emptyset$  if  $m > n$ , and contains the unique element  $\text{id}_m$  if  $m = n$ .

**Definition 1** A globular set is a presheaf on  $\mathbf{O}$ .

In other words, a globular set is a functor from  $\mathbf{O}^{\text{op}}$  to  $\mathbf{Sets}$ . Globular sets and natural transformations form a category  $\mathbf{Glob}$ . The Yoneda embedding

$$\mathbf{O} \rightarrow \mathbf{Glob}$$

takes each integer  $n$  to the *standard globe*  $O[n]$ . We still denote by  $s_n, t_n : O[n] \rightarrow O[n + 1]$  the morphisms of globular sets representing the corresponding arrows from  $n$  to  $n + 1$ .

Let  $X$  be a globular set and  $p$  an integer, the set  $X(p)$  will be denoted by  $X_p$ , and its elements called *cells of dimension  $p$*  or  *$p$ -cells*. Hence  $O[n]$  has exactly two  $p$ -cells for  $p < n$ , exactly one  $n$ -cell, and no  $p$ -cells for  $p > n$ . Let  $\partial O[n]$  be the globular set with the same cells as  $O[n]$  except for  $(\partial O[n])_n = \emptyset$ , and

$$i_n : \partial O[n] \rightarrow O[n]$$

the canonical injection:  $\partial O[n]$  has two  $p$ -cells for  $p < n$  and no other cells.

Let us point out a few facts about  $i_n$ :

- $s_n \circ i_n = t_n \circ i_n$ ;
- there are unique maps  $\widehat{s}_n$  and  $\widehat{t}_n$  such that  $s_n = i_{n+1} \circ \widehat{s}_n$  and  $t_n = i_{n+1} \circ \widehat{t}_n$ ;

- the following diagram is a pushout:

$$\begin{array}{ccc}
\partial O[n] & \xrightarrow{i_n} & O[n] \\
i_n \downarrow & & \downarrow \widehat{s}_n \\
O[n] & \xrightarrow{\widehat{t}_n} & \partial O[n+1]
\end{array}$$

Now let  $X$  be a globular set, Yoneda's lemma yields a natural equivalence

$$X_n \cong \mathbf{Glob}(O[n], X) \quad (1)$$

hence in particular  $s_n, t_n$  give rise to a double sequence of maps

$$\sigma_n, \tau_n : X_n \Leftarrow X_{n+1}$$

satisfying the *boundary conditions*:

$$\begin{aligned}
\sigma_n \circ \sigma_{n+1} &= \sigma_n \circ \tau_{n+1}, \\
\tau_n \circ \sigma_{n+1} &= \tau_n \circ \tau_{n+1}.
\end{aligned}$$

Whenever  $m < n$ , we set  $\sigma_{m,n} = \sigma_m \circ \cdots \circ \sigma_{n-1}$  and  $\tau_{m,n} = \tau_m \circ \cdots \circ \tau_{n-1}$ . Let  $0 \leq i < n$ , we say that the  $n$ -cells  $x, y \in X_n$  are  *$i$ -composable* if  $\tau_{i,n}x = \sigma_{i,n}y$ , a relation we denote by  $x \triangleright_i y$ .

If  $u \in X_n$  and  $\sigma_{n-1}(u) = x$ ,  $\tau_{n-1}(u) = y$ ,  $x$  and  $y$  are respectively the *source* and the *target* of  $u$ , which we simply denote by  $u : x \rightarrow y$ . Likewise, if  $\sigma_{i,n}u = x$  and  $\tau_{i,n}u = y$ , we shall write  $u : x \rightarrow_i y$ . In case  $u : x \rightarrow y$  and  $v : x \rightarrow y$ , we say that  $u, v$  are *parallel*, which we denote by  $u \parallel v$  (see figure 2). Any two 0-cells are also considered to be parallel. Let

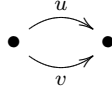


Figure 2: parallel cells

$\mathcal{P}_n(X)$  denote the set of ordered pairs of parallel  $n$ -cells in  $X$ . We get a natural equivalence

$$\mathcal{P}_n(X) \cong \mathbf{Glob}(\partial O[n+1], X) \quad (2)$$

similar to (1). The equivalences (1) and (2) assert that, for each  $n$ , the functors  $X \mapsto X_n$  and  $X \mapsto \mathcal{P}_n(X)$  from  $\mathbf{Glob}$  to  $\mathbf{Sets}$  are representable, the representing objects being respectively  $O[n]$  and  $\partial O[n+1]$ .

## 2.2 Complexes

Recall that an  $\infty$ -category is a globular set  $C$  endowed with

- a *product*  $u *_n v : x \rightarrow z$  defined for all  $u : x \rightarrow y$  and  $v : y \rightarrow z$  in  $C_n$ ;
- a *product*  $u *_i v : x *_i y \rightarrow z *_i t$  defined for all  $u : x \rightarrow z$  and  $v : y \rightarrow t$  in  $C_n$  with  $i < n - 1$  and  $u \triangleright_i v$ ;
- a *unit*  $1_{n+1}(x) : x \rightarrow x$  defined for all  $x \in C_n$ .

These operations satisfy the conditions of *associativity*, *left and right unit*, and *exchange*:

- $(x *_i y) *_i z = x *_i (y *_i z)$  for all  $x \triangleright_i y \triangleright_i z$  in  $C_n$  with  $i < n$ ;
- $1_{n,i}(x) *_i u = u = u *_i 1_{n,i}(y)$  for all  $u : x \rightarrow_i y$  in  $C_n$  with  $i < n$ , where  $1_{n,i} = 1_n \circ 1_{n-2} \circ \cdots \circ 1_{i+1}$ ;

- $(x *_i y) *_j (z *_i t) = (x *_j z) *_i (y *_j t)$  for all  $x, y, z, t \in C_n$  with  $i < j < n$  and  $x \triangleright_i y$ ,  $x \triangleright_j z$ ,  $y \triangleright_j t$ .

Throughout this work, *complex* means  $\infty$ -category. Let  $C, D$  be complexes. A *morphism*  $f : C \rightarrow D$  is a morphism of the underlying globular sets preserving units and products. Complexes and morphisms build a category **Compl**, and there is an obvious forgetful functor **Compl**  $\rightarrow$  **Glob**. Its left adjoint **Glob**  $\rightarrow$  **Compl** associates to each globular set the *free complex* generated by it. Note that **Glob** is a topos of presheaves and that the forgetful functor **Compl**  $\rightarrow$  **Glob** is finitary monadic over **Glob**. Hence **Compl** is complete and cocomplete, and we shall take limits and colimits in **Compl** without further explanations (see also [Bat98, Str00]).

By restricting a complex  $C$  to its cells of dimension  $\leq n$ , we get an *n-category*

$$C|_n : C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n.$$

This *n-category* can be extended to a complex  $C^{(n)}$  by adjoining units to  $C|_n$  in all dimensions  $> n$ :

$$C^{(n)} : C_0 \leftarrow \cdots \leftarrow C_n \leftarrow C_n \leftarrow \cdots .$$

Let us call  $C^{(n)}$  the *n-skeleton* of  $C$ . It will be convenient to define  $C^{(-1)}$  as the initial complex  $0$  with no cells. There is a canonical inclusion

$$j_n : C^{(n)} \rightarrow C^{(n+1)}.$$

Here again  $j_{-1}$  denotes the unique morphism  $0 \rightarrow C^{(0)}$ .

The following result is an easy consequence of the definitions:

**Lemma 1** *Any complex  $C$  is the colimit of its n-skeleta:*

$$C^{(-1)} \xrightarrow{j_{-1}} C^{(0)} \xrightarrow{j_0} \cdots \xrightarrow{j_{n-1}} C^{(n)} \xrightarrow{j_n} \cdots$$

## 2.3 Polygraphs

Let us describe a process of attaching  $n+1$ -cells to an *n-category*  $C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n$ . Let  $S_{n+1}$  be a set, and  $\sigma_n, \tau_n : C_n \leftarrow S_{n+1}$  a graph where  $\sigma_n, \tau_n$  satisfy the boundary conditions  $\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n$  and  $\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n$ . We build the free  $n+1$ -category  $C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow S_{n+1}^*$ , where  $S_{n+1}^*$  consists of formal compositions of elements of  $S_{n+1}$ , including identities on cells of  $C_n$ , and subject to the equations of units, associativity and exchange. We refer to [Bur93] or [Mét03] for formal definitions.

Now *n-polygraphs* and free generated *n-categories* are defined by simultaneous induction on  $n$ :

- a 0-polygraph is a set  $S_0$ , generating the 0-category (i.e. set)  $S_0^* = S_0$ ;
- given an *n-polygraph*  $S_0^* \leftarrow S_1, \dots, S_{n-1}^* \leftarrow S_n$  with the free *n-category*  $S_0^* \leftarrow \cdots \leftarrow S_n^*$  it generates, an  $n+1$ -polygraph is determined by a graph  $\sigma_n, \tau_n : S_n^* \leftarrow S_{n+1}$  satisfying the boundary conditions, and the free  $n+1$ -category generated by it is  $S_0^* \leftarrow S_1^* \leftarrow \cdots \leftarrow S_n^* \leftarrow S_{n+1}^*$ .

In particular, a 1-polygraph is simply a graph, and the notion of free 1-category generated by it coincides with the usual notion of a free category generated by a graph.

**Definition 2** *A polygraph is an infinite sequence*

$$S : S_0^* \leftarrow S_1, S_1^* \leftarrow S_2, \dots, S_n^* \leftarrow S_{n+1}, \dots$$

*whose first  $n$  items define an n-polygraph for each  $n$ .*

A *free complex* is a complex generated by a polygraph, that is of the form

$$S^* : S_0^* \leftarrow S_1^* \leftarrow \cdots \leftarrow S_n^* \leftarrow S_{n+1}^* \cdots .$$

Let  $S, T$  be polygraphs. A morphism  $f : S \rightarrow T$  amounts to a sequence of maps  $f_n : S_n \rightarrow T_n$  such that, for all  $\xi : x \rightarrow y$  in  $S_n$ ,  $f_n(\xi) : f_{n-1}^*(x) \rightarrow f_{n-1}^*(y)$ , where  $f_n^*$  is the unique extension of  $f_n$  which is compatible with products and units.

We denote by **Pol** the category of polygraphs and morphisms. The functor

$$\begin{aligned} \mathbf{Pol} &\rightarrow \mathbf{Compl}, \\ S &\mapsto S^*, \end{aligned}$$

is left-adjoint to a forgetful functor

$$\begin{aligned} \mathbf{Compl} &\rightarrow \mathbf{Pol}, \\ C &\mapsto |C|. \end{aligned}$$

A detailed description of  $C \mapsto |C|$  is given in [Mét03], where this functor is called  $P$ .

Remark that any globular set  $X$  can be viewed as a particular polygraph and that this identification makes **Glob** a full subcategory of **Pol**. Moreover the free complex generated by a globular set is the same as the free complex generated by the corresponding polygraph. However most free complexes generated by polygraphs cannot be generated by globular sets alone.

For instance the globular sets  $O[n]$  and  $\partial O[n]$  can be viewed as polygraphs, and generate complexes  $O[n]^*$  and  $\partial O[n]^*$ . Remark that in this case, the free construction does not create new non-trivial cells. Therefore, from now on, we drop the “\*” in the notation of these complexes. Likewise,  $i_n$  will denote a morphism of globular sets, polygraphs, or complexes according to the context. Note also that the natural equivalences (1) and (2) extend to **Compl**:

$$C_n \cong \mathbf{Compl}(O[n], C) \quad (3)$$

$$\mathcal{P}_n(C) \cong \mathbf{Compl}(\partial O[n+1], C) \quad (4)$$

Let  $S$  be a polygraph,  $S^*$  the free complex it generates, and  $n$  an integer. By  $\sum_{S_n} \partial O[n]$  (resp.  $\sum_{S_n} O[n]$ ), we mean the direct sum of copies of  $\partial O[n]$  (resp.  $O[n]$ ) indexed by the elements of  $S_n$ . As a consequence of (4), the source and target maps  $S_{n-1}^* \leftarrow S_n$  determine a morphism

$$\rho : \sum_{S_n} \partial O[n] \rightarrow (S^*)^{(n-1)}.$$

Then the following result is merely a reformulation of the definition of polygraphs:

**Lemma 2** *The diagram*

$$\begin{array}{ccc} \sum_{S_n} \partial O[n] & \xrightarrow{\rho} & (S^*)^{(n-1)} \\ \sum_{S_n} i_n \downarrow & & \downarrow j_n \\ \sum_{S_n} O[n] & \longrightarrow & (S^*)^{(n)} \end{array}$$

is a pushout in **Compl**.

### 3 Two classes of morphisms

Let **C** be a category, and  $f : A \rightarrow B$ ,  $g : C \rightarrow D$  morphisms.  $f$  is *left-orthogonal* to  $g$  (or, equivalently,  $g$  is *right-orthogonal* to  $f$ ) if, for each pair of morphisms  $u : A \rightarrow C$ ,  $v : B \rightarrow D$  such that  $g \circ u = v \circ f$ , there exists an  $h : B \rightarrow C$  making the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & \nearrow h & \downarrow g \\ B & \xrightarrow{v} & D \end{array}$$

For any class  $\mathcal{M}$  of morphisms in **C**,  ${}^\perp \mathcal{M}$  (resp.  $\mathcal{M}^\perp$ ) denotes the class of morphisms in **C** which are left-(resp. right-) orthogonal to all morphisms in  $\mathcal{M}$ .

### 3.1 Trivial fibrations

Let  $\mathcal{I}$  be the set  $\{i_n | n \in \mathbb{N}\}$  as morphisms in **Compl**.

**Definition 3** A morphism of complexes is a trivial fibration if and only if it belongs to  $\mathcal{I}^\perp$ .

In other words,  $p : C \rightarrow D$  is a trivial fibration if for all  $n$ ,  $f : \partial O[n] \rightarrow C$ , and  $g : O[n] \rightarrow D$  such that  $p \circ f = g \circ i_n$ , there is an  $h : O[n] \rightarrow C$  making the following diagram commutative:

$$\begin{array}{ccc} \partial O[n] & \xrightarrow{f} & C \\ i_n \downarrow & \nearrow h & \downarrow p \\ O[n] & \xrightarrow{g} & D \end{array}$$

**Definition 4** Let  $C$  be a complex. A polygraphic resolution of  $C$  is a pair  $(S, p)$  where  $S$  is a polygraph and  $p : S^* \rightarrow C$  is a trivial fibration.

It was shown in [Mét03] that, for each complex  $C$ , the counit of the adjunction  $(\cdot)^* \dashv |\cdot|$ ,

$$\epsilon_C : |C|^* \rightarrow C,$$

is a trivial fibration. Hence  $(|C|, \epsilon_C)$  is a polygraphic resolution of  $C$ , and we get the following result, which will play an essential part in section 4 below :

**Proposition 1** Each complex  $C$  has a polygraphic resolution.

### 3.2 Cofibrations

**Definition 5** A morphism of complexes is a cofibration if and only if it is left-orthogonal to all trivial fibrations.

Hence the class of cofibrations is exactly  ${}^\perp(\mathcal{I}^\perp)$ . Immediate examples of cofibrations are the  $i_n$ 's themselves. The following lemma summarizes standard properties of maps defined by left-orthogonality conditions.

**Lemma 3** Let  $\mathbf{C}$  be a category, and  $\mathcal{M}$  an arbitrary class of morphisms of  $\mathbf{C}$ . Let  $\mathcal{L} = {}^\perp \mathcal{M}$ . Then

- $\mathcal{L}$  is stable by direct sums: if  $f_i : X_i \rightarrow Y_i$ ,  $i \in I$  is a family of maps of  $\mathcal{L}$  with direct sum  $f = \sum_{i \in I} f_i : \sum_{i \in I} X_i \rightarrow \sum_{i \in I} Y_i$ , then  $f \in \mathcal{L}$ ;
- $\mathcal{L}$  is stable by pushout: whenever  $f \in \mathcal{L}$  and

$$\begin{array}{ccc} X & \longrightarrow & Z \\ f \downarrow & & \downarrow g \\ Y & \longrightarrow & T \end{array}$$

is a pushout square in  $\mathbf{C}$ , then  $g \in \mathcal{L}$ ;

- suppose

$$X_0 \xrightarrow{l_0} \dots \xrightarrow{l_{n-1}} X_n \xrightarrow{l_n} \dots$$

is a sequence of maps  $l_n \in \mathcal{L}$ , with colimit  $(X, m_n : X_n \rightarrow X)$ . Then  $m_0 : X_0 \rightarrow X$  belongs to  $\mathcal{L}$ .

**Proof.** We leave the first two claims as exercises. As for the third point, let  $f : Y \rightarrow Z$  be a morphism in  $\mathcal{M}$ , and  $u : X_0 \rightarrow Y$ ,  $v : X \rightarrow Z$  such that the following diagram commutes:

$$\begin{array}{ccc} X_0 & \xrightarrow{u} & Y \\ m_0 \downarrow & & \downarrow f \\ X & \xrightarrow{v} & Z \end{array}$$

Let us define  $v_n = v \circ m_n$  for each  $n \geq 0$ . Thus, for each  $n \geq 0$ ,  $v_{n+1} \circ l_n = v \circ m_{n+1} \circ l_n = v \circ m_n = v_n$ , so that  $v_n : X_n \rightarrow Z$  determines an inductive cone on the base  $(X_n)$  to the vertex  $Z$ . Let us define a family of maps  $u_n : X_n \rightarrow Y$  satisfying the following equations:

$$f \circ u_n = v_n, \quad (5)$$

$$u_{n+1} \circ l_n = u_n. \quad (6)$$

Let  $n = 0$ . Define  $u_0 = u$ . We get  $f \circ u_0 = f \circ u = v \circ m_0 = v_0$ , and (5) holds. Thus  $f \circ u_0 = v_1 \circ l_0$ , and because  $f \in \mathcal{M}$  and  $l_0 \in \mathcal{L}$ , there is an  $u_1 : X_1 \rightarrow Y$  such that  $u_1 \circ l_0 = u_0$ , so that (6) holds. Suppose now that (5) and (6) hold for an  $n \geq 0$ . By the induction hypothesis, the following diagram commutes:

$$\begin{array}{ccc} X_n & \xrightarrow{u_n} & Y \\ l_n \downarrow & & \downarrow f \\ X_{n+1} & \xrightarrow{v_{n+1}} & Z \end{array}$$

with  $f \in \mathcal{M}$  and  $l_n \in \mathcal{L}$ . Hence there is a  $u_{n+1} : X_{n+1} \rightarrow Y$  such that  $f \circ u_{n+1} = v_{n+1}$  and  $u_{n+1} \circ l_n = u_n$ , and our equations hold for  $n + 1$ . In particular, (6) means that  $(u_n)$  determines an inductive cone on the base  $(X_n)$  to the vertex  $Y$ . As  $X$  is the colimit of the  $X_n$ 's, there is a morphism  $h : X \rightarrow Y$  such that, for each  $n \geq 0$ ,  $h \circ m_n = u_n$ . In particular,  $h \circ m_0 = u_0 = u$ . Also, for each  $n \geq 0$ ,  $f \circ h \circ m_n = f \circ u_n = v_n = v \circ m_n$ . Uniqueness of connecting morphisms show that  $f \circ h = v$ . Hence the following diagram is commutative

$$\begin{array}{ccc} X_0 & \xrightarrow{u} & Y \\ m_0 \downarrow & \nearrow h & \downarrow f \\ X & \xrightarrow{v} & Z \end{array}$$

and we have shown that  $m_0 \in \mathcal{L}$ , as required.  $\triangleleft$

**Definition 6** A complex  $C$  is cofibrant if  $0 \rightarrow C$  is a cofibration.

**Proposition 2** Free complexes are cofibrant.

**Proof.** Let  $S$  be a polygraph and  $C = S^*$ . By lemma 2, for each  $n \geq -1$ , the canonical inclusion  $j_n : C^{(n)} \rightarrow C^{(n+1)}$  is a pushout of  $\sum_{S_n} i_n$ . Now lemma 3 applies in the particular case where  $\mathcal{L}$  is the class of cofibrations: by the first point,  $\sum_{S_n} i_n$  is a cofibration, and by the second point, so is  $j_n$ . By lemma 1,  $C$  is a colimit of the sequence

$$C^{(-1)} \xrightarrow{j_{-1}} C^{(0)} \xrightarrow{j_0} \dots \xrightarrow{j_{n-1}} C^{(n)} \xrightarrow{j_n} \dots$$

hence the third point of lemma 3 applies, with  $X_n = C^{(n-1)}$  and  $l_n = j_{n-1}$ , so that  $0 \rightarrow C$  is a cofibration. In other words,  $C$  is cofibrant.  $\triangleleft$

## 4 Main result

The main goal of this work is to establish the converse of proposition 2:

**Theorem 1** Any cofibrant complex is isomorphic to a free one.

Let **Fcompl** denote the full subcategory of **Compl** whose objects are the free complexes  $S^*$  generated by polygraphs. Then, theorem 1 reduces to the following statement:

**Theorem 2** **Fcompl** is Cauchy-complete.

Recall that a category  $\mathbf{C}$  is *Cauchy-complete* if all its idempotents split, that is, for each object  $C$ , and each endomorphism  $h : C \rightarrow C$  such that  $h \circ h = h$ , there is an object  $D$ , together with morphisms  $r : C \rightarrow D$ ,  $u : D \rightarrow C$ , satisfying

$$\begin{aligned} u \circ r &= h, \\ r \circ u &= \text{id}. \end{aligned}$$

Theorem 2 will be proved in annex B. Let us assume the result for the moment, and let  $C$  be a cofibrant complex. By proposition 1,  $C$  has a free resolution  $p : S^* \rightarrow C$ , with  $S^*$  an object of  $\mathbf{Fcompl}$ . Because  $C$  is cofibrant, and  $p$  is a trivial fibration, the identity morphism  $\text{id}_C : C \rightarrow C$  lifts through  $p$ , whence a morphism  $q : C \rightarrow S^*$  such that  $p \circ q = \text{id}_C$ . Let  $h = q \circ p$ ,  $h \circ h = q \circ p \circ q \circ p = q \circ \text{id}_C \circ p = q \circ p = h$ , hence  $h$  is an idempotent endomorphism of  $S^*$ . By using Cauchy completeness, we get a polygraph  $T$ , and morphisms  $r : S^* \rightarrow T^*$ ,  $u : T^* \rightarrow S^*$  such that  $r \circ u = \text{id}_{T^*}$  and  $u \circ r = h$ . Now, let  $f = p \circ u : T^* \rightarrow C$  and  $g = r \circ q : C \rightarrow T^*$ . We get

$$\begin{aligned} g \circ f &= r \circ q \circ p \circ u \\ &= r \circ h \circ u \\ &= r \circ u \circ r \circ u \\ &= \text{id}_{T^*} \circ \text{id}_{T^*} \\ &= \text{id}_{T^*}. \end{aligned}$$

Likewise

$$\begin{aligned} f \circ g &= p \circ u \circ r \circ q \\ &= p \circ h \circ q \\ &= p \circ q \circ p \circ q \\ &= \text{id}_C \circ \text{id}_C \\ &= \text{id}_C. \end{aligned}$$

Hence  $f : T^* \rightarrow C$  is an isomorphism with inverse  $g = f^{-1}$  so that  $C$  is isomorphic to a free object, as required.

## References

- [Bat98] Michael Batanin. Computads for finitary monads on globular sets. *Contemp. Math.*, 230:37–57, 1998.
- [Bur93] Albert Burroni. Higher-dimensional word problems with applications to equational logic. *Theoretical Computer Science*, 115:43–62, 1993.
- [Gui06a] Yves Guiraud. The three dimensions of proofs. *Annals of Pure and Applied Logic*, 141(1–2):266–295, 2006.
- [Gui06b] Yves Guiraud. Two polygraphic presentations of Petri nets. *Theoretical Computer Science*, 360(1–3):124–146, 2006.
- [LM06] Yves Lafont and François Métayer. Polygraphic resolutions and homology of monoids. submitted, 2006.
- [Mét03] François Métayer. Resolutions by polygraphs. *Theory and Applications of Categories*, 11(7):148–184, 2003. <http://www.tac.mta.ca/tac/>.
- [Pow91] John Power. An  $n$ -categorical pasting theorem. In *Category Theory, Proc. Int. Conf., Como/Italy 1990*, number 1488 in Lect. Notes Math., pages 326–358, 1991.
- [Str76] Ross Street. Limits indexed by category-valued 2-functors. *Journal of Pure and Applied Algebra*, 8:149–181, 1976.
- [Str00] Ross Street. The petit topos of globular sets. *Journal of Pure and Applied Algebra*, 154(1–3):299–315, 2000.

# A Contexts

## A.1 Indeterminates

Let  $C$  be a complex, and  $n \geq 1$ . An  $n$ -type is an ordered pair  $(x, y)$  of parallel cells in  $C_{n-1}$ , that is an element of  $\mathcal{P}_{n-1}(C)$ . By (4),  $n$ -types amount to morphisms  $\theta : \partial O[n] \rightarrow C$ . We shall use the same notations for both sides of the natural equivalences (3) and (4).

**Definition 7** *The type of an  $n$ -cell  $x \in C_n$  is the pair  $(\sigma_{n-1}x, \tau_{n-1}x)$ .*

Hence the type of an  $n$ -cell is a particular  $n$ -type.

Given an  $n$ -type  $\theta$ , we may adjoin to  $C$  an *indeterminate  $n$ -cell of type  $\theta$*  by taking the following pushout in **Compl**:

$$\begin{array}{ccc} \partial O[n] & \xrightarrow{\theta} & C \\ i_n \downarrow & & \downarrow j_\theta \\ O[n] & \xrightarrow{\mathbf{x}} & C[\mathbf{x}] \end{array}$$

We let boldface variables  $\mathbf{x}, \mathbf{y}, \dots$  range over indeterminates.

Let  $\mathbf{x}$  be an indeterminate  $n$ -cell of type  $\theta$  and  $z : O[n] \rightarrow C$  an  $n$ -cell in  $C$ . To say that  $z$  is of type  $\theta$  amounts to the commutativity of the following diagram:

$$\begin{array}{ccc} \partial O[n] & \xrightarrow{\theta} & C \\ i_n \downarrow & & \downarrow \text{id} \\ O[n] & \xrightarrow{z} & C \end{array}$$

The pushout property gives a unique morphism  $\text{sub}_z : C[\mathbf{x}] \rightarrow C$  such that  $\text{sub}_z \circ \mathbf{x} = z$  and  $\text{sub}_z \circ j_\theta = \text{id}$ .  $\text{sub}_z$  is nothing but the operation of substituting the cell  $z$  for  $\mathbf{x}$  (see figure 3). Now  $n$ -cells of  $C[\mathbf{x}]$  are formal composites of elements in  $C_n \cup \{\mathbf{x}\}$ . Different expressions may

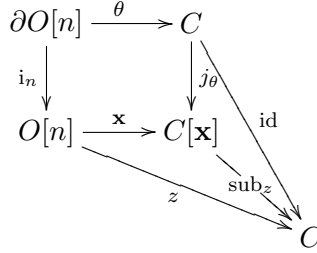


Figure 3: substitution

denote the same cell: however all those expressions contain the same number of occurrences of  $\mathbf{x}$ .

**Definition 8** *An  $n$ -context over  $\mathbf{x}$  is an  $n$ -cell of  $C[\mathbf{x}]$  having exactly one occurrence of  $\mathbf{x}$ .*

We denote  $n$ -contexts over  $\mathbf{x}$  by  $c[\mathbf{x}], d[\mathbf{x}], \dots$ . An  $n$ -cell  $z$  of  $C$  is *adapted* to the context  $c[\mathbf{x}]$  if it has the same type as  $\mathbf{x}$ . Contexts are subject to the following operations:

- for each  $n$ -context  $c[\mathbf{x}] : O[n] \rightarrow C[\mathbf{x}]$  and each adapted  $n$ -cell  $z$ , we denote by  $c[z]$  the new  $n$ -cell of  $C$  obtained by substituting  $z$  for  $\mathbf{x}$ , in other terms  $c[z] = \text{sub}_z \circ c[\mathbf{x}]$ ;

- let  $u : C \rightarrow D$  be a morphism of complexes and  $c[\mathbf{x}] : O[n] \rightarrow C[\mathbf{x}]$  an  $n$ -context of  $C$ . Define a new indeterminate  $\mathbf{y} : O[n] \rightarrow D[\mathbf{y}]$  by the following pushout square

$$\begin{array}{ccc} \partial O[n] & \xrightarrow{u \circ \theta} & D \\ i_n \downarrow & & \downarrow j_{u \circ \theta} \\ O[n] & \xrightarrow{\mathbf{y}} & D[\mathbf{y}] \end{array} .$$

This determines a unique morphism

$$\hat{u} : C[\mathbf{x}] \rightarrow D[\mathbf{y}]$$

such that  $j_{u \circ \theta} \circ u = \hat{u} \circ j_\theta$  and  $\hat{u} \circ \mathbf{x} = \mathbf{y}$  (see figure 4). Thus  $\hat{u} \circ c[\mathbf{x}]$  is an  $n$ -context

$$\begin{array}{ccccc} \partial O[n] & \xrightarrow{\theta} & C & & \\ i_n \downarrow & & \downarrow j_\theta & \searrow u & \\ O[n] & \xrightarrow{\mathbf{x}} & C[\mathbf{x}] & & D \\ & \searrow \mathbf{y} & \searrow \hat{u} & & \downarrow j_{\theta \circ u} \\ & & & & D[\mathbf{y}] \end{array}$$

Figure 4: context transformation

over  $\mathbf{y}$ , denoted by  $c^u[\mathbf{y}]$ .

Note that, if  $z$  is an  $n$ -cell adapted to  $c[\mathbf{x}]$ , then  $u(z)$  is adapted to  $c^u[\mathbf{y}]$  and

$$u(c[z]) = c^u[u(z)]. \quad (7)$$

## A.2 Thin contexts

Let us introduce a few additional terminology about cells and contexts. If  $S$  is a polygraph, the elements of  $S_n^*$  are the cells of *dimension*  $n$ , or  $n$ -cells. The *generators of dimension*  $n$ , or  $n$ -*generators* are the elements of  $S_n$ . Each  $n$ -generator  $\alpha$  determines an  $n$ -cell  $\alpha^*$ . Such cells are called *atomic*. All 0-cells are atomic, and if  $n > 0$ , each  $n$ -cell may be expressed as a composition of atomic cells and units on  $n-1$ -cells. For each  $n$ -cell  $x$ , and generator  $\alpha$ , the number of occurrences of  $\alpha^*$  in an expression of  $x$  only depends on  $x$ , not on the particular expression. We call this number the *weight of  $x$  at  $\alpha$* , and denote it by  $w_\alpha(x)$ . The *total weight of  $x$*  is

$$w(x) = \sum_{\alpha \in S_n} w_\alpha(x).$$

The same definitions hold for contexts, where we take into account all generators but the indeterminate. Thus, for instance.  $w(\mathbf{x}) = 0$  for any indeterminate  $\mathbf{x}$ .

**Definition 9** An  $n$ -context  $c[\mathbf{x}]$  is thin if its total weight is zero.

Now, if  $x \in S_n^*$ , either  $w(x) > 0$  or there is a cell  $y \in S_{n-1}^*$  such that  $x = 1_n(y)$ . More generally, if  $w(x) = 0$ , there is a unique integer  $p < n$  with the following property:

- there is a  $p$ -cell  $z$  in  $S^*$  such that  $w(z) > 0$  and  $x = 1_{n,p}(z)$

Let us call  $p$  the *thickness of  $x$* , and denote it by  $p = \text{th}(x)$ . If  $w(x) \neq 0$ , we define  $\text{th}(x) = n$ . The same definitions immediately apply to contexts. In particular, an  $n$ -context  $c[\mathbf{x}]$  is thin if and only if  $\text{th}(c[\mathbf{x}]) < n$ . We finally associate to each cell  $x$  an integer *size*( $x$ ) by:

- if  $w(x) \neq 0$ ,  $\text{size}(x) = w(x)$ ;

- if  $w(x) = 0$ ,  $p = \text{th}(x)$ , and  $z$  is the unique cell in  $S_p^*$  such that  $x = 1_{n,p}(z)$ , then  $\text{size}(x) = w(z)$ .

In other words, the size of a cell  $x$  is the number of generators of maximal dimension needed to express  $x$ . The size of contexts is defined accordingly. Thus, the only contexts of size zero are just indeterminates. We call those contexts *trivial*.

**Lemma 4** *If  $n > 1$  and  $c[\mathbf{x}]$  is a thin  $n$ -context, there is an  $n-1$ -context  $d[\mathbf{y}]$  such that  $\text{size}(d[\mathbf{y}]) = \text{size}(c[\mathbf{x}])$  and for each adapted  $n$ -cell  $z$ ,*

$$\sigma_{n-1}(c[z]) = d[\sigma_{n-1}(z)].$$

**Proof.** Let  $\mathbf{x}$  be an  $n$ -indeterminate of type  $\theta = (x, y)$ . We define a family  $(\mathcal{C}_i)_{0 \leq i \leq n}$  of sets of  $n$ -contexts over  $\mathbf{x}$  by:

- $\mathcal{C}_0 = \{\mathbf{x}\}$ ;
- $\mathcal{C}_{i+1} = \{a *_i c[\mathbf{x}] *_i b\} \cup \{a *_i c[\mathbf{x}]\} \cup \{c[\mathbf{x}] *_i b\}$ , where  $c[\mathbf{x}] \in \mathcal{C}_i$ , and  $a, b$  are  $n$ -cells of  $S^*$  such that  $a \triangleright_i c[\mathbf{x}] \triangleright_i b$ ,  $\text{th}(a) > i$  and  $\text{th}(b) > i$ .

Note that whenever  $\text{th}(a) \leq i$  (resp.  $\text{th}(b) \leq i$ ),  $a *_i c[\mathbf{x}] = c[\mathbf{x}]$  (resp.  $c[\mathbf{x}] *_i b = c[\mathbf{x}]$ ). Also the exchange rule allows to perform compositions along higher dimensions outside those along lower dimensions. Hence  $\bigcup_{0 \leq i \leq n} \mathcal{C}_i$  contains all  $n$ -contexts on  $\mathbf{x}$ . As contexts in  $\mathcal{C}_n$  cannot be thin, all thin contexts belong to  $\bigcup_{0 \leq i \leq n-1} \mathcal{C}_i$ . Thus the lemma reduces to the following statement:

- given  $n > 1$ ,  $i \in \{0, \dots, n-1\}$ , and a thin  $n$ -context  $c[\mathbf{x}] \in \mathcal{C}_i$ , there is an  $n-1$ -context  $d[\mathbf{y}]$  such that  $\text{size}(d[\mathbf{y}]) = \text{size}(c[\mathbf{x}])$  and, for each adapted  $n$ -cell  $z$ ,  $\sigma_{n-1}(c[z]) = d[\sigma_{n-1}(z)]$ .

We prove this by induction on  $i \in \{0, \dots, n-1\}$ .

- If  $i = 0$ ,  $c[\mathbf{x}] = \mathbf{x}$  and we take  $d[\mathbf{y}] = \mathbf{y}$  where  $\mathbf{y}$  is an  $n-1$ -indeterminate of type  $\phi = (\sigma_{n-2}(x), \tau_{n-2}(x))$ .  $\text{size}(d[\mathbf{y}]) = \text{size}(c[\mathbf{x}]) = 0$  and  $\sigma_{n-1}(c[z]) = \sigma_{n-1}(z) = d[\sigma_{n-1}(z)]$ .
- Suppose  $0 < i \leq n-1$  and the property holds for  $i-1$ . Let  $c[\mathbf{x}] \in \mathcal{C}_i$  be a thin context. Then  $c[\mathbf{x}]$  is of the form  $a *_i c'[\mathbf{x}] *_i b$  or  $a *_i c'[\mathbf{x}]$  or  $c'[\mathbf{x}] *_i b$ , where  $c'[\mathbf{x}] \in \mathcal{C}_{i-1}$ . We only treat the first case, the other two being very similar. Because  $c[\mathbf{x}]$  is thin, so is  $c'[\mathbf{x}]$ , and  $w(a) = w(b) = 0$ . Hence

$$\begin{aligned} \text{size}(a) &= \text{size}(\sigma_{n-1}(a)), \\ \text{size}(b) &= \text{size}(\sigma_{n-1}(b)). \end{aligned}$$

By the induction hypothesis, there is an  $n-1$ -context  $d'[\mathbf{y}]$  such that  $\text{size}(d'[\mathbf{y}]) = \text{size}(c'[\mathbf{x}])$  and, for each adapted  $n$ -cell  $z$ ,  $\sigma_{n-1}(c'[z]_\theta) = d'[\sigma_{n-1}(z)]_\phi$ . As  $i-1 < n-1$ , we may define

$$d[\mathbf{y}] = \sigma_{n-1}(a) *_i d'[\mathbf{y}] *_i \sigma_{n-1}(b).$$

We get

$$\begin{aligned} \text{size}(d[\mathbf{y}]) &= \text{size}(\sigma_{n-1}(a)) + \text{size}(d'[\mathbf{y}]) + \text{size}(\sigma_{n-1}(b)) \\ &= \text{size}(a) + \text{size}(d'[\mathbf{y}]) + \text{size}(b) \\ &= \text{size}(a) + \text{size}(c'[\mathbf{x}]) + \text{size}(b) \\ &= \text{size}(c[\mathbf{x}]) \end{aligned}$$

and we get, for each adapted  $n$ -cell  $z$ ,

$$\sigma_{n-1}(c[z]) = d[\sigma_{n-1}(z)].$$

◁

**Lemma 5** *Let  $c[\mathbf{x}]$  be an  $n$ -context and  $z$  an adapted  $n$ -cell. If  $c[z] = z$ , then  $c[\mathbf{x}]$  is trivial.*

**Proof.** By induction on the dimension  $n$ . If  $n = 1$ , all contexts are trivial and we are done. Suppose now  $n > 1$  and the result holds in dimension  $n - 1$ . Let  $c[\mathbf{x}]$  be an  $n$ -context and  $z$  an adapted  $n$ -cell such that

$$c[z] = z \tag{8}$$

If  $w(c[\mathbf{x}]) > 0$ , then either  $w(z) > 0$  and  $\text{size}(c[z]) > \text{size}(z)$ , or  $w(z) = 0$  and  $\text{th}(c[z]) = n > \text{th}(z)$ . In both cases,  $c[z] \neq z$ , a contradiction, and we are done. Otherwise,  $c[\mathbf{x}]$  is thin, and lemma 4 gives an  $n-1$ -context  $d[\mathbf{y}]$  having the same size as  $c[\mathbf{x}]$  and satisfying  $\sigma_{n-1}(c[z]) = d[\sigma_{n-1}(z)]$ . Hence, taking the source on both sides of (8), we get

$$d[\sigma_{n-1}(z)] = \sigma_{n-1}(z).$$

Thus, by the induction hypothesis,  $d[\mathbf{y}]$  is a trivial  $n-1$ -context, hence  $\text{size}(d[\mathbf{y}]) = 0$ . Therefore  $\text{size}(c[\mathbf{x}]) = 0$  as well, and  $c[\mathbf{x}]$  is trivial.  $\triangleleft$

The following technical lemma will be crucial in the proof of theorem 2:

**Lemma 6** *Let  $c[\mathbf{x}]$  be a thin  $n$ -context, and  $z$  an adapted  $n$ -cell. If  $c[z]$  is parallel to  $z$ , then  $c[z] = z$ .*

**Proof.** Suppose  $c[\mathbf{x}]$  is a thin  $n$ -context, and  $z$  is an adapted  $n$ -cell such that  $c[z] \parallel z$ . If  $n = 1$ , thin contexts are trivial and the result is immediate. Otherwise,  $n > 1$  and by lemma 4, there is an  $n-1$ -context  $d[\mathbf{y}]$  such that  $\text{size}(d[\mathbf{y}]) = \text{size}(c[\mathbf{x}])$  and  $\sigma_{n-1}(c[z]) = d[\sigma_{n-1}(z)]$ . As  $c[z]$  is parallel to  $z$ , this implies  $d[\sigma_{n-1}(z)] = \sigma_{n-1}(z)$ , and by lemma 5,  $d[\mathbf{y}]$  is a trivial context. Now  $\text{size}(d[\mathbf{y}]) = \text{size}(c[\mathbf{x}]) = 0$  so that  $c[\mathbf{x}]$  is trivial, and  $c[z] = z$ .  $\triangleleft$

## B Cauchy completeness

This section is devoted to the proof of theorem 2. Thus, let  $S$  be a polygraph, and  $h : S^* \rightarrow S^*$  an idempotent morphism in **Compl**. We need to build a polygraph  $T$ , together with morphisms  $u : T^* \rightarrow S^*$  and  $r : S^* \rightarrow T^*$  such that

$$r \circ u = \text{id}, \tag{9}$$

$$u \circ r = h. \tag{10}$$

We shall define  $T$ ,  $u$  and  $r$  inductively on the dimension. In dimension 0,

$$T_0 = \{h(x) \mid x \in S_0^* = S_0\},$$

$u$  is the inclusion  $T_0^* = T_0 \rightarrow S_0^* = S_0$ , and for each  $x \in S_0$ ,  $r(x) = h(x)$ . The equations (9) and (10) are clearly satisfied.

Suppose now that  $n > 0$  and  $T$ ,  $u$ ,  $r$  have been defined up to dimension  $n-1$ , and satisfy the required conditions. We shall extend the  $n-1$  polygraph  $T$  to an  $n$ -polygraph, and the morphisms  $u$ ,  $r$  of  $n-1$ -complexes to morphisms of  $n$ -complexes still satisfying the above equations.

$\triangleright$  *Step 1.* Let us split  $S_n$  in three subsets  $S_n^0$ ,  $S_n^1$  and  $S_n^2$ , according to the value of  $h(\alpha^*)$ , for  $\alpha \in S_n$ :

- $S_n^0 = \{\alpha \in S_n \mid w(h(\alpha^*)) = 0\}$ , hence  $S_n^0$  contains the generators  $\alpha$  such that  $h(\alpha^*)$  is degenerate;
- $S_n^1$  contains the generators  $\alpha \in S_n$  such that  $w_\alpha(h(\alpha^*)) = 1$  and  $w_\beta(h(\alpha^*)) = 0$  if  $\beta \notin S_n^0 \cup \{\alpha\}$ ;
- $S_n^2 = S_n \setminus S_n^0 \cup S_n^1$ .

We may now define a set  $T_n$  by:

$$T_n = \{h(\alpha^*) \mid \alpha \in S_n^1\}$$

By definition, we get an inclusion map

$$v : T_n \rightarrow S_n^*.$$

such that

$$h \circ v = v. \quad (11)$$

Indeed, elements of  $T_n$  belong to the image of the idempotent  $h$ , hence are fixed by  $h$ .

We now define a graph  $\sigma^T, \tau^T : T_{n-1}^* \leftarrow T_n$  by

$$\sigma^T = r \circ \sigma_{n-1} \circ v \quad (12)$$

$$\tau^T = r \circ \tau_{n-1} \circ v \quad (13)$$

where  $\sigma_{n-1}, \tau_{n-1}$  are the source and target maps in  $S^*$  and  $r$  is given by the induction hypothesis:

$$\begin{array}{ccc} T_{n-1}^* & \xleftarrow{\sigma^T, \tau^T} & T_n \\ r \uparrow & & \downarrow v \\ S_{n-1}^* & \xleftarrow{\sigma_{n-1}, \tau_{n-1}} & S_n^* \end{array} .$$

By using the fact that  $r$  is a morphism up to dimension  $n-1$ , we see that for each  $\theta \in T_n$ ,  $\sigma^T(\theta) \parallel \tau^T(\theta)$  and the boundary conditions are satisfied. Thus  $T$  extends to an  $n$ -polygraph and the free  $n-1$ -complex  $T^*$  extends to a free  $n$ -complex. We still denote these extensions by  $T, T^*$ , and the source and target maps  $T_{n-1}^* \leftarrow T_n^*$  by  $\sigma^T$  and  $\tau^T$ .

On the other hand, the following diagram commutes

$$\begin{array}{ccc} T_{n-1}^* & \xleftarrow{\sigma^T} & T_n \\ u \downarrow & & \downarrow v \\ S_{n-1}^* & \xleftarrow{\sigma_{n-1}} & S_n^* \end{array} .$$

because

$$\begin{aligned} u \circ \sigma^T &= u \circ r \circ \sigma_{n-1} \circ v, \\ &= h \circ \sigma_{n-1} \circ v, \\ &= \sigma_{n-1} \circ h \circ v, \\ &= \sigma_{n-1} \circ v. \end{aligned}$$

Likewise

$$u \circ \tau^T = u \circ r \circ \tau_{n-1} \circ v.$$

Hence  $v : T_n \rightarrow S_n^*$  gives rise to  $u : T_n^* \rightarrow S_n^*$ , extending  $u$  to a morphism of  $n$ -complexes  $T^* \rightarrow S^*$ .

To sum up, we have extended  $T$  and  $u$  up to dimension  $n$ . Remark that the only property of  $T_n$  we needed so far is that its elements are fixed by  $h$ .

▷ *Step 2.* We introduce an auxiliary  $n$ -polygraph  $U$  by

- $U$  is identical to  $S$  up to dimension  $n-1$ ;
- $U_n = S_n^0 + S_n^1$  and the source and target maps  $U_{n-1}^* \leftarrow U_n$  simply restrict those on  $S_n$ .

Thus we get an inclusion monomorphism of  $n$ -polygraphs  $\iota : U \rightarrow S$ , generating a monomorphism of  $n$ -complexes  $\iota^* : U^* \rightarrow S^*$ . The restrictions of  $\sigma_{n-1}$  and  $\tau_{n-1}$  to  $U_n^*$  will be denoted by  $\sigma^U$  and  $\tau^U$ , as well as the corresponding maps on generators:  $U_{n-1}^* \leftarrow U_n$ .

**Lemma 7** *There are morphisms of  $n$ -complexes*

$$h' : U^* \rightarrow U^*, \quad k : S^* \rightarrow U^*,$$

such that the following diagram commutes:

$$\begin{array}{ccc} U^* & \xrightarrow{\iota^*} & S^* \\ h' \downarrow & \swarrow k & \downarrow h \\ U^* & \xrightarrow{\iota^*} & S^* \end{array}$$

**Proof.** The existence of  $h'$  making the outer square commutative follows from the remark that  $U^*$  is stable by  $h$ , so that  $h'$  is simply the restriction of  $h$  to  $U^*$ .

As for  $k$ , the statement reduces to the fact that all  $n$ -cells of the form  $y = h(x)$  in  $S^*$  can be expressed by generators taken from  $U_n$ . Thus, let  $y = h(x)$  an  $n$ -cell of  $S^*$ . The (occurrences) of generators  $\gamma \in S_n$  such that  $w_\gamma(y) > 0$  may be arranged in a list

$$\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$$

where  $\alpha_i \notin S_n^0$  and  $\beta_j \in S_n^0$ . Notice that repetitions are possible. Let  $y_i = h(\alpha_i^*)$  for each  $i \in \{1, \dots, p\}$ . As  $h(y) = y$  and  $w(h(\beta_j^*)) = 0$  for each  $j \in \{1, \dots, q\}$ , we get

$$\sum_{\gamma \notin S_n^0} w_\gamma(y) = \sum_{i=1}^p \sum_{\gamma \notin S_n^0} w_\gamma(y_i) \quad (14)$$

But  $h(y_i) = y_i = h(\alpha_i^*)$  and the generators of  $y_i$  cannot be all in  $S_n^0$ , otherwise  $w(h(y_i)) = 0$ , in contradiction with  $\alpha_i \notin S_n^0$ . Hence, for each  $i \in \{1, \dots, p\}$ , there is at least one  $\gamma \notin S_n^0$ , such that  $w_\gamma(y_i) > 0$ . Therefore, the left hand side of (14) is equal to  $p$ , whereas the right hand side has  $p$  terms, all of which are  $\geq 1$ . This is possible only if

$$\sum_{\gamma \notin S_n^0} w_\gamma(y_i) = 1$$

for each  $i \in \{1, \dots, p\}$ . Let  $\delta_i$  be the only generator in  $S_n \setminus S_n^0$  such that  $w_{\delta_i}(y_i) = 1$ . The occurrences of  $n$ -generators in  $y_i$  are exactly those in  $h(y_i)$ , hence in  $h(\delta_i^*)$ . It follows that  $w_{\delta_i}(h(\delta_i^*)) = 1$  and  $w_\gamma(h(\delta_i^*)) = 0$  for each  $\gamma \notin S_n^0 \cup \{\delta_i\}$ . This means exactly that  $\delta_i \in S_n^1$ . Therefore  $y$  can be expressed by using as  $n$ -generators

$$\delta_1, \dots, \delta_p, \beta_1, \dots, \beta_q,$$

all in  $S_n^0 \cup S_n^1 = U_n$ . Thus for each  $x \in S_n^*$ , there is a unique  $y \in U_n^*$  such that  $\iota^*(y) = h(x)$ . Hence a morphism  $k : S^* \rightarrow U^*$  such that  $\iota^* \circ k = h$ . Finally  $\iota^* \circ k \circ \iota^* = h \circ \iota^* = \iota^* \circ h'$ , and because  $\iota^*$  is a monomorphism,  $k \circ \iota^* = h'$ .  $\triangleleft$

If  $x \in T_n^*$ ,  $u(x) \in S_n^*$  can be expressed by generators from  $U_n$ , hence there is a morphism  $u' : T^* \rightarrow U^*$  such that  $u = \iota^* \circ u'$ . Of course  $u'$  coincides with  $u$  in all dimensions  $i < n$ .

$\triangleright$  *Step 3.* We now define a morphism  $r' : U^* \rightarrow T^*$  which coincides with  $r$  in dimensions  $i < n$ . All we need is a map

$$\rho : U_n \rightarrow T_n^*$$

satisfying the boundary conditions. Thus, let  $\alpha \in U_n$ , we distinguish two cases, according as  $\alpha \in S_n^0$  or  $\alpha \in S_n^1$ .

$\diamond$  *Case 1.* Let  $\alpha \in S_n^0$ . There is a unique  $y \in S_{n-1}^*$  such that  $h(\alpha^*) = 1_n(y)$ . Now  $r(y) \in T_{n-1}^*$ , so that we may define  $\rho(\alpha) = 1_n(r(y))$ . The boundary conditions are straightforward in this case.

◊ *Case 2.* Let  $\alpha \in S_n^1$ . There is a unique generator  $\theta \in T_n$  such that  $h(\alpha^*) = v(\theta)$ . We define  $\rho(\alpha) = \theta^*$ . By using the induction hypothesis on  $r$  and  $u$ , we get

$$\begin{aligned}
\sigma^T(\rho(\alpha)) &= \sigma^T(\theta^*) \\
&= r(\sigma_{n-1}(v(\theta))) \\
&= r(\sigma_{n-1}(h(\alpha^*))) \\
&= r(h(\sigma_{n-1}(\alpha^*))) \\
&= r(u(r(\sigma_{n-1}(\alpha^*)))) \\
&= r(\sigma_{n-1}(\alpha^*)) \\
&= r'(\sigma^U(\alpha))
\end{aligned}$$

Hence  $\sigma^T(\rho(\alpha)) = r'(\sigma^U(\alpha))$  and likewise  $\tau^T(\rho(\alpha)) = r'(\tau^U(\alpha))$ , and the boundary conditions are satisfied.

Thus  $\rho$  gives rise to a morphism of complexes  $r' : U^* \rightarrow t^*$  extending  $r$  up to dimension  $n$ .

▷ *Step 4.* Having defined  $u' : T^* \rightarrow U^*$  and  $r' : U^* \rightarrow T^*$ , we first note that  $u' \circ r' = h'$ , which directly follows from our definition of  $r'$ . We now prove the following lemma:

**Lemma 8**  $r' \circ u' = \text{id}$ .

**Proof.**  $r' \circ u'$  is an endomorphism of the complex  $T^*$ . We know by the induction hypothesis that  $r' \circ u' = r \circ u = \text{id}$  in all dimensions  $i < n$ . Thus, it suffices to show that, for each generator  $\theta \in T_n$ ,

$$r'(u'(\theta^*)) = \theta^* \tag{15}$$

This follows from two facts:

- the two members of (15) are parallel cells:

$$\sigma^T(r'(u'(\theta^*))) = r'(u'(\sigma^T(\theta^*)))$$

because  $r', u'$  are morphisms. But  $\sigma^T(\theta^*)$  has dimension  $n-1$ , where, by the induction hypothesis,  $r' \circ u' = \text{id}$ , so that the above equation becomes

$$\sigma^T(r'(u'(\theta^*))) = \sigma^T(\theta^*)$$

and likewise

$$\tau^T(r'(u'(\theta^*))) = \tau^T(\theta^*).$$

- there is a *thin*  $n$ -context  $c[\mathbf{x}]$  in  $T^*$  such that

$$r'(u'(\theta^*)) = c[\theta^*].$$

In fact, by definition of  $T_n$ , there is a generator  $\alpha \in S_n^1$  such that  $u'(\theta^*) = h(\alpha^*)$ . As  $h(\alpha^*)$  contains a single occurrence of  $\alpha^*$ , there is an  $n$ -context  $d[\mathbf{y}]$  in  $U^*$  such that  $u'(\theta^*) = d[\alpha^*]$ . Now by applying (7) of section A.1,

$$\begin{aligned}
r'(d[\alpha^*]) &= d^{r'}[r'(\alpha^*)] \\
&= d^{r'}[\rho(\alpha)] \\
&= d^{r'}[\theta^*]
\end{aligned}$$

Define  $c[\mathbf{x}] = d^{r'}[\mathbf{x}]$ . All generators of  $d[\alpha^*]$  but  $\alpha$  itself belong to  $S_n^0$ , hence are sent to identities by  $r'$ . Therefore  $c[\mathbf{x}]$  is thin, and we are done.

$c[\mathbf{x}]$  is a thin context such that  $c[\theta^*] \parallel \theta^*$ . By lemma 6,  $c[\theta^*] = \theta^*$  and (15) is proved. ◁

▷ *Step 5.* We complete the argument by defining  $r = r' \circ k$ . Hence  $r$  is a morphism  $S^* \rightarrow T^*$  and

$$\begin{aligned} u \circ r &= \iota^* \circ u' \circ r' \circ k, \\ &= \iota^* \circ h' \circ k, \\ &= \iota^* \circ k \circ \iota^* \circ k, \\ &= h \circ h, \\ &= h. \end{aligned}$$

Also

$$\begin{aligned} r \circ u &= r' \circ k \circ \iota^* \circ u', \\ &= r' \circ h' \circ u', \\ &= r' \circ u' \circ r' \circ u', \\ &= \text{id} \circ \text{id}, \\ &= \text{id}. \end{aligned}$$

Thus (9) and (10) hold in dimension  $n$  and we are done.