Polygraphic resolutions and homology of monoids

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Abstract

We prove that for any monoid M, the homology defined by the second author by means of polygraphic resolutions coincides with the homology classically defined by means of resolutions by free $\mathbb{Z}M$-modules.

1 Introduction

Since the work of Squier and others [Ani86, Squ87, Kob90], we know that monoids presented by a finite, terminating and confluent rewriting system satisfy a homological finiteness condition. This has two consequences:

- the possibility to prove negative results, e.g. examples of monoids having a decidable word problem, but no presentation satisfying the above conditions;
- on the positive side, the construction of explicit resolutions from such presentations. See for example [DL03] for a recent application of similar methods to compute the homology of gaussian groups.

Now rewriting systems quite naturally lead to n-categories, as follows. Let M be a monoid presented by a system (Σ, R) of generators and rewrite rules. If Σ^* denotes the set of words on the alphabet Σ, $R \subset \Sigma^* \times \Sigma^*$ is a set of ordered pairs of words. A rewrite rule $\zeta : x \to y$ applies to any word uxv with $u, v \in \Sigma^*$, defining a reduction step $u\zeta v : uxv \to uyv$. Thus R generates a set R^* of *reduction paths* between words, whose elements are composable sequences of one-step reductions, up to suitable commutation rules (see [Laf06] for a detailed survey). These data fit together in a 2-category

$$\top \Leftarrow \Sigma^* \Leftarrow R^*$$

where $\top$ denotes the singleton. It has a unique object, words as arrows and reduction paths as 2-arrows. Here $\Leftarrow$ denotes the source and target maps: all words clearly have the same source and target, namely the single element of $\top$, and a reduction path from w to w' has of course source w and target w'. Words compose by concatenation, while reduction paths are subject to *two* sorts of composition, either "parallel" or "sequential". What we get exactly is a free 2-category generated by a *computad* [Str76].
At the next dimension, consider a set $P \subset R^* \times R^*$ of pairs of *parallel* reduction paths, i.e. with the same source and the same target. The smallest equivalence relation on R^* containing P and passing to the context is the 2-*congruence generated* by P. In case the relation of parallelism itself is generated by a finite set D, we say that the underlying monoid M is of *finite derivation type*. It turns out that the latter property holds for all monoids presented by finite, confluent and terminating rewriting systems [SOK94, Laf95]. In n-categorical language, P generates a set P^* of 3-arrows extending the above 2-category to a 3-category:

$$\top \Leftarrow \Sigma^* \Leftarrow R^* \Leftarrow P^*.$$

Note that there are now three ways of composing the elements of P^*. We look here for sets P such that each pair (x, y) of parallel paths in R^* can be filled by at least one $u : x \to y$ in P^*.
This point of view was systematized by the second author [Mét03]. Objects of study are now arbitrary ∞-categories, not just monoids; $(\top, \Sigma, R, D)$ becomes an infinite sequence $(S_0, S_1, \ldots, S_n, \ldots)$ defining n-*computads* [Pow91] or n-*polygraphs* [Bur93], a terminology we shall adopt here.
An ∞-polygraph, or simply *polygraph* S, generates a free ∞-category S^*, generalizing the above situation. There is an abelianization functor taking each polygraph S to a chain complex $\mathbb{Z}S$ of abelian groups, thus defining a homology

$$\mathrm{H}_*(S) =_{\mathrm{def}} \mathrm{H}_*(\mathbb{Z}S). \tag{1}$$

Now let C be an ∞-category, and S a polygraph. A *polygraphic resolution of* C *by* S is a morphism $S^* \to C$ satisfying some lifting properties (see section 4). But the homology $\mathrm{H}_*(S)$ only depends on C [Mét03], so that we may define a "polygraphic homology" of C by

$$\mathrm{H}^{\mathrm{pol}}_*(C) =_{\mathrm{def}} \mathrm{H}_*(S). \tag{2}$$

A monoid M can be seen as a particular ∞-category, with degenerate cells but in dimension 1. Thus, for $C = M$, (2) defines the polygraphic homology of M, whence an immediate question:

> does $\mathrm{H}^{\mathrm{pol}}_*(M)$ coincide with the usual homology of M, defined by means of resolutions of $\mathbb{Z}$ by free $\mathbb{Z}M$-modules?

A positive answer in the case of groups was given by the first author, and the goal of this article is to extend the result to arbitrary monoids. The proof is based on the notion of *unfolding*, defined in section 6, an ∞-category built upon a polygraphic resolution $S^* \to M$ and from which we recover the usual homology of M by abelianization. As many properties of unfoldings are derived from those of resolutions, we first recall the results of [Mét03] in sections 4 and 5, postponing the detailed proofs to annexes A, B and C. Thus our text is essentially selfcontained. In many places, the main definitions of [Mét03] are reformulated and somewhat simplified.

In particular, we borrow the terminology of trivial fibration from model categories, for obvious similarity reasons; beyond the analogy, this points towards a model structure on ∞-categories, yet to be discovered, in the spirit of what has been done for 1- and 2-categories [Tho80, JT91, Lac04, WHPT04].

This work is part of a general program aiming at a homotopical theory of computations, whose further developments include

- a general finiteness conjecture [Laf06]: is it true that a monoid M presented by a finite, terminating and confluent rewriting system always has a polygraphic resolution $S^* \to M$ where S_i is finite in each dimension?
- the study of other structures expressible by polygraphs, as proof systems [Gui06b], Petri nets [Gui06c] and term algebras [Mal04]. In the last case, the polygraphic homology is likely to be degenerate; however, resolutions still bear many relevant informations and could lead to new, refined, invariants;
- potential applications to the theory of directed homotopy. See [Gou03] for a survey.

2 Non abelian complexes

Definition 1 *A* (non abelian) complex *is a (strict)* ∞*-category* $C : C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_n \Leftarrow C_{n+1} \cdots$

In this infinite sequence, $C_n \Leftarrow C_{n+1}$ stands for the *source map* $C_n \overset{\sigma_n}{\leftarrow} C_{n+1}$ and for the *target map* $C_n \overset{\tau_n}{\leftarrow} C_{n+1}$. We define $\sigma_{i,n} = \sigma_i \circ \sigma_{i+1} \circ \cdots \circ \sigma_{n-1}$ and $\tau_{i,n} = \tau_i \circ \tau_{i+1} \circ \cdots \circ \tau_{n-1}$, and we introduce the following notations:

- if $x, y \in C_n$ and $u \in C_{n+1}$, then $u : x \to y$ means $\sigma_n(u) = x$ and $\tau_n(u) = y$;
- if $x, y \in C_n$ with $n > 0$, then $x \parallel y$ means $\sigma_{n-1}(x) = \sigma_{n-1}(y)$ and $\tau_{n-1}(x) = \tau_{n-1}(y)$;
- if $x, y \in C_i$ and $u \in C_n$ with $i < n$, then $u : x \to_i y$ means $\sigma_{i,n}(u) = x$ and $\tau_{i,n}(u) = y$;
- if $x, y \in C_n$ with $i < n$, then $x \rhd_i y$ means $\tau_{i,n}(x) = \sigma_{i,n}(y)$.

The *boundary conditions* $\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n$ and $\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n$ hold for each $n > 0$. In other words, we have $x \parallel y$ for all $u : x \to y$ in C_{n+1} (see figure 1). We also write $x \parallel y$ whenever $x, y \in C_0$.

$$\begin{tikzcd}
\cdot \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow u"] & \cdot
\end{tikzcd}$$

Figure 1: boundary conditions

In addition to this structure of ∞-*graph*, there is:

- a *product* $u *_n v : x \to z$ defined for all $u : x \to y$ and $v : y \to z$ in C_{n+1} (so that $u \triangleright_n v$);
- a *product* $u *_i v : x *_i y \to z *_i t$ defined for all $u : x \to z$ and $v : y \to t$ in C_{n+1} with $i < n$ and $u \triangleright_i v$;
- a *unit* $1_{n+1}(x) : x \to x$ defined for all $x \in C_n$.

All those operations satisfy the conditions of *associativity*, *left and right unit*, and *exchange*:

- $(x *_i y) *_i z = x *_i (y *_i z)$ for all $x \triangleright_i y \triangleright_i z$ in C_n with $i < n$;
- $1_{n,i}(x) *_i u = u = u *_i 1_{n,i}(y)$ for all $u : x \to_i y$ in C_n with $i < n$, where $1_{n,i} = 1_n \circ 1_{n-2} \circ \cdots \circ 1_{i+1}$;
- $(x *_i y) *_j (z *_i t) = (x *_j z) *_i (y *_j t)$ for all $x, y, z, t \in C_n$ with $i < j < n$ and $x \triangleright_i y$, $x \triangleright_j z$, $y \triangleright_j t$ (which implies $z \triangleright_i t$ by the boundary conditions, see figure 2).

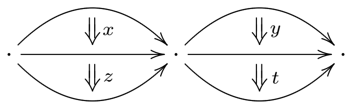

Figure 2: exchange

By restricting this definition to a finite sequence $C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n$, we get the notion of *$n$-category*. Conversely, any such n-category is converted into a complex by concatenating with the infinite stationary sequence $C_n \Leftarrow C_n \cdots C_n \Leftarrow C_n \cdots$ where $\sigma_i = \tau_i = \mathrm{id}_{C_n}$ for all $i \geq n$. In particular, we get the following examples:

- a *set* $S : S \Leftarrow S \Leftarrow S \cdots S \Leftarrow S \cdots$
- a *monoid* $M : \top \Leftarrow M \Leftarrow M \cdots M \Leftarrow M \cdots$
- a *category* $C : C_0 \Leftarrow C_1 \Leftarrow C_1 \cdots C_1 \Leftarrow C_1 \cdots$
- an *abelian monoid* $A : \top \Leftarrow \top \Leftarrow A \Leftarrow A \cdots A \Leftarrow A \cdots$
- a *2-monoid* (or *strict monoidal category*) $C : \top \Leftarrow C_1 \Leftarrow C_2 \Leftarrow C_2 \cdots C_2 \Leftarrow C_2 \cdots$
- a *2-category* $C : C_0 \Leftarrow C_1 \Leftarrow C_2 \Leftarrow C_2 \cdots C_2 \Leftarrow C_2 \cdots$

Here, $\top$ stands for the *singleton*, which is the terminal object in the category of sets. Note that we use the same notation for a monoid M, its underlying set, and its associated complex.

Definition 2 *A complex C such that $C_0 = \top$ is called a* monoidal complex.

In that case, we write xy for $x *_0 y$, which is defined for all $x, y \in C_n$ with $n > 0$, and 1 for the corresponding unit in C_n. Similarly, a n-category $C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n$ such that $C_0 = \top$ is called a *n-monoid*.

Definition 3 *If C and D are complexes, a* morphism *$f : C \to D$ is an infinite sequence of maps $f_n : C_n \to D_n$ which are compatible with sources, targets, products and units:*

- *$f_{n+1}(u) : f_n(x) \to f_n(y)$ for all $u : x \to y$ in C_{n+1};*
- *$f_n(x *_i y) = f_n(x) *_i f_n(y)$ for all $x \triangleright_i y$ in C_n with $i < n$;*
- *$f_{n+1}(1_{n+1}(x)) = 1_{n+1}(f_n(x))$ for all $x \in C_n$.*

Note that if C is a monoidal complex and M is a monoid, then a morphism $f : C \to M$ is just a map $f_1 : C_1 \to M$ satisfying the following three conditions:

$$f_1(xy) = f_1(x)f_1(y) \text{ for all } x, y \in C_1, \qquad f_1(1) = 1, \qquad f_1(x) = f_1(y) \text{ for all } u : x \to y \text{ in } C_2.$$

Indeed, we have $f_n = f_1 \circ \sigma_{1,n} = f_1 \circ \tau_{1,n}$ for each $n > 1$, and all conditions are consequences of the above three. In particular, $f_n(x) = f_n(y)$ for all $x \parallel y$ in C_n with $n > 1$.

Note also that the singleton $\top : \top \Leftarrow \top \Leftarrow \top \cdots \top \Leftarrow \top \cdots$ is the terminal object in this category of complexes.

3 Polygraphs and free complexes

A *graph* $S_0 \Leftarrow S_1$ consists of two sets S_0, S_1 and two maps $S_0 \overset{\sigma_0}{\leftarrow} S_1$ and $S_0 \overset{\tau_0}{\leftarrow} S_1$. It generates a *free category* $S_0 \Leftarrow S_1^*$, where S_1^* is the set of *paths* in the graph $S_0 \Leftarrow S_1$.

Similarly, if $n > 0$ and $C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n$ is a n-category, then any graph $C_n \Leftarrow S_{n+1}$ satisfying the boundary conditions $\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n$ and $\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n$ generates a *free* $n+1$*-category* $C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n \Leftarrow S_{n+1}^*$, where S_{n+1}^* consists of formal compositions of elements of S_{n+1}.

Definition 4 *[Bur93] The notion of* n-polygraph *is defined by induction on* n*:*

- *A 0-polygraph is a set* S_0 *that we also write* S_0^**.*
- *A 1-polygraph is a graph* $S_0 = S_0^* \Leftarrow S_1$*.*
- *A 2-polygraph is given by a graph (or 1-polygraph)* $S_0^* \Leftarrow S_1$ *together with a graph* $S_1^* \Leftarrow S_2$ *satisfying the boundary conditions* $\sigma_0 \circ \sigma_1 = \sigma_0 \circ \tau_1$ *and* $\tau_0 \circ \sigma_1 = \tau_0 \circ \tau_1$*.*
- *In general, a* $n+1$*-polygraph is given by a* n*-polygraph* $S_0^* \Leftarrow S_1, S_1^* \Leftarrow S_2, \ldots, S_{n-1}^* \Leftarrow S_n$ *together with a graph* $S_n^* \Leftarrow S_{n+1}$ *satisfying the boundary conditions* $\sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n$ *and* $\tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n$*.*

The elements of S_n are called n*-generators*. Here are two basic cases:

- An *alphabet* $S_1 = \{\xi_1, \xi_2, \ldots\}$ defines a graph $\top \Leftarrow S_1$ with only one vertex. The free category generated by this graph is $\top \Leftarrow S_1^*$, where S_1^* is the free monoid generated by S_1.
- A *rewriting system* on S_1^*, given by the set of rules $S_2 = \{x_1 \overset{\zeta_1}{\to} y_1, x_2 \overset{\zeta_2}{\to} y_2, \ldots\}$, defines a graph $S_1^* \Leftarrow S_2$. We get a 2-polygraph, since the boundary conditions are trivially satisfied, and the free 2-category generated by this 2-polygraph is $\top \Leftarrow S_1^* \Leftarrow S_2^*$, where S_2^* is the set of reductions quotiented by the exchange relation.

Therefore, a n-polygraph can be considered as a *higher-dimensional rewriting system* (syntactical interpretation) or as a *directed CW-complex* (geometric interpretation). Various examples of 3-polygraphs corresponding to higher dimensional rewriting systems are given in [Laf03]. See also [Gui06a, Gui06b, Gui06c].

Definition 5 *[Bur93] A* polygraph *is an infinite sequence* $S_0^* \Leftarrow S_1, S_1^* \Leftarrow S_2, \ldots, S_n^* \Leftarrow S_{n+1}, \ldots$ *whose first* n *items define a* n*-polygraph for each* n*. It generates a free complex* $S^* : S_0^* \Leftarrow S_1^* \Leftarrow S_2^* \cdots S_n^* \Leftarrow S_{n+1}^* \cdots$

In particular, note that the singleton $\top$ is the free complex Ω^* defined by $\Omega_0 = \top$ and $\Omega_n = \emptyset$ for each $n > 0$.

Definition 6 *[Mét03] If* S *and* T *are polygraphs, a* morphism of polygraphs $f : S \to T$ *is given by an infinite sequence of maps* $f_n : S_n \to T_n$ *satisfying the following condition:*

- $f_{n+1}(\xi) : f_n^*(x) \to f_n^*(y)$ *for all* $\xi : x \to y$ *in* S_{n+1}*.*

Here, f_n^* stands for the obvious extension of f_n which is compatible with products and units. So we get a functor from the category of polygraphs to the category of complexes mapping S to S^* and $f : S \to T$ to $f^* : S^* \to T^*$. It is the left adjoint of some forgetful functor [Mét03].

A morphism of the form $f^* : S^* \to T^*$ is called *atomic*. Not all morphisms between free complexes are atomic. In fact, morphisms of polygraphs and atomic morphisms only appear in appendix A.

4 Polygraphic resolutions

Definition 7 *A morphism* $p : C \to D$ *is a* trivial fibration *if* $p_0 : C_0 \to D_0$ *is onto and* p *has the* lifting property:

- *if* $x \parallel y$ *in* C_n *and* $v : p_n(x) \to p_n(y)$ *in* D_{n+1}*, there is some* $u : x \to y$ *in* C_{n+1} *such that* $p_{n+1}(u) = v$*.*

As a consequence, each $p_n : C_n \to D_n$ is onto and p has the *stretching property*:

- if $x \parallel y$ in C_n and $p_n(x) = p_n(y)$, there is some $u : x \to y$ in C_{n+1} such that $p_{n+1}(u) = 1_{n+1}(p_n(x))$.

Conversely, if each p_n is onto and p has the stretching property, then p is a trivial fibration [Mét03].

Definition 8 *A complex C is* exact *if the canonical morphism $\pi : C \to \top$ is a trivial fibration.*

This means that C_0 is not empty, and C has the *filling property*: if $x \parallel y$ in C_n, there is some $u : x \to y$ in C_{n+1}.

Proposition 1 *[Mét03] Any free complex S^* is* cofibrant: *for any trivial fibration $p : C \to D$ and for any morphism $g : S^* \to D$, there is some morphism $f : S^* \to C$ such that $g = p \circ f$.*

It suffices indeed to define $f_n(\xi)$ for each $\xi \in S_n$, using the fact that p is a trivial fibration.

In fact, the converse of this proposition holds: any cofibrant complex is free [Mét06].

Definition 9 *[Mét03] A* (polygraphic) resolution of *C is a trivial fibration $p : S^* \to C$ where S^* is free.*

Polygraphic resolutions are the analogues of free resolutions in a category of modules.

Theorem 1 *[Mét03]*

1. *Any complex C has a resolution $p : S^* \to C$.*
2. *If $p : S^* \to C$ and $q : T^* \to C$ are resolutions, there is some morphism $f : S^* \to T^*$ such that $p = q \circ f$.*
3. *Two such morphisms are homotopic.*

The first point is straightforward: S_n and p_n are defined by induction on n, starting from $S_0 = C_0$ and $p_0 = \mathrm{id}_{C_0}$. For any $x \parallel y$ in S^*_n and $v : p_n(x) \to p_n(y)$ in C_{n+1}, we introduce a $n{+}1$-generator $\xi : x \to y$ and we define $p_{n+1}(\xi) = v$, so that p is a resolution by construction. The second point follows immediately from proposition 1. The third point is the crucial one: it uses the *homotopy relation* $f \sim g$. See appendices A and B.

Corollary 1 *Two resolutions $p : S^* \to C$ and $q : T^* \to C$ are homotopically equivalent.*

This means that there are some morphisms $f : S^* \to T^*$ and $g : T^* \to S^*$ such that $q \circ f = p$, $p \circ g = q$, $g \circ f \sim \mathrm{id}_{S^*}$ and $f \circ g \sim \mathrm{id}_{T^*}$.

Note that any monoid M has a *monoidal resolution*, that is a resolution $p : S^* \to M$ such that $S^*_0 = S_0 = \top$. Such a resolution contains a *presentation* of M, where S_1 is the set of generators and S_2 is the set of relations. Conversely, any *symmetric presentation* of M can be extended to a monoidal resolution of M.

5 Abelianization and homology

If $S^* : S^*_0 \Leftarrow S^*_1 \Leftarrow S^*_2 \cdots S^*_n \Leftarrow S^*_{n+1} \cdots$ is a free complex and $\xi \in S_n$, we write $[\xi]$ for the corresponding generator in the *free $\mathbb{Z}$-module* $\mathbb{Z}S_n$ generated by S_n, and if $n > 0$, we extend this notation to S^*_n as follows:

$$[x *_i y] = [x] + [y] \text{ for all } x \triangleright_i y \text{ in } S^*_n \text{ with } i < n, \qquad [1_n(x)] = 0 \text{ for all } x \in S^*_{n-1}.$$

In other words, $[x]$ counts the number of all occurrences of each n-generator in x. The fact that it is well defined follows from the universal property of S^*_n and the definition of a suitable n-category. See appendix D.

For each $n \geq 0$, we define a $\mathbb{Z}$-linear map $\mathbb{Z}S_n \xleftarrow{\partial_n} \mathbb{Z}S_{n+1}$ by $\partial_n[\xi] = [y] - [x]$ for all $\xi : x \to y$ in S_{n+1}.

Lemma 1 $\partial_n[u] = [y] - [x]$ *for all $u : x \to y$ in S^*_{n+1}.*

This is easily proved by induction on $u \in S^*_{n+1}$. Using this and the boundary conditions, we get $\partial_n \circ \partial_{n+1} = 0$.

Definition 10 *[Mét03] The* abelianization of a free complex *$S^* : S^*_0 \Leftarrow S^*_1 \Leftarrow S^*_2 \cdots S^*_n \Leftarrow S^*_{n+1} \cdots$ is the abelian complex of free $\mathbb{Z}$-modules $\mathbb{Z}S : \mathbb{Z}S_0 \xleftarrow{\partial_0} \mathbb{Z}S_1 \xleftarrow{\partial_1} \mathbb{Z}S_2 \cdots \mathbb{Z}S_n \xleftarrow{\partial_n} \mathbb{Z}S_{n+1} \cdots$*

If $f : S^* \to T^*$ is a morphism, we define a $\mathbb{Z}$-linear map $f^{\mathrm{ab}}_n : \mathbb{Z}S_n \to \mathbb{Z}T_n$ by $f^{\mathrm{ab}}_n[\xi] = [f_n(\xi)]$ for all $\xi \in S_n$.

Lemma 2 $f^{\mathrm{ab}}_n[x] = [f_n(x)]$ *for all $x \in S^*_n$.*

This is easily proved by induction on $x \in S^*_n$. Using this, we get $\partial_n \circ f^{\mathrm{ab}}_{n+1} = f^{\mathrm{ab}}_n \circ \partial_n$ for each n.

Definition 11 *[Mét03] The* abelianization of a morphism $f : S^* \to T^*$ *is the morphism of abelian complex* $f^{\mathrm{ab}} : \mathbb{Z}S \to \mathbb{Z}T$ *defined by the infinite sequence* $f^{\mathrm{ab}}_n : \mathbb{Z}S_n \to \mathbb{Z}T_n$.

Note that we get a functor, since $\mathrm{id}^{\mathrm{ab}}_{S^*} = \mathrm{id}_{\mathbb{Z}S}$, and $(g \circ f)^{\mathrm{ab}} = g^{\mathrm{ab}} \circ f^{\mathrm{ab}}$ for any $f : S^* \to T^*$ and $g : T^* \to U^*$.

Proposition 2 *[Mét03] If* $f, g : S^* \to T^*$ *are homotopic morphisms, so are* $f^{\mathrm{ab}}, g^{\mathrm{ab}} : \mathbb{Z}S \to \mathbb{Z}T$.

This crucial result is proved in appendix C. By corollary 1, we get:

Corollary 2 *If* $p : S^* \to C$ *and* $q : T^* \to C$ *are two resolutions of* C*, then* $\mathbb{Z}S$ *and* $\mathbb{Z}T$ *have the same homology.*

This means that the homology groups of $\mathbb{Z}S$, defined by $\mathrm{H}_0(\mathbb{Z}S) = \mathbb{Z}S_0 / \operatorname{im} \partial_0$ and $\mathrm{H}_n(\mathbb{Z}S) = \ker \partial_{n-1} / \operatorname{im} \partial_n$ for each $n > 0$, do not depend on the choice of the resolution $p : S^* \to C$.

Definition 12 *[Mét03] The* homology of a complex C *is the homology of* $\mathbb{Z}S$ *for any resolution* $p : S^* \to C$.

Corollary 3 *If* S^* *is an exact free complex, then the following augmented complex of free* $\mathbb{Z}$*-modules is exact:*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}S_0 \xleftarrow{\partial_0} \mathbb{Z}S_1 \xleftarrow{\partial_1} \mathbb{Z}S_2 \cdots \mathbb{Z}S_n \xleftarrow{\partial_n} \mathbb{Z}S_{n+1} \cdots$$

Here, $\varepsilon = \pi^{\mathrm{ab}}_0$ where $\pi : S^* \to \Omega^* = \top$ is the canonical morphism. In other words, $\varepsilon(\xi) = 1$ for all $\xi \in S_0$.

6 Unfolding a morphism

If M is a monoid and S is a set, we write $M \cdot S$ for the cartesian product $M \times S$ whose elements are written $\lambda \cdot x$, and the *free (left) action* of M on the set $M \cdot S$ is defined by $\lambda \cdot (\mu \cdot x) = \lambda\mu \cdot x$ for all $\lambda, \mu \in M$ and $x \in S$. In particular, we identify $M \cdot \top$ with M, where the action of M on itself is defined by $\lambda \cdot \mu = \lambda\mu$ for all $\lambda, \mu \in M$.

Let $f : C \to M$ be a morphism, where M is a monoid and C is a monoidal complex, so that $M \cdot C_0 = M \cdot \top = M$. We shall define a new complex $M \cdot C : M \Leftarrow M \cdot C_1 \Leftarrow M \cdot C_2 \cdots M \cdot C_n \Leftarrow M \cdot C_{n+1} \cdots$

First, we write $\overline{x} = f_n(x) \in M$ for all $x \in C_n$ with $n > 0$, and we define the structure of ∞-graph as follows:

- $\lambda \cdot x : \lambda \to \lambda\overline{x}$ in $M \cdot C_1$ for all $\lambda \in M$ and $x \in C_1$;
- $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y$ in $M \cdot C_{n+1}$ for all $\lambda \in M$ and $u : x \to y$ in C_{n+1} with $n > 0$.

As consequences, we get:

- if $\lambda \in M$ and $x, y \in C_1$, then $\lambda \cdot x \parallel \mu \cdot y$ if and only if $\lambda = \mu$ and $\lambda\overline{x} = \lambda\overline{y}$;
- if $\lambda \in M$ and $x, y \in C_n$ with $n > 1$, then $\lambda \cdot x \parallel \mu \cdot y$ if and only if $\lambda = \mu$ and $x \parallel y$.

In particular, for any $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y$ in $M \cdot C_2$, we have $u : x \to y$ in C_2 and $\overline{x} = \overline{y}$ since $f : C \to M$ is a morphism, so that $\lambda \cdot x \parallel \lambda \cdot y$. The other boundary conditions follow directly from the boundary conditions for C.

More generally, we have $\lambda \cdot x : \lambda \to_0 \lambda\overline{x}$ in $M \cdot C_n$ for all $\lambda \in M$ and $x \in C_n$ with $n > 0$, and $\lambda \cdot u : \lambda \cdot x \to_i \lambda \cdot y$ in $M \cdot C_n$ for all $\lambda \in M$ and $u : x \to_i y$ in C_n with $n > i > 0$. As consequences, we get:

- if $x, y \in C_n$ with $n > 0$, then $\lambda \cdot x \triangleright_0 \mu \cdot y$ if and only if $\lambda\overline{x} = \mu$;
- if $x, y \in C_n$ with $n > i > 0$, then $\lambda \cdot x \triangleright_i \mu \cdot y$ if and only if $\lambda = \mu$ and $x \triangleright_i y$.

Using this, we define products and units as follows:

- $(\lambda \cdot x) *_0 (\lambda\overline{x} \cdot y) = \lambda \cdot xy$ for all $\lambda \in M$ and $x, y \in C_n$ with $n > 0$;
- $(\lambda \cdot x) *_i (\lambda \cdot y) = \lambda \cdot (x *_i y)$ for all $\lambda \in M$ and $x \triangleright_i y$ in C_n with $n > i > 0$;
- $1_{n+1}(\lambda \cdot x) = \lambda \cdot 1_{n+1}(x)$ for all $\lambda \in M$ and $x \in C_n$. In particular, $1_1(\lambda) = \lambda \cdot 1$ for all $\lambda \in M$.

It is easy to see that those operations satisfy the conditions of associativity, left and right unit, and exchange. Furthermore, we have an obvious morphism $\tilde{f} : M \cdot C \to C$ defined by $\tilde{f}_n(\lambda \cdot x) = x$ for all $\lambda \in M$ and $x \in C_n$.

Definition 13 *The complex* $M \cdot C : M \Leftarrow M \cdot C_1 \Leftarrow M \cdot C_2 \cdots M \cdot C_n \Leftarrow M \cdot C_{n+1} \cdots$ *is called the* unfolding of the morphism $f : C \to M$*, and* $\tilde{f} : M \cdot C \to C$ *is called its* folding morphism.

In fact, $M \cdot C$ is a M*-complex,* which means that its structure of complex is compatible with the action of M.

Proposition 3 *If* G *is a group and* $p : C \to G$ *is a trivial fibration, then its unfolding* $G \cdot C$ *is an exact complex.*

Indeed, G is not empty, and using the fact that p is a trivial fibration, we prove the filling property for each $G \cdot C_n$:

- if $\lambda, \mu \in G$, there is some $x \in C_1$ such that $\overline{x} = \lambda^{-1}\mu$, and we get $\lambda \cdot x : \lambda \to \lambda\overline{x} = \mu$ in $G \cdot C_1$;
- if $\lambda \cdot x \parallel \mu \cdot y$ where $\lambda, \mu \in G$ and $x, y \in C_1$, we have $\lambda = \mu$ and $\lambda\overline{x} = \lambda\overline{y}$, so that $\overline{x} = \overline{y}$ by left cancellation. Therefore, there is some $u : x \to y$ in C_2, and we get $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y = \mu \cdot y$ in $G \cdot C_2$;
- if $\lambda \cdot x \parallel \mu \cdot y$ where $\lambda, \mu \in G$ and $x, y \in C_n$ with $n > 1$, we have $\lambda = \mu$ and $x \parallel y$, so that $\overline{x} = \overline{y}$. Therefore, there is some $u : x \to y$ in C_{n+1}, and we get $\lambda \cdot u : \lambda \cdot x \to \lambda \cdot y = \mu \cdot y$ in $G \cdot C_{n+1}$.

This proposition does not hold for an arbitrary monoid. In fact, the converse holds: if the unfolding of $f : C \to M$ is an exact complex, then M is a group and f is a trivial fibration. However, we have a weaker property:

Proposition 4 *If* $p : C \to M$ *is a trivial fibration, then its unfolding* $M \cdot C$ *has a filling property relative to* $1 \cdot C$*:*

- *if* $1 \cdot x \parallel \mu \cdot y$ *where* $\mu \in M$ *and* $x, y \in C_n$*, there is some* $1 \cdot u : 1 \cdot x \to \mu \cdot y$ *in* $M \cdot C_{n+1}$.

No extra assumption on the monoid M is needed here, since $\lambda = 1$ has a right inverse and is left cancelable.

7 The free case

Here, we consider the unfolding $M \cdot S^* : M \Leftarrow M \cdot S_1^* \Leftarrow M \cdot S_2^* \cdots M \cdot S_n^* \Leftarrow M \cdot S_{n+1}^* \cdots$ of a morphism $f : S^* \to M$, where $S_0^* = S_0 = \top$, so that $M \cdot S_0^* = M \cdot \top = M$. We shall see that $M \cdot S^*$ is a free complex.

For any $n > 0$, the canonical injection of $M \cdot S_n$ into $M \cdot S_n^*$ defines a graph $M \cdot S_{n-1}^* \Leftarrow M \cdot S_n$, which satisfies the boundary conditions $\sigma_{n-2} \circ \sigma_{n-1} = \sigma_{n-2} \circ \tau_{n-1}$ and $\tau_{n-2} \circ \sigma_{n-1} = \tau_{n-2} \circ \tau_{n-1}$ (for $n > 1$). We get a free n-category $M \Leftarrow M \cdot S_1^* \Leftarrow M \cdot S_2^* \cdots M \cdot S_{n-2}^* \Leftarrow M \cdot S_{n-1}^* \Leftarrow (M \cdot S_n)^*$ and a map $\varphi : (M \cdot S_n)^* \to M \cdot S_n^*$ such that $\sigma_{n-1} \circ \varphi = \sigma_{n-1}$ and $\tau_{n-1} \circ \varphi = \tau_{n-1}$, which is compatible with products and units.

If $x \in (M \cdot S_n)^*$ and $y \in M \cdot S_n^*$, we can write $x \parallel y$ even though x and y do not belong to the same complex, since their sources and targets do.

If $\lambda \in M$ and $\xi \in S_n$, we write $\langle \lambda \cdot \xi \rangle$ for the corresponding n-generator in $(M \cdot S_n)^*$. More generally, if $\lambda \in M$ and $x \in S_n^*$, we define $\langle \lambda \cdot x \rangle$ in $(M \cdot S_n)^*$ such that $\langle \lambda \cdot x \rangle \parallel \lambda \cdot x$ as follows:

- $\langle \lambda \cdot xy \rangle = \langle \lambda \cdot x \rangle *_0 \langle \lambda\overline{x} \cdot y \rangle$ for all $\lambda \in M$ and $x, y \in S_n^*$;
- $\langle \lambda \cdot x *_i y \rangle = \langle \lambda \cdot x \rangle *_i \langle \lambda \cdot y \rangle$ for all $\lambda \in M$ and $x \triangleright_i y$ in S_n^* with $n > i > 0$;
- $\langle \lambda \cdot 1_n(x) \rangle = 1_n(\lambda \cdot x)$ for all $\lambda \in M$ and $x \in S_{n-1}^*$.

In other words, $\langle \lambda \cdot x \rangle$ is a decomposition of $\lambda \cdot x$ as a formal product of n-generators in $M \cdot S_n$. The fact that it is well defined follows from the universal property of S_n^* and the definition of a suitable n-category. See appendix E. To sum up, we have defined a map $\psi : M \cdot S_n^* \to (M \cdot S_n)^*$ such that $\sigma_{n-1} \circ \psi = \sigma_{n-1}$ and $\tau_{n-1} \circ \psi = \tau_{n-1}$, which is obviously compatible with products and units.

By construction, $\psi(\varphi\langle \lambda \cdot \xi \rangle) = \langle \lambda \cdot \xi \rangle$ for all $\lambda \in M$ and $\xi \in S_n$, so that $\psi \circ \varphi$ is the identity on $(M \cdot S_n)^*$. Furthermore, we have $\varphi\langle \lambda \cdot x \rangle = \lambda \cdot x$ for all $\lambda \in M$ and $x \in S_n^*$: this is easily proved by induction on $x \in S_n^*$. Hence $\varphi \circ \psi$ is the identity on $M \cdot S_n^*$, and we can identify $M \cdot S_n^*$ with $(M \cdot S_n)^*$.

By abelianization of the complex $M \cdot S^* = (M \cdot S)^*$, we get a complex of free $\mathbb{Z}$-modules:

$$\mathbb{Z}(M \cdot S) : \mathbb{Z}M \xleftarrow{\partial_0} \mathbb{Z}(M \cdot S_1) \xleftarrow{\partial_1} \mathbb{Z}(M \cdot S_2) \cdots \mathbb{Z}(M \cdot S_n) \xleftarrow{\partial_n} \mathbb{Z}(M \cdot S_{n+1}) \cdots$$

Furthermore, the free $\mathbb{Z}$-module $\mathbb{Z}(M \cdot S_n)$ can be identified with the free $\mathbb{Z}M$-module $\mathbb{Z}M \cdot S_n$.

Lemma 3 $[\lambda \cdot x] = \lambda \cdot [1 \cdot x]$ *for all* $\lambda \in M$ *and* $x \in S_n^*$.

This is easily proved by induction on $x \in S_n^*$. Using this, we get the fact that all ∂_n are $\mathbb{Z}M$-linear. In other words, we have $\partial_n(\lambda \cdot \xi) = \lambda \cdot \partial_n(1 \cdot \xi)$ for all $\lambda \in M$ and $\xi \in S_n$.

Note also that the folding morphism $\tilde{f} : \mathbb{Z}M \cdot S \to \mathbb{Z}S$ is $\mathbb{Z}M$-linear if we consider the *trivial action* of M on $\mathbb{Z}S$. In other words, we have $\tilde{f}_n^{\mathrm{ab}}[\lambda \cdot \xi] = [\xi]$ for all $\lambda \in M$ and $\xi \in S_n$. To sum up, we have the following result:

Proposition 5 *$M \cdot S^*$ is a free complex of the form $(M \cdot S)^*$. Its abelianization is a complex of free $\mathbb{Z}M$-modules:*

$$\mathbb{Z}M \cdot S : \mathbb{Z}M \xleftarrow{\partial_0} \mathbb{Z}M \cdot S_1 \xleftarrow{\partial_1} \mathbb{Z}M \cdot S_2 \cdots \mathbb{Z}M \cdot S_n \xleftarrow{\partial_n} \mathbb{Z}M \cdot S_{n+1} \cdots$$

Furthermore, the abelian complex $\mathbb{Z}S$ is obtained by trivializing the action of M in $\mathbb{Z}M \cdot S$.

Now we can state the main result of this paper:

Theorem 2 *If $p : S^* \to M$ is a monoidal resolution of M and $M \cdot S$ is its unfolding, then $\mathbb{Z}M \cdot S$ is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}M$-modules. In other words, the following augmented complex of $\mathbb{Z}M$-modules is exact:*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}M \xleftarrow{\partial_0} \mathbb{Z}M \cdot S_1 \xleftarrow{\partial_1} \mathbb{Z}M \cdot S_2 \cdots \mathbb{Z}M \cdot S_n \xleftarrow{\partial_n} \mathbb{Z}M \cdot S_{n+1} \cdots$$

Here, ε is defined by $\varepsilon(\lambda) = 1$ for all $\lambda \in M$. It is $\mathbb{Z}M$-linear if we consider the trivial action of M on $\mathbb{Z}$.

Corollary 4 *The homology of a monoid M coincides with the homology of the (non abelian) complex M.*

In the case of groups, theorem 2 follows from proposition 3 and corollary 3.

We consider now a monoidal resolution $p : S^* \to M$ where M is an arbitrary monoid. Proposition 4 asserts that the canonical morphism $\pi : M \cdot S^* \to \top$ is a $(\mathcal{X}, \mathcal{Y})$*-fibration*, where $\mathcal{X} = 1 \cdot S^*$ is the *right ideal* consisting of all cells of the form $1 \cdot x$ with $x \in S_n^*$ for some n, and $\mathcal{Y} = M \cdot S^*$ is the *left ideal* consisting of all cells in $M \cdot S^*$. See appendix B for the corresponding definitions.

Note that $\mathcal{X}$ is just the set of cells in $M \cdot S^*$ whose 0-dimensional source is $1 \in M \cdot S_0^* = M \cdot \top = M$. We have a *canonical inclusion* $\iota : \top \to M \cdot S^*$ which maps the vertex of $\top$ to 1, and the following conditions hold trivially:

$$\iota \circ \pi(M \cdot S^*) \subset \mathcal{X}, \qquad \mathrm{id}_{M \cdot S^*}(M \cdot S^*) \subset \mathcal{Y}, \qquad \pi \circ \iota \circ \pi = \pi \circ \mathrm{id}_{M \cdot S^*}.$$

Since $M \cdot S^*$ is free, proposition 7 of appendix B applies, so that $\iota \circ \pi \rightsquigarrow \mathrm{id}_{M \cdot S^*}$, which implies $\iota \circ \pi \sim \mathrm{id}_{M \cdot S^*}$. By proposition 2, the $\mathbb{Z}M$-linear maps $(\iota \circ \pi)^{\mathrm{ab}} = \iota^{\mathrm{ab}} \circ \pi^{\mathrm{ab}}$ and $\mathrm{id}_{M \cdot S^*}^{\mathrm{ab}} = \mathrm{id}_{\mathbb{Z}M \cdot S}$ are algebraically homotopic. Since $\pi_0^{\mathrm{ab}} = \varepsilon$ and $\pi_n^{\mathrm{ab}} = 0$ for each $n > 0$, the augmented complex of theorem 2 is exact, and we are done.

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A Cylinders

Our definition of homotopy will be based on the construction of a functor

$$C \mapsto C^I$$

which to each complex C associates a new complex C^I consisting intuitively of paths in C.

We first describe a family of polygraphs: for each integer n, the *n-cylinder*, denoted by $\mathbf{n}$ is defined by its sets of generators $\mathbf{n}_i$, together with source an target maps $\sigma_i, \tau_i : \mathbf{n}_i^* \Leftarrow \mathbf{n}_{i+1}$ in each dimension $i \geq 0$. Figure 3 represents the n-cylinder for $n = 0, 1, 2$. 2-cylinders appear early in the litterature [Bén67]; general cylinders were considered in connection with tensor products on ∞-categories [Cra95].

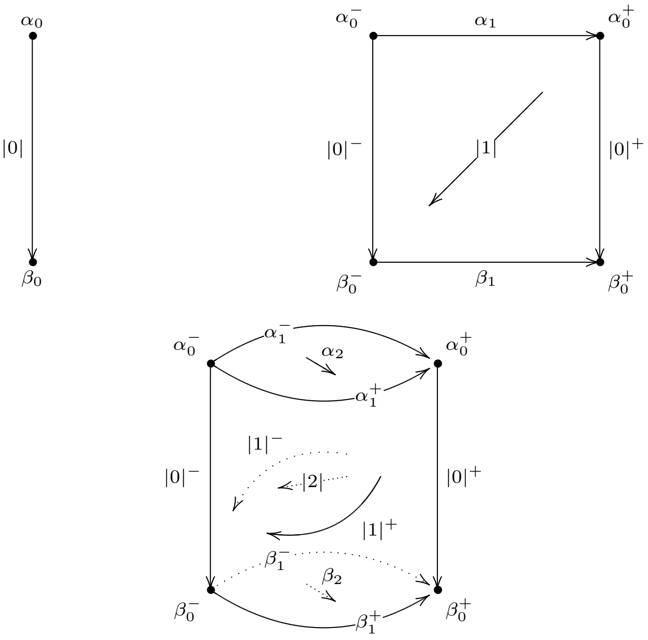

Figure 3: n-cylinder for $n = 0, 1, 2$

The present construction takes a different approach and is equivalent to Burroni's [Bur00]. The following tables display the generators of $\mathbf{n}$, as well as the expression of their source and target in each dimension:

- For $n = 0$, we get

dimension	generators
0	α_0 β_0
1	$\|0\| : \alpha_0 \to \beta_0$

- for $n = 1$,

dimension	generators
0	α_0^- α_0^+ β_0^- β_0^+
1	$\|0\|^- : \alpha_0^- \to \beta_0^-$ $\|0\|^+ : \alpha_0^+ \to \beta_0^+$ $\alpha_1 : \alpha_0^- \to \alpha_0^+$ $\beta_1 : \beta_0^- \to \beta_0^+$
2	$\|1\| : \alpha_1 *_0 \|0\|^+ \to \|0\|^- *_0 \beta_1$

- for $n > 1$,

dimension	generators
0	α_0^- α_0^+ β_0^- β_0^+
$1 \leq i \leq n-1$	$\|i-1\|^- : \alpha^-_{i-1} *_0 \|0\|^+ *_1 \cdots *_{i-2} \|i-2\|^+ \to \|i-2\|^- *_{i-2} \cdots *_1 \|0\|^- *_0 \beta^-_{i-1}$ $\|i-1\|^+ : \alpha^+_{i-1} *_0 \|0\|^+ *_1 \cdots *_{i-2} \|i-2\|^+ \to \|i-2\|^- *_{i-2} \cdots *_1 \|0\|^- *_0 \beta^+_{i-1}$ $\alpha^-_i : \alpha^-_{i-1} \to \alpha^+_{i-1}$ $\alpha^+_i : \alpha^-_{i-1} \to \alpha^+_{i-1}$ $\beta^-_i : \beta^-_{i-1} \to \beta^+_{i-1}$ $\beta^+_i : \beta^-_{i-1} \to \beta^+_{i-1}$
n	$\|n-1\|^- : \alpha^-_{n-1} *_0 \|0\|^+ *_1 \cdots *_{n-2} \|n-2\|^+ \to \|n-2\|^- *_{n-2} \cdots *_1 \|0\|^- *_0 \beta^-_{n-1}$ $\|n-1\|^+ : \alpha^+_{n-1} *_0 \|0\|^+ *_1 \cdots *_{n-2} \|n-2\|^+ \to \|n-2\|^- *_{n-2} \cdots *_1 \|0\|^- *_0 \beta^+_{n-1}$ $\alpha_n : \alpha^-_{n-1} \to \alpha^+_{n-1}$ $\beta_n : \beta^-_{n-1} \to \beta^+_{n-1}$
$n+1$	$\|n\| : \alpha_n *_0 \|0\|^+ *_1 \cdots *_{n-1} \|n-1\|^+ \to \|n-1\|^- *_{n-1} \cdots *_1 \|0\|^- *_0 \beta_n$

We dispense of parentheses and identity symbols by assuming that $*_i$ has precedence over $*_j$ if $i < j$ and by denoting x for $1_{j,i}(x)$ if x is of dimension i and $j > i$. For example, all cells appearing in the expression of $\sigma_n\,|n|$ have dimension n: among these, only α_n and $|n-1|^+$ are not identities.
Because a polygraph is entirely determined by its generators and maps σ, τ, the above tables define at most one family of polygraphs. We still need a coherence result, which amounts to the following lemma.

Lemma 4 *For each $n \geq 0$, the polygraph $\mathbf{n}$ is well defined.*

We first prove the existence of $\mathbf{n}$ by induction on n. $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$ are easily seen to be well defined. Suppose now $n > 2$ and $\mathbf{m}$ is a well defined polygraph for each $m < n$, we show that $\mathbf{n}$ is also well defined:

- For each $i \leq n-2$, $\mathbf{n}_i$ is exactly $(\mathbf{n}-\mathbf{1})_i$, with the same source and target maps, hence $\mathbf{n}$ is well defined up to dimension $n-2$.
- $\mathbf{n}_{n-1}$ is obtained from $(\mathbf{n}-\mathbf{1})_{n-1}$ by splitting α_{n-1} in two copies α^-_{n-1}, α^+_{n-1} having the same source and target, and likewise for β_{n-1}, whereas $|n-2|^-$ and $|n-2|^+$ are left unchanged. Hence $\mathbf{n}$ is now well defined up to dimension $n-1$.
- The source and target formulas defining $|n-1|^-$ and $|n-1|^+$ are the same as those defining $|n-1|$ in $\mathbf{n}-\mathbf{1}$, but for the signs on α_{n-1} and β_{n-1}, hence they are coherent. Also the previous point shows that $\alpha^-_{n-1} \parallel \alpha^+_{n-1}$ and $\beta^-_{n-1} \parallel \beta^+_{n-1}$, so that the source and target formulas defining α_n and β_n are coherent. Thus $\mathbf{n}$ is well defined up to dimension n.
- It remains to show that the last cell $|n| \in \mathbf{n}_{n+1}$ may be attached to $\mathbf{n}^*_n$ according to the given source and target formulas: this amounts to check that
  - $u = \alpha_n *_0 |0|^+ *_1 \cdots *_{n-1} |n-1|^+$ and $v = |n-1|^- *_{n-1} \cdots *_1 |0|^- *_0 \beta_n$ are well defined cells.
  - $u \parallel v$.

  As for the first point, we prove by induction on $1 \leq i \leq n-1$ that $u_i = \alpha_n *_0 |0|^+ *_1 \cdots *_{i-1} |i-1|^+$ is well defined and that $u_i \rhd_i |i|^+$ whenever $i < n-1$, by using equations already satisfied in lower dimensions, and likewise for v. The second point amounts to evaluate $\sigma_{n-1}(u)$, $\sigma_{n-1}(v)$, $\tau_{n-1}(u)$ and $\tau_{n-1}(v)$:

$$\begin{aligned}
\sigma_{n-1}(u) &= \sigma_{n-1}(\alpha_n *_0 |0|^+ *_1 \cdots *_{n-2} |n-2|^+) \\
&= \alpha^-_{n-1} *_0 |0|^+ *_1 \cdots *_{n-2} |n-2|^+ \\
&= \sigma_{n-1}|n-1|^- \\
&= \sigma_{n-1}(v),
\end{aligned}$$

  and likewise for targets, by using our convention on identities.

This ends the proof of the lemma.
Moreover, there are morphisms of polygraphs $\mathrm{s}_n, \mathrm{t}_n : \mathbf{n} \to \mathbf{n+1}$ satisfying the coboundary conditions

$$\begin{array}{rcl} \mathrm{s}_{n+1} \circ \mathrm{s}_n & = & \mathrm{t}_{n+1} \circ \mathrm{s}_n, \\ \mathrm{s}_{n+1} \circ \mathrm{t}_n & = & \mathrm{t}_{n+1} \circ \mathrm{t}_n. \end{array}$$

$\mathrm{s}_n : \mathbf{n} \to \mathbf{n+1}$ is easily defined on each dimension:

- for $0 \leq i \leq n-1$, s_n is the identity on $\mathbf{n}_i = (\mathbf{n+1})_i$;
- for $i = n$, $\mathrm{s}_n(\alpha_n) = \alpha_n^-$, $\mathrm{s}_n(\beta_n) = \beta_n^-$, and $\mathrm{s}_n(x) = x$ for $x \in \{|n-1|^-, |n-1|^+\}$;
- for $i = n+1$, $\mathrm{s}_n|n| = |n|^- \in (\mathbf{n+1})_{n+1}$.

The definition of t_n is the same, except for the change of sign. The coboundary relations are straightforward.
Each n-cylinder $\mathbf{n}$ generates a complex $\mathbf{n}^*$. For any pair of complexes C, D we denote by **Compl**(D, C) the set of morphisms $f : D \to C$. Thus for each integer n, we get a set

$$C_n^I = \mathbf{Compl}(\mathbf{n}^*, C)$$

and the maps s_n, t_n give rise to s_n^* and t_n^* from $\mathbf{n}^*$ to $\mathbf{n+1}^*$, hence to

$$\sigma_n, \tau_n : C_n^I \Leftarrow C_{n+1}^I$$

defined by $\sigma_n(x) = x \circ \mathrm{s}_n^*$ and $\tau_n(x) = x \circ \mathrm{t}_n^*$. Because of the coboundary conditions, the σ_n's and τ_n's satisfy the boundary condition, making C^I an ∞-graph or globular set.

Now C^I also has a structure of complex. Let $0 \leq i < n$, we define

$$\mathrm{s}_{i,n}, \mathrm{t}_{i,n} : \mathbf{i} \to \mathbf{n}$$

by $\mathrm{s}_{i,n} = \mathrm{s}_{n-1} \circ \cdots \circ \mathrm{s}_i$ and $\mathrm{t}_{i,n} = \mathrm{t}_{n-1} \circ \cdots \circ \mathrm{t}_i$. Because $S \mapsto S^*$ is a left-adjoint, it preserves colimits, and there are pushout diagrams:

$$\begin{tikzcd} \mathbf{i} \arrow[r, "\mathrm{s}_{i,n}"] \arrow[d, "\mathrm{t}_{i,n}"'] & \mathbf{n} \arrow[d] \\ \mathbf{n} \arrow[r] & \mathbf{n} +_i \mathbf{n} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{i}^* \arrow[r, "\mathrm{s}_{i,n}^*"] \arrow[d, "\mathrm{t}_{i,n}^*"'] & \mathbf{n}^* \arrow[d] \\ \mathbf{n}^* \arrow[r] & \mathbf{n}^* +_i \mathbf{n}^* \end{tikzcd}$$

The pushout $\mathbf{n} +_i \mathbf{n}$ can be concretely described by generators: starting with two copies of the n-cylinder, whose generators we denote by $\alpha_{1,j}^\pm$, $\beta_{1,j}^\pm$, $|j|_1^\pm$ and $\alpha_{2,j}^\pm$, $\beta_{2,j}^\pm$, $|j|_2^\pm$ respectively, we take the disjoint union of both sets of generators in each dimension and perform the following identifications:

- for $0 \leq j < i$

$$\begin{array}{rcl} \alpha_{1,j}^+ & = & \alpha_{2,j}^+, \\ \alpha_{1,j}^- & = & \alpha_{2,j}^-, \\ \beta_{1,j}^+ & = & \beta_{2,j}^+, \\ \beta_{1,j}^- & = & \beta_{2,j}^-, \\ |j|_1^+ & = & |j|_2^+, \\ |j|_1^- & = & |j|_2^-; \end{array}$$

- for $j = i$

$$\begin{array}{rcl} \alpha_{1,i}^+ & = & \alpha_{2,i}^-, \\ \beta_{1,i}^+ & = & \beta_{2,i}^-, \\ |i|_1^+ & = & |i|_2^-. \end{array}$$

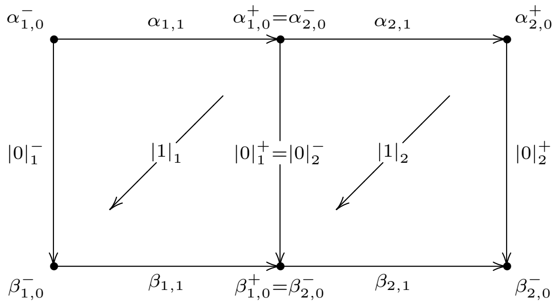

Figure 4: $\mathbf{1} +_0 \mathbf{1}$

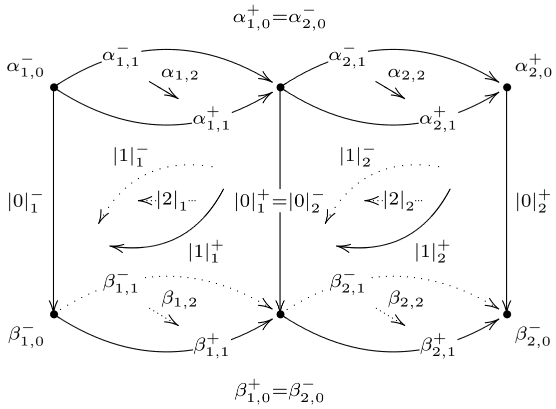

Figure 5: $\mathbf{2} +_0 \mathbf{2}$

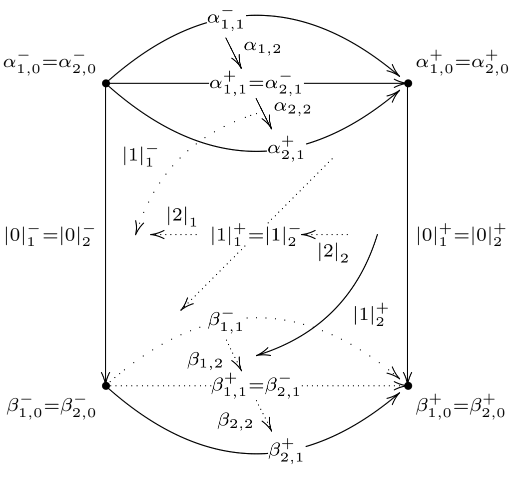

Figure 6: $\mathbf{2} +_1 \mathbf{2}$

The resulting polygraph is well defined because each time we identify two generators, their sources and targets are already identified in lower dimensions. Figures 4, 5 and 6 show $\mathbf{1} +_0 \mathbf{1}$, $\mathbf{2} +_0 \mathbf{2}$ and $\mathbf{2} +_1 \mathbf{2}$ respectively.
It is now possible to define a *comultiplication*

$$\gamma_{i,n} : \mathbf{n}^* \to \mathbf{n}^* +_i \mathbf{n}^*.$$

The value of $\gamma_{i,n}(\xi)$ for generators ξ of $\mathbf{n}$ is given by the following tables:

j	ξ	$\gamma_{i,n}(\xi)$
$0 \leq j < i$	α_j^- α_j^+ β_j^- β_j^+	$\alpha_{1,j}^- = \alpha_{2,j}^-$ $\alpha_{1,j}^+ = \alpha_{2,j}^+$ $\beta_{1,j}^- = \beta_{2,j}^-$ $\beta_{1,j}^+ = \beta_{2,j}^+$
$j = i$	α_i^- α_i^+ β_i^- β_i^+	$\alpha_{1,i}^-$ $\alpha_{2,i}^+$ $\beta_{1,i}^-$ $\beta_{2,i}^+$

j	ξ	$\gamma_{i,n}(\xi)$
$i < j < n$	α_j^- α_j^+ β_j^- β_j^+	$\alpha_{1,j}^- *_i \alpha_{2,j}^-$ $\alpha_{1,j}^+ *_i \alpha_{2,j}^+$ $\beta_{1,j}^- *_i \beta_{2,j}^-$ $\beta_{1,j}^+ *_i \beta_{2,j}^+$
$j = n$	α_n β_n	$\alpha_{1,n} *_i \alpha_{2,n}$ $\beta_{1,n} *_i \beta_{2,n}$

j	ξ	$\gamma_{i,n}(\xi)$
$0 \leq j < i$	$\lvert j\rvert^-$ $\lvert j\rvert^+$	$\lvert j\rvert_1^- = \lvert j\rvert_2^-$ $\lvert j\rvert_1^+ = \lvert j\rvert_2^+$
$j = i$	$\lvert i\rvert^-$ $\lvert i\rvert^+$	$\lvert i\rvert_1^-$ $\lvert i\rvert_2^+$
$i+1 = j < n$	$\lvert j\rvert^-$ $\lvert j\rvert^+$	$(\alpha_{1,i+1}^- *_0 \lvert 0\rvert_1^+ *_1 \cdots *_{i-1} \lvert i-1\rvert_1^+ *_i \lvert j\rvert_2^-) *_{i+1}$ $(\lvert j\rvert_1^- *_i \lvert i-1\rvert_2^- *_{i-1} \cdots *_1 \lvert 0\rvert_2^- *_0 \beta_{2,i+1}^-)$ $(\alpha_{1,i+1}^+ *_0 \lvert 0\rvert_1^+ *_1 \cdots *_{i-1} \lvert i-1\rvert_1^+ *_i \lvert j\rvert_2^+) *_{i+1}$ $(\lvert j\rvert_1^+ *_i \lvert i-1\rvert_2^- *_{i-1} \cdots *_1 \lvert 0\rvert_2^- *_0 \beta_{2,i+1}^+)$
$i+1 < j < n$	$\lvert j\rvert^-$ $\lvert j\rvert^+$	$(\alpha_{1,i+1}^- *_0 \lvert 0\rvert_1^+ *_1 \cdots *_{i-1} \lvert i-1\rvert_1^+ *_i \lvert j\rvert_2^-) *_{i+1}$ $(\lvert j\rvert_1^- *_i \lvert i-1\rvert_2^- *_{i-1} \cdots *_1 \lvert 0\rvert_2^- *_0 \beta_{2,i+1}^+)$ $(\alpha_{1,i+1}^- *_0 \lvert 0\rvert_1^+ *_1 \cdots *_{i-1} \lvert i-1\rvert_1^+ *_i \lvert j\rvert_2^+) *_{i+1}$ $(\lvert j\rvert_1^+ *_i \lvert i-1\rvert_2^- *_{i-1} \cdots *_1 \lvert 0\rvert_2^- *_0 \beta_{2,i+1}^+)$
$i+1 < j = n$	$\lvert n\rvert$	$(\alpha_{1,i+1}^- *_0 \lvert 0\rvert_1^+ *_1 \cdots *_{i-1} \lvert i-1\rvert_1^+ *_i \lvert n\rvert_2) *_{i+1}$ $(\lvert n\rvert_1 *_i \lvert i-1\rvert_2^- *_{i-1} \cdots *_1 \lvert 0\rvert_2^- *_0 \beta_{2,i+1}^+)$
$i+1 = j = n$	$\lvert n\rvert$	$(\alpha_{1,n} *_0 \lvert 0\rvert_1^+ *_1 \cdots *_{n-2} \lvert n-2\rvert_1^+ *_{n-1} \lvert n\rvert_2) *_n$ $(\lvert n\rvert_1 *_{n-1} \lvert n-2\rvert_2^- *_{n-2} \cdots *_1 \lvert 0\rvert_2^- *_0 \beta_{2,n})$

The source and target relations in $\mathbf{n}$ together with the above identifications of cells show that $\gamma_{i,n}$ is well defined.
Remark: the morphism $\gamma_{i,n}$ is not atomic.
From $\gamma_{i,n}$, we immediately get a partial composition operator on C_n^I: let $x, y \in C_n^I$ such that $x \triangleright_i y$, that is $\sigma_{i,n}(y) = \tau_{i,n}(x)$, which amounts to $y \circ \mathrm{s}_{i,n}^* = x \circ \mathrm{t}_{i,n}^*$, whence a unique morphism $[x,y] : \mathbf{n}^* +_i \mathbf{n}^*$ making the following diagram commutative:

$$\begin{tikzcd}
\mathbf{i}^* \arrow[r, "\mathrm{s}^*_{i,n}"] \arrow[d, "\mathrm{t}^*_{i,n}"'] & \mathbf{n}^* \arrow[d] \arrow[rdd, "y"] & \\
\mathbf{n}^* \arrow[r] \arrow[rrd, "x"'] & \mathbf{n}^* +_i \mathbf{n}^* \arrow[rd, "{[x,y]}"] & \\
 & & C
\end{tikzcd}$$

We now define

$$x *_i y = [x,y] \circ \gamma_{n,i},$$

which is again an element of C_n^I.
Likewise, there is a unique morphism $\iota_{n+1} : (\mathbf{n+1})^* \to \mathbf{n}^*$ collapsing α_{n+1}, β_{n+1} and $|n+1|$ to identities on α_n, β_n and $|n|$ respectively. To each $x \in C_n^I$ we associate

$$1_{n+1}(x) = x \circ \iota_{n+1},$$

an element of C^I_{n+1}.
The operations $*_i$ and maps 1_n satisfy the conditions of left and right unit, associativity and exchange. This was shown in detail in [Mét03], where C^I is given a slightly different but equivalent definition, and named HC: we identify C^I_n with $(HC)_n$ by sending each $x \in C^I_n$ to the 5-tuple $(x \circ \sigma^*_{n-1}, x \circ \tau^*_{n-1}, x(\alpha_n), x(\beta_n), x(|n|)) \in (HC)_n$.
As a consequence, the globular set C^I has the expected structure of complex.
Finally we have two morphisms

$$a, b : C^I \to C$$

defined by $a_n(x) = x(\alpha_n)$ and $b_n(x) = x(\beta_n)$ for each $x \in C^I_n$, that is $x : \mathbf{n}^* \to C$.

B Homotopy

We may now define a homotopy relation among morphisms $f, g : D \to C$, where C, D are two complexes. By the previous section, there is a complex C^I equipped with maps $a, b : C^I \to C$, leading to the following definition:

Definition 14 *A* homotopy *from f to g is a morphism $h : D \to C^I$ such that $f = a \circ h$ and $g = b \circ h$.*

Let us denote by $f \rightsquigarrow g$ the existence of such a homotopy. Remark that $\rightsquigarrow$ is a reflexive, but *not* a symmetric relation. We denote by $\sim$ its symmetric and transitive closure: hence $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$. When $f \sim g$, we say that *f and g are homotopic*.
The third point of theorem 1, section 4, is an immediate consequence of the following proposition:

Proposition 6 *Let $p : C \to D$ be a trivial fibration, and f, g maps from S^* to C. If $p \circ f = p \circ g$, then $f \rightsquigarrow g$.*

For each map $p : C \to D$ there is a map $p \times p : C \times C \to D \times D$. Let $C \times_p C$ be the subcomplex of C consisting of pairs of cells (x, y) in C such that $p(x) = p(y)$. We get a canonical inclusion $j : C \times_p C \to C \times C$ as well as a map $q : C \times_p C \to D$ defined by $q\,(x, y) = p(x) = p(y)$, making the following diagram a pullback square:

$$\begin{tikzcd}
C \times_p C \arrow[r, "j"] \arrow[d, "q"'] & C \times C \arrow[d, "p\times p"] \\
D \arrow[r, "\Delta"'] & D \times D
\end{tikzcd}$$

where Δ is the diagonal map $x \mapsto (x, x)$.
We now define a subset E of C^I consisting of cells $x \in C^I$ satisfying the following *collapsing conditions*:

- $p(a(x)) = p(b(x))$;
- if $x : \mathbf{n}^* \to C$ and $0 \leq i \leq n$, $p_{i+1}(x(|i|^{\pm}))$ is a degenerate cell of D_{i+1}.

The stability of E by source and target maps, as well as compositions and identities, makes E a subcomplex of C^I, and we get a canonical inclusion $k : E \to C^I$. Also for each $x \in E$, $(a(x), b(x))$ belongs to $C \times_p C$ because of the first condition, hence a map $r : E \to C \times_p C, x \mapsto (a(x), b(x))$ and a commutative square:

$$\begin{tikzcd}
E \arrow[r, "k"] \arrow[d, "r"'] & C^I \arrow[d, "(a{,}b)"] \\
C \times_p C \arrow[r, "j"'] & C \times C
\end{tikzcd}$$

Lemma 5 *If p is a trivial fibration, so is r.*

Suppose p is a trivial fibration. Let $(x, y) \in (C \times_p C)_0$. Because $p_0(x) = p_0(y)$, there is a cell $z : x \to y$ in C_1 such that $p_1(z) = 1_1(p_0(x)) = 1_1(p_0(y))$. Now $u : \mathbf{0}^* \to C$ defined by $u_0(\alpha_0) = x$, $u_0(\beta_0) = y$ and $u_1\,|0| = z$ belongs to E_0 and $r_0(u) = (x, y)$, hence r_0 is surjective.
Let $n > 0$, u, v parallel cells in E_{n-1} such that there is a $z : r_{n-1}(u) \to r_{n-1}(v)$ in $(C \times_p C)_n$. Let $r_{n-1}(u) = x = \left(x^1, x^2\right)$, $r_{n-1}(v) = y = \left(y^1, y^2\right)$ and $z = \left(z^1, z^2\right)$. Remark that $z^i : x^i \to y^i$ for $i \in \{1, 2\}$ and that $p_n(z^1) = p_n(z^2)$.

We need to find a $w : u \to v$ in E_n satisfying $r_n(w) = z$. Now such a w is a map $\mathbf{n}^* \to C$ so that we must define w on the generating cells of $\mathbf{n}$. By abuse of language, we identify everywhere each generator in $\mathbf{n}_i$ with its canonical image in $\mathbf{n}_i^*$.
Let $\xi \in \mathbf{n}_i$ be a generator. In lower dimensions, $w(\xi)$ is entirely determined by u and v. Precisely:

- If $0 \leq i \leq n-2$, or $\xi \in \{|n-2|^-, |n-2|^+\} \subset \mathbf{n}_{n-1}$, then ξ is already a generator of $\mathbf{n-1}$, and $u(\xi) = v(\xi)$ because $u \parallel v$. Moreover s_{n-1} and t_{n-1} act trivially on those cells, so that we may set $w(\xi) = u(\xi) = v(\xi)$ in this case;
- if ξ is α_{n-1}^- (resp. β_{n-1}^-, $|n-1|^-$), it is $\mathrm{s}_{n-1}(\alpha_{n-1})$ (resp. $\mathrm{s}_{n-1}(\beta_{n-1})$, $\mathrm{s}_{n-1}(|n-1|)$). Thus $w(\xi)$ is $u(\alpha_{n-1})$ (resp. $u(\beta_{n-1})$, $u(|n-1|)$);
- if ξ is α_{n-1}^+ (resp. β_{n-1}^+, $|n-1|^+$), it is $\mathrm{t}_{n-1}(\alpha_{n-1})$ (resp. $\mathrm{t}_{n-1}(\beta_{n-1})$, $\mathrm{t}_{n-1}(|n-1|)$). Thus $w(\xi)$ is $v(\alpha_{n-1})$ (resp. $v(\beta_{n-1})$, $v(|n-1|)$);

The next case relies on the existence of $z : r_{n-1}(u) \to r_{n-1}(v)$:

- If $\xi = \alpha_n \in \mathbf{n}_n$, $w(\alpha_{n-1}^-) = u(\alpha_{n-1})$ and $w(\alpha_{n-1}^+) = v(\alpha_{n-1})$ are respectively $a_{n-1}(u) = x^1$ and $a_{n-1}(v) = y^1$. Take $w(\xi) = w(\alpha_n) = z^1$, with $z^1 : x^1 \to y^1$ in C_n defined above. Likewise, if $\xi = \beta_n$, we define $w(\xi) = w(\beta_n) = z^2$, with $z^2 : x^2 \to y^2$. Recall that $p(z^1) = p(z^2)$.

At this point, $w : \mathbf{n}^* \to C$ is well defined and satisfies the collapsing conditions up to dimension n. We finally turn to the last case, which is the core of the proof:

- Suppose $\xi = |n|$. The source and target of ξ are by definition $\sigma_n\,|n| = \alpha_n *_0 |0|^+ *_1 \cdots *_{n-1} |n-1|^+$ and $\tau_n\,|n| = |n-1|^- *_{n-1} \cdots *_1 |0|^- *_0 \beta_n$. By the above construction of w in dimensions $\leq n$, we get

  $$\begin{array}{rcl} w(\sigma_n(\xi)) & = & w(\alpha_n) *_0 w|0|^+ *_1 \cdots *_{n-1} w|n-1|^+, \\ w(\tau_n(\xi)) & = & w|n-1|^- *_{n-1} \cdots *_1 w|0|^- *_0 w(\beta_n), \end{array}$$

  and the collapsing conditions, together with $p(z^1) = p(z^2)$ imply

  $$\begin{array}{rcl} p(w(\sigma_n(\xi))) & = & p(w(\alpha_n)) \\ & = & p(z^1) \\ & = & p(z^2) \\ & = & p(w(\beta_n)) \\ & = & p(w(\tau_n(\xi))) \end{array}$$

  hence $w(\sigma_n(\xi))$ and $w(\tau_n(\xi))$ are two parallel cells in C_n having the same image in D_n by p. Because p is a trivial fibration, there is a cell $\hat{z} : w(\sigma_n(\xi)) \to w(\tau_n(\xi))$ in C_{n+1} such that $p(\hat{z}) = 1_{n+1}(p(z^1)) = 1_{n+1}(p(z^2))$. Hence we may extend w up to dimension $n+1$ by $w(\xi) = \hat{z}$.

Thus w is a well defined cell in C_n^I satisfying the second collapsing condition. Now, by definition, $a_n(w) = w(\alpha_n) = z^1$ and $b_n(w) = w(\beta_n) = z^2$, so that the first collapsing condition holds. To sum up, $w : u \to v$ belongs to E_n and $r_n(w) = z$, as required. This ends the proof of lemma 5.
Proposition 6 follows immediately: from $f, g : S^* \to C$ we get $(f, g) : S^* \to C \times C$. Because $p \circ f = p \circ g$, (f, g) factorizes through j as $j \circ l$, with $l : S^* \to C \times_p C$. But S^* is cofibrant, and by lemma 5, r is a trivial fibration, whence a $\hat{l} : S^* \to E$ such that $r \circ \hat{l} = l$. Now the following diagram commutes:

$$\begin{tikzcd}
 & E \arrow[r, "k"] \arrow[d, "r"'] & C^I \arrow[d, "{(a,b)}"] \\
S^* \arrow[ur, "\hat{l}"] \arrow[r, "l"'] & C \times_p C \arrow[r, "j"'] & C \times C
\end{tikzcd}$$

and $h = k \circ \hat{l}$ yields the expected homotopy from f to g.
Proposition 6 settles the case of groups. The case of monoids, however, requires a refined version. We first introduce some terminology.

Definition 15 *Let C be a complex. A* right-ideal *$\mathcal{X} \subset C$ is a set of cells, closed by source, target and identities, such that whenever $x \in \mathcal{X}$, $y \in C$ and $x \triangleright_i y$, then $x *_i y \in \mathcal{X}$.*

Hence a right-ideal is in particular a subcomplex of C. Likewise, we get the obvious notion of *left-ideal*.

Definition 16 *Let $(\mathcal{X},\mathcal{Y})$ a pair consisting of a right-ideal $\mathcal{X}$ and a left-ideal $\mathcal{Y}$, and $p : C \to D$ a morphism. p is a* trivial fibration relative to $(\mathcal{X},\mathcal{Y})$ *or short $(\mathcal{X},\mathcal{Y})$-fibration if:*

- *p_0 is surjective;*
- *if $x \in \mathcal{X}_n$, $y \in \mathcal{Y}_n$, $x \parallel y$, and there is $u : p(x) \to p(y)$ in D_{n+1}, then there is a $z : x \to y$ in C_{n+1} such that $p(z) = u$.*

Thus we may state a relativized version of proposition 6:

Proposition 7 *Let $p : C \to D$ be a $(\mathcal{X},\mathcal{Y})$-fibration, and f, g maps from S^* to C such that $f(S^*) \subset \mathcal{X}$ and $g(S^*) \subset \mathcal{Y}$. If $p \circ f = p \circ g$, then $f \rightsquigarrow g$.*

We adapt the above proof as follows: we first replace $C \times_p C$ by $\mathcal{X} \times_p \mathcal{Y}$, the set of pairs (x, y) such that $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $p(x) = p(y)$, and the complex E by its subcomplex $E' = \{x \in E | a(x) \in \mathcal{X}, b(x) \in \mathcal{Y}\}$. We now have $j' : \mathcal{X} \times_p \mathcal{Y} \to C \times C$, $k' : E' \to C^I$ and $r' : E' \to \mathcal{X} \times_p \mathcal{Y}$ making the following diagram commutative:

$$\begin{tikzcd}
E' \arrow[r, "k'"] \arrow[d, "r'"'] & C^I \arrow[d, "(a,b)"] \\
\mathcal{X} \times_p \mathcal{Y} \arrow[r, "j'"'] & C \times C
\end{tikzcd}$$

Then a relativized version of lemma 5 still holds:

Lemma 6 *If p is a $(\mathcal{X},\mathcal{Y})$-fibration, then r' is a trivial fibration.*

The proof goes as above, by constructing $w : \mathbf{n}^* \to C$, except that all pairs of cells previously in $C \times_p C$ now need to be in $\mathcal{X} \times_p \mathcal{Y}$. This is immediate up to dimension n. As for $w\,|n|$, just notice that $w(\alpha_n) \in \mathcal{X}$ and $w(\beta_n) \in \mathcal{Y}$, so that

$$\begin{aligned}
w(\sigma_n(\xi)) &= w(\alpha_n) *_0 w|0|^+ *_1 \cdots *_{n-1} w|n-1|^+ \in \mathcal{X}, \\
w(\tau_n(\xi)) &= w|n-1|^- *_{n-1} \cdots *_1 w|0|^- *_0 w(\beta_n) \in \mathcal{Y},
\end{aligned}$$

because $\mathcal{X}$ is a right-ideal, and $\mathcal{Y}$ a left-ideal. But p is a $(\mathcal{X},\mathcal{Y})$-fibration, so that we still have a cell $\hat{z} : w(\sigma_n(\xi)) \to w(\tau_n(\xi))$ satisfying the expected conditions.
Proposition 7 follows as above: because $f(S^*) \subset \mathcal{X}$ and $g(S^*) \subset \mathcal{Y}$ and $p \circ f = p \circ g$, (f, g) factorizes through j' as $j' \circ l'$ with $l' : S^* \to \mathcal{X} \times_p \mathcal{Y}$. Since r' is trivial fibration, there is a $\hat{l}'$ such that $r' \circ \hat{l}' = l'$. Finally, $h' = k' \circ \hat{l}'$ is a homotopy from f to g.

C Abelianization and homotopy

We finally prove proposition 2 of section 5:

> If $f, g : S^* \to T^*$ are homotopic morphisms, so are $f^{\mathrm{ab}}, g^{\mathrm{ab}} : \mathbb{Z}S \to \mathbb{Z}T$.

It suffice to prove that, if $f, g : S^* \to T^*$ and $f \rightsquigarrow g$, then f^{ab} and g^{ab} are (algebraically) homotopic. Suppose then that there is a map $h : S^* \to (T^*)^I$ such that $a \circ h = f$ and $b \circ h = g$. Using h, we build an algebraic homotopy between f^{ab} and g^{ab}, that is a family of $\mathbb{Z}$-linear maps $k_n : \mathbb{Z}S_n \to \mathbb{Z}T_{n+1}$ such that

$$g_n^{\mathrm{ab}} - f_n^{\mathrm{ab}} = \partial_n \circ k_n + k_{n-1} \circ \partial_{n-1}. \tag{3}$$

Thus, let $\xi \in S_n$, $h_n(\xi) : \mathbf{n}^* \to T^*$, so that $h_n(\xi)\,|n| \in T^*_{n+1}$ and we may define

$$k_n[\xi] = [h_n(\xi)\,|n|]. \tag{4}$$

We first remark that the defining equation (4) extends to non-atomic cells $u \in S^*_n$, so that

$$k_n[u] = [h_n(u)\,|n|]. \tag{5}$$

This is a small but crucial point: let us prove it by structural induction on u.

- If u is a generator, (5) is simply (4);
- if u is of the form $1_n(v)$ with $v \in S^*_{n-1}$, the left hand side of (5) vanishes, and because h is a morphism of complexes,

  $$\begin{aligned} h_n(1_n(v)) &= 1_n(h_{n-1}(v)) \\ &= h_{n-1}(v) \circ \iota_n, \end{aligned}$$

  but $\iota_n\, |n| = 1_n\, |n-1|$, so that $h_n(u)\, |n|$ is degenerate and $[h_n(u)\, |n|] = 0$;
- if u is of the form $v *_i w$, where v and w satisfy the induction hypothesis, $h_n(u) = h_n(v) *_i h_n(w)$, using again the fact that h is a morphism of complexes. Now the composition formulas in T^{*I} show that the only non-degenerate cells in $(h_n(v) *_i h_n(w))\, |n|$ are $h_n(v)\, |n|$ and $h_n(w)\, |n|$ so that

  $$\begin{aligned} [(h_n(v) *_i h_n(w))\, |n|] &= [h_n(v)\, |n|] + [h_n(w)\, |n|] \\ &= k_n[v] + k_n[w] \\ &= k_n([v] + [w]) \\ &= k_n[v *_i w] \\ &= k_n[u]. \end{aligned}$$

Hence the result.
Let us now compute $\partial_n \circ k_n(\xi)$: by using the expressions of the source and target of $|n|$, and applying the morphism $h_n(\xi)$, we get

$$\begin{aligned} \sigma_n(h_n(\xi)\, |n|) &= h_n(\xi)(\alpha_n) *_0 h_n(\xi)|0|^+ *_1 \cdots *_{n-1} h_n(\xi)|n-1|^+, \\ \tau_n(h_n(\xi)\, |n|) &= h_n(\xi)|n-1|^- *_{n-1} \cdots *_1 h_n(\xi)|0|^- *_0 h_n(\xi)(\beta_n). \end{aligned}$$

When linearizing, all degenerate cells vanish, so that

$$\partial_n[h_n(\xi)\, |n|] \quad = \quad ([h_n(\xi)(\beta_n)] - [h_n(\xi)(\alpha_n)]) - ([h_n(\xi)|n-1|^+] - [h_n(\xi)|n-1|^-]).$$

By definition, $h_n(\xi)(\beta_n) = b_n \circ h_n(\xi) = g_n(\xi)$ and $h_n(\xi)(\alpha_n) = a_n \circ h_n(\xi) = f_n(\xi)$.
On the other hand

$$\begin{aligned} h_n(\xi)|n-1|^+ &= h_n(\xi)(\mathrm{t}_{n-1}\, |n-1|) \\ &= h_{n-1}(\tau_{n-1}(\xi))\, |n-1|\, . \end{aligned}$$

Likewise

$$\begin{aligned} h_n(\xi)|n-1|^- &= h_n(\xi)(\mathrm{s}_{n-1}\, |n-1|) \\ &= h_{n-1}(\sigma_{n-1}(\xi))\, |n-1|\, , \end{aligned}$$

and by using (5) we get

$$\begin{aligned} [h_n(\xi)|n-1|^+] - [h_n(\xi)|n-1|^-] &= [h_{n-1}(\tau_{n-1}(\xi))\, |n-1|] - [h_{n-1}(\sigma_{n-1}(\xi))\, |n-1|] \\ &= k_{n-1}[\tau_{n-1}(\xi)] - k_{n-1}[\sigma_{n-1}(\xi)] \\ &= k_{n-1} \circ \partial_{n-1}[\xi]. \end{aligned}$$

Finally

$$\partial_n \circ k_n[\xi] = (g_n^{\mathrm{ab}} - f_n^{\mathrm{ab}})[\xi] - k_{n-1} \circ \partial_{n-1}[\xi],$$

which immediately gives (3), and ends the proof.

D Counting generators

If A is an (additive) abelian monoid, any n-category $C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n$ extends to a $n{+}1$-category $C_0 \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n \Leftarrow \langle A : C_n \rangle$ as follows:

- $\langle A : C_n\rangle$ is the set of formal cells $\langle a : x \to y\rangle$ where $a \in A$ and $x \parallel y$ in C_n;
- $\langle a : x \to y\rangle : x \to y$ for all $a \in A$ and $x \parallel y$ in C_n;
- $\langle a : x \to y\rangle *_n \langle b : y \to z\rangle = \langle a + b : x \to z\rangle$ for all $a, b \in A$ and $x \parallel y \parallel z$ in C_n;
- $\langle a : x \to z\rangle *_i \langle b : y \to t\rangle = \langle a + b : x *_i y \to z *_i t\rangle$ for all $a, b \in A$ and $x, y, z, t \in C_n$ such that $i < n$ and $x \parallel z$, $y \parallel t$, $x \rhd_i y$ (which implies $z \rhd_i t$ by the boundary conditions);
- $1_{n+1}(x) = \langle 0 : x \to x\rangle$ for all $x \in C_n$.

It is easy to see that those operations satisfy the conditions of left and right unit, associativity, and exchange.

In particular, if $S_0^* \Leftarrow S_1, S_1^* \Leftarrow S_2, \ldots, S_{n-1}^* \Leftarrow S_n, S_n^* \Leftarrow S_{n+1}$ is a $n{+}1$-polygraph, we have a n-category $S_0^* \Leftarrow S_1^* \Leftarrow S_2^* \cdots S_{n-1}^* \Leftarrow S_n^*$ and an injection of S_{n+1} into $\langle \mathbb{Z}S_{n+1} : S_n^*\rangle$ mapping $\xi : x \to y$ to $\langle [\xi] : x \to y\rangle$. By the universal property of S_{n+1}^*, we get a map $\rho : S_{n+1}^* \to \langle \mathbb{Z}S_{n+1} : S_n^*\rangle$ such that $\sigma_n \circ \rho = \sigma_n$ and $\tau_n \circ \rho = \tau_n$, which is compatible with products and units.

This means that $\rho(u) = \langle [u] : x \to y\rangle$ for all $u : x \to y$ in S_{n+1}^*, where the map $u \mapsto [u]$ extends the one defined on S_{n+1} and satisfies the following properties:

$$[u *_i v] = [u] + [v] \text{ for all } u \rhd_i v \text{ in } S_{n+1}^* \text{ with } i < n+1, \qquad [1_{n+1}(x)] = 0 \text{ for all } x \in S_n^*.$$

E Decomposition

If $C : C_0 \Leftarrow C_1$ is a category, we define a monoid $\hat{C}$ as follows:

- $\hat{C}$ is the set of families $(u_x)_{x \in C_0}$ such that $u_x \in C_1$ and $\sigma_0(u_x) = x$ for all $x \in C_0$;
- the product of two such families $(u_x)_{x \in C_0}$ and $(v_x)_{x \in C_0}$ is the family $(w_x)_{x \in C_0}$ defined by $w_x = u_x *_0 v_y$ where $y = \tau_0(u_x)$ for all $x \in C_0$.

It is easy to see that this product is associative, with unit $(1_1(x))_{x \in C_0}$.

In particular, if M is a monoid, S_1 is an alphabet and $f : S_1^* \to M$ is a morphism of monoid, we have a category $C : M \Leftarrow (M \cdot S_1)^*$ and a canonical injection of S_1 into $\hat{C}$ mapping $\xi \in S_1$ to the family $\langle \lambda \cdot \xi\rangle_{\lambda \in M}$. By the universal property of S_1^*, we get a morphism $\rho : S_1^* \to \hat{C}$.

This means that $\rho(x)$ is the family $\langle \lambda \cdot x\rangle_{\lambda \in M}$ for all $x \in S_1^*$, where the map $\lambda \cdot x \mapsto \langle \lambda \cdot x\rangle$ extends the one defined on $M \cdot S_1$ and satisfies the following properties:

- $\sigma_0\langle \lambda \cdot x\rangle = \lambda$ for all $\lambda \in M$ and $x \in S_1^*$;
- $\langle \lambda \cdot xy\rangle = \langle \lambda \cdot x\rangle *_0 \langle \mu \cdot y\rangle$ where $\mu = \tau_0\langle \lambda \cdot x\rangle$ for all $\lambda \in M$ and $x, y \in S_1^*$;
- $\langle \lambda \cdot 1\rangle = 1_1(\lambda)$ for all $\lambda \in M$.

Furthermore, we have $\tau_0\langle \lambda \cdot x\rangle = \lambda\overline{x}$ for all $\lambda \in M$ and $x \in S_1^*$: this is easily proved by induction on $x \in S_1^*$. Hence, we get the expected properties for $\langle \lambda \cdot x\rangle$ in case $x \in S_1^*$.

Now, we consider a monoid M, a n-monoid $C : \top \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n$ with $n > 0$, a morphism $f : C \to M$, and a $n{+}1$-category $D : M \Leftarrow M \cdot C_1 \Leftarrow M \cdot C_2 \cdots M \cdot C_{n-1} \Leftarrow M \cdot C_n \Leftarrow D_{n+1}$ extending the n-monoid $M \cdot C$. Then, C extends to a $n{+}1$-monoid $\top \Leftarrow C_1 \Leftarrow C_2 \cdots C_{n-1} \Leftarrow C_n \Leftarrow \hat{D}$ defined as follows:

- $\hat{D}$ is the set of formal cells $\langle (u_\lambda)_{\lambda \in M} : x \to y\rangle$ where $x \parallel y$ in C_n and $(u_\lambda)_{\lambda \in M}$ is a family such that $u_\lambda : \lambda \cdot x \to \lambda \cdot y$ in D_{n+1} for all $\lambda \in M$;
- $\langle (u_\lambda)_{\lambda \in M} : x \to y\rangle : x \to y$ for $x \parallel y$ in C_n;
- $\langle (u_\lambda)_{\lambda \in M} : x \to z\rangle\langle (v_\lambda)_{\lambda \in M} : y \to t\rangle = \langle (u_\lambda *_0 v_{\lambda\overline{x}})_{\lambda \in M} : xy \to zt\rangle$ for $x \parallel z$ and $y \parallel t$ in C_n;
- $\langle (u_\lambda)_{\lambda \in M} : x \to z\rangle *_i \langle (v_\lambda)_{\lambda \in M} : y \to t\rangle = \langle (u_\lambda *_i v_\lambda)_{\lambda \in M} : x *_i y \to z *_i t\rangle$, for $x, y, z, t \in C_n$ such that $n > i > 0$ and $x \parallel z$, $y \parallel t$, $x \rhd_i y$ (which implies $z \rhd_i t$ by the boundary conditions);
- $\langle (u_\lambda)_{\lambda \in M} : x \to y\rangle *_n \langle (v_\lambda)_{\lambda \in M} : y \to z\rangle = \langle (u_\lambda *_n v_\lambda)_{\lambda \in M} : x \to z\rangle$ for $x \parallel y \parallel z$ in C_n;

- $1_{n+1}(x) = \langle (1_{n+1}(\lambda \cdot x))_{\lambda \in M} : x \to x \rangle$.

It is easy to see that those operations satisfy the conditions of left and right unit, associativity, and exchange.

In particular, consider a $n{+}1$-polygraph $\top \Leftarrow S_1, S_1^* \Leftarrow S_2, \ldots, S_{n-1}^* \Leftarrow S_n, S_n^* \Leftarrow S_{n+1}$, and assume that C is the n-monoid $\top \Leftarrow S_1^* \Leftarrow S_2^* \cdots S_{n-1}^* \Leftarrow S_n^*$ and D_{n+1} is $(M \cdot S_{n+1})^*$. Then, we have an injection of S_{n+1} into $\hat{D}$ mapping $\xi : x \to y$ to $\langle \langle \lambda \cdot \xi \rangle_{\lambda \in M} : x \to y \rangle$. By the universal property of S_{n+1}^*, we get a map $\rho : S_{n+1}^* \to \hat{D}$ such that $\sigma_n \circ \rho = \sigma_n$ and $\tau_n \circ \rho = \tau_n$, which is compatible with products and units.

This means that $\rho(u) = \langle \langle \lambda \cdot u \rangle_{\lambda \in M} : x \to y \rangle$ for all $u : x \to y$ in S_{n+1}^*, where the map $\lambda \cdot u \mapsto \langle \lambda \cdot u \rangle$ extends the one defined on $M \cdot S_{n+1}$ and satisfies the following properties:

- $\langle \lambda \cdot u \rangle : \lambda \cdot x \to \lambda \cdot y$ for all $\lambda \in M$ and $u : x \to y$ in S_{n+1}^*;
- $\langle \lambda \cdot uv \rangle = \langle \lambda \cdot u \rangle *_0 \langle \lambda \overline{u} \cdot v \rangle$ for all $\lambda \in M$ and $u, v \in S_{n+1}^*$.
- $\langle \lambda \cdot u *_i v \rangle = \langle \lambda \cdot u \rangle *_i \langle \lambda \cdot v \rangle$ for all $\lambda \in M$ and $u \rhd_i v$ in S_{n+1}^* with $n + 1 > i > 0$;
- $\langle \lambda \cdot 1_{n+1}(x) \rangle = 1_{n+1}(\lambda \cdot x)$ for all $\lambda \in M$ and $x \in S_n^*$.

Hence, we get the expected properties for $\langle \lambda \cdot u \rangle$ in case $u \in S_{n+1}^*$.