Homotopical semantics of parallel composition in CCS

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Parallel composition in CCS

- Combinatorial semantics using a kind of coskeleton construction on labelled precubical sets (slight modification of a construction due to K. Worytkiewicz)
- Categorical semantics using algebraic relations on labelled flows (a very well-known idea in theoretical computer science: notion of commutative face)
- Equivalence of the two semantics using a realization functor from labelled precubical sets to labelled flows (Homotopical construction)

Example of labelled 2-cubes

- $\Sigma = \{a, b, c, \dots\} \cup \{\overline{a}, \overline{b}, \overline{c}, \dots\} \cup \{\tau\}$ with $\overline{\overline{a}} = a$
- Arbitrary total ordering $(\Sigma, \leqslant)$: label for concurrent execution of a and b = label for concurrent execution of b and a

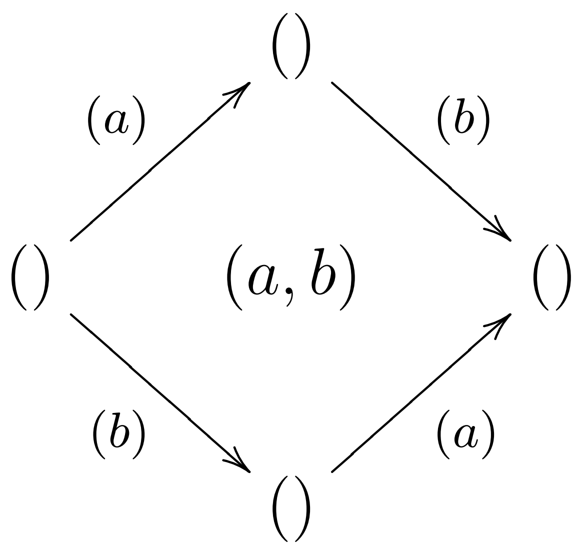


Figure 1: Concurrent execution of a and b with $a \leqslant b$

Example of labelled flows

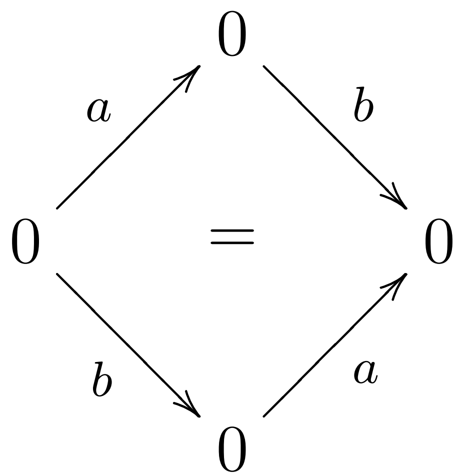


Figure 2: Concurrent execution of a and b

HDA paradigm

- One n-transition (concurrent execution of n actions) corresponding to one full n-cube

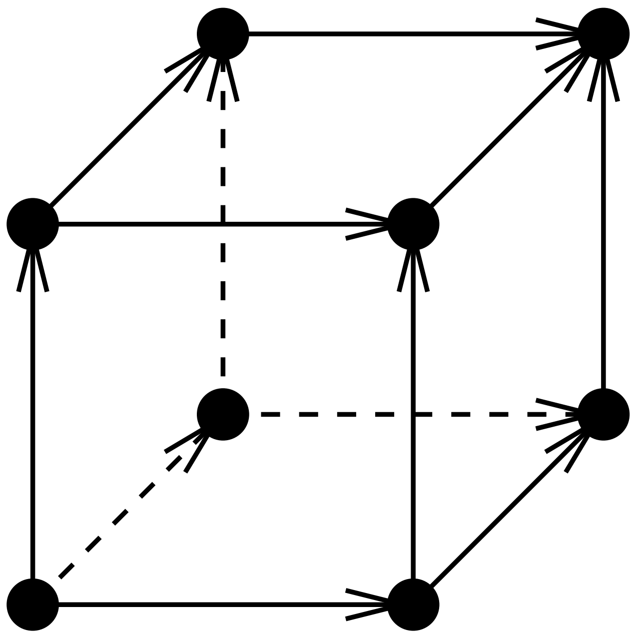
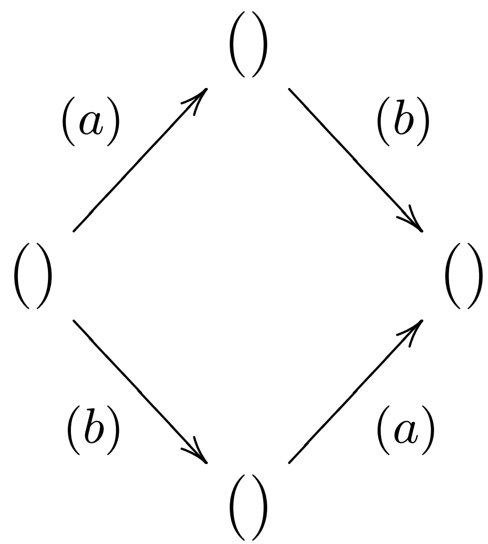


Figure 3: The full 3-cube

Example of impossible HDA

- Start from an empty labelled 2-cube



- Add two squares (combinatorial semantics)
- Impossible HDA since either a and b run sequentially (empty case), or a and b run concurrently (full case)
- Note: impossible to add two algebraic relations

Concurrency and synchronization

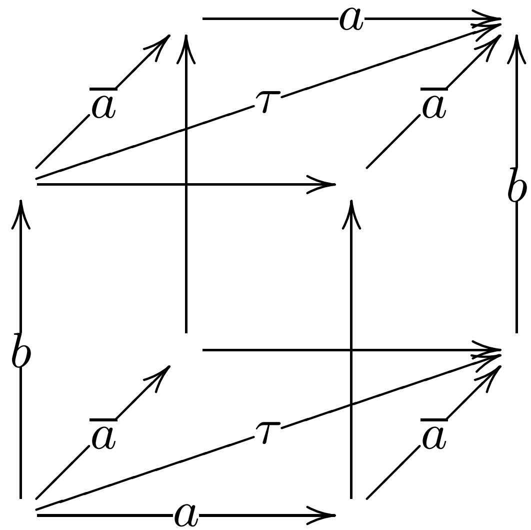


Figure 4: 1-dimensional paths of $(a.nil|b.nil)|\overline{a}.nil$

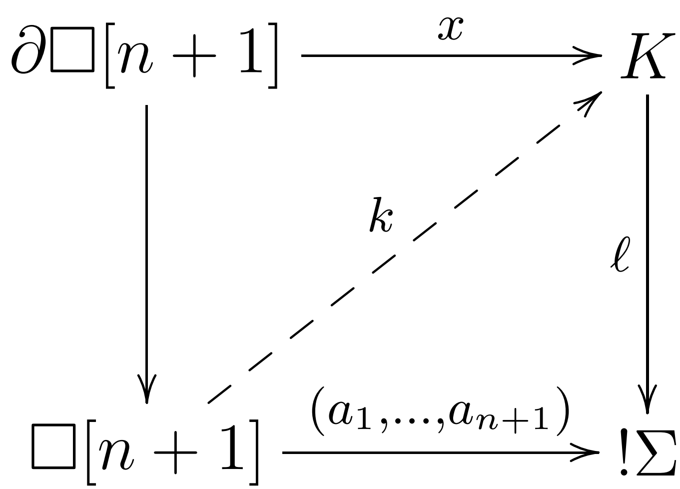
- Combinatorics: filling out the shells without forgetting HDA paradigm
- Categorical: adding algebraic relations on morphisms

Labelled precubical set

- Precubical set K: cubical set without degeneracy maps
- Precubical set of labels $!\Sigma$ (E. Goubault)
  - $(!\Sigma)_0 = \{()\}$
  - for $n \geqslant 1$,
    $(!\Sigma)_n = \{(a_1, \ldots, a_n) \in \Sigma \times \ldots \times \Sigma, a_1 \leqslant \ldots \leqslant a_n\}$
  - $\partial_i^0(a_1, \ldots, a_n) = \partial_i^1(a_1, \ldots, a_n) = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$
- Labelled precubical set: $\ell : K \to !\Sigma$
- Two opposite faces have same labelling

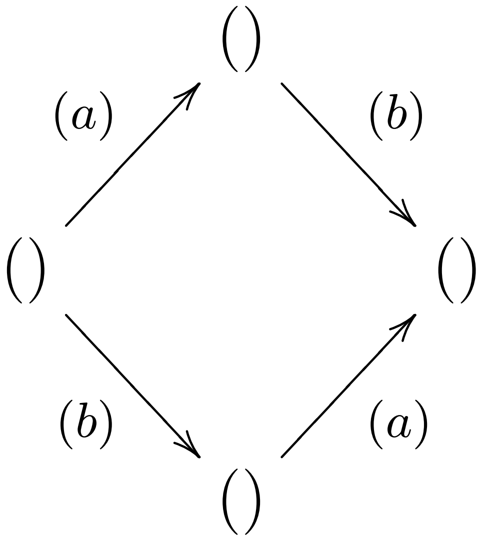
Labelled n-shell

- Labelled precubical set $\ell : K \to !\Sigma$
- $n \geqslant 1$, $(n+1)$-cube $\Box[n+1]$, boundary of the $(n+1)$-cube $\partial\Box[n+1]$
- Labelled n-shell of K : commutative diagram



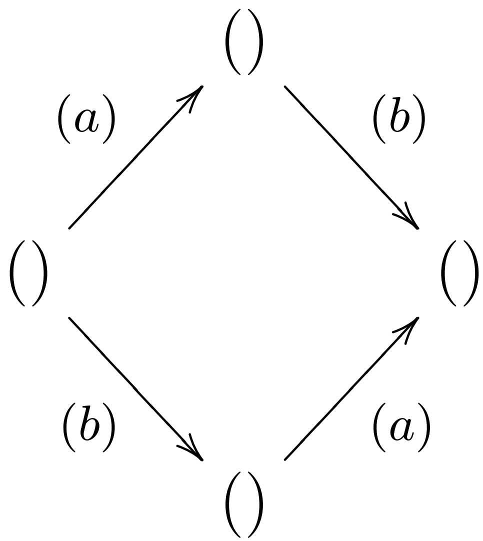
- Filling out shell: adding lift k

All 1-shells of an empty 2-cube



- 2! non-degenerate labelled 1-shells ($p!$ cardinal of the group of automorphisms of $\{0 < 1\}^p$)
- 2 degenerate labelled 1-shells if $a = b$
- Usual coskeleton functor generates too many cubes !!

Non-twisted labelled shell

- There exists a notion of non-twisted labelled shells on labelled precubical sets K with $K_0 = \{0,1\}^p$ for some $p \geqslant 2$ such that:
- 

- 1 non-twisted labelled 1-shell (one of the two non-degenerate ones)

Non-twisted labelled shell

- Let $K_0 = [p] = \{0,1\}^p$.
- $$\begin{array}{ccc} \partial\Box[n+1] & \xrightarrow{x} & K \\ \downarrow & & \downarrow \ell \\ \Box[n+1] & \longrightarrow & !\Sigma. \end{array}$$

  non-twisted if
- $\exists[q], \exists\phi, \exists\psi, x_0 : [n+1] \xrightarrow{\phi} [q] \xrightarrow{\psi} [p]$ where
  - $\psi \in \Box$
  - $\phi : (\epsilon_1, \ldots, \epsilon_{n+1}) \mapsto (\epsilon_{i_1}, \ldots, \epsilon_{i_q})$, $1 = i_1 \leqslant \ldots \leqslant i_q = n+1$, $\{1, \ldots, n+1\} \subset \{i_1, \ldots, i_q\}$
- Why the repetition of coordinates in ϕ ? Answer later

Labelled coskeleton construction

- A 1-dimensional labelled precubical set K with $K_0 = \{0,1\}^p$
- $K^{(1)} = K$ and

$$
\begin{array}{ccc}
\displaystyle\bigsqcup_{\text{non-twisted labelled } n\text{-shells}} \partial\square[n+1] & \longrightarrow & K^{(n)} \\
\downarrow & & \downarrow \\
\displaystyle\bigsqcup_{\text{non-twisted labelled } n\text{-shells}} \square[n+1] & \longrightarrow & K^{(n+1)}.
\end{array}
$$

$$
\boxed{\mathrm{COSK}^{CCS}(K) := \varinjlim_{n} K^{(n)}}
$$

Property of coskeleton construction

- $n \geqslant 2$
- $\square[n]$ labelled n-cube
- $\square[n]_{\leqslant 1}$ 1-dimensional part
- Bijection between the set of non-twisted p-shells of $\square[n]$ and the $(p+1)$-cube of $\square[n]$ for $p \geqslant 1$

$$\boxed{\mathrm{COSK}^{CCS}(\square[n]_{\leqslant 1}) \cong \square[n]}$$

Parallel composition (local situation)

- Labelled m-cube $\square[m]$
- Labelled n-cube $\square[n]$
- Use of the 1-dimensional operational semantics:
  - $Z_0 := \square[m]_0 \times \square[n]_0 \cong [m+n]$
  - $Z_1 := (\square[m]_1 \times \square[n]_0) \oplus (\square[m]_0 \times \square[n]_1) \oplus \{(x,y) \in \square[m]_1 \times \square[n]_1, \ell(x) = \overline{\ell(y)}\}$
  - if $\ell(x) = \overline{\ell(y)}$, then $\ell(x,y) = \tau$
- Filling out all non-twisted shells:

$$\boxed{\square[m] \otimes_\sigma \square[n] := \mathrm{COSK}^{CCS}(Z)}$$

Parallel composition (global situation)

- Two labelled precubical sets K and L
- $$\boxed{K \otimes_\sigma L := \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \square[m] \otimes_\sigma \square[n]}$$
- $$\boxed{K \otimes_\sigma \square[0] \cong K}$$
- If no synchronizations, then $K \otimes_\sigma L \cong K \otimes L$ (tensor product of precubical sets) since
$$\boxed{K \otimes_\sigma L \cong \varinjlim_{\square[m]\to K} \varinjlim_{\square[n]\to L} \square[m+n]}$$

Labelled flow

- Flow X: small category without identities enriched over compactly generated topological spaces
- Set X^0 of states/objects of X
- Compactly generated space $\mathbb{P}_{\alpha,\beta}X$ of morphisms/non-constant execution paths from α to β of X^0
- Flow of labels $?\Sigma$
  - $(?\Sigma)^0 = \{0\}$
  - $\mathbb{P}(?\Sigma)$ non-unitary associative free monoid generated by Σ and $ab = ba$
- Labelled flow: $\ell : X \to ?\Sigma$

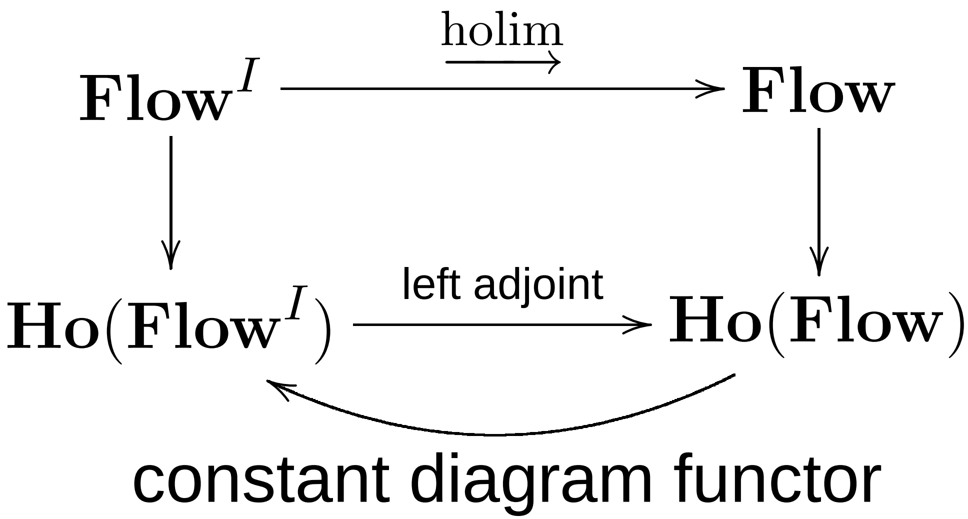
Realizing poset

- $(A, \leqslant)$ poset
- Element of A = State
- Execution path from α to β iff $\alpha < \beta$
- Note $\mathbb{P}_{\alpha,\alpha}A = \varnothing$ for all $\alpha \in A$ (A as flow without loops)
- Partial ordering models observable time ordering

Functor $\{$poset+strictly increasing map$\} \rightarrow \mathbf{Flow}$

Weak S-homotopy and realization

- $f : X \xrightarrow{\simeq} Y$ weak S-homotopy iff f^0 bijection and $\mathbb{P}f$ weak homotopy equivalence

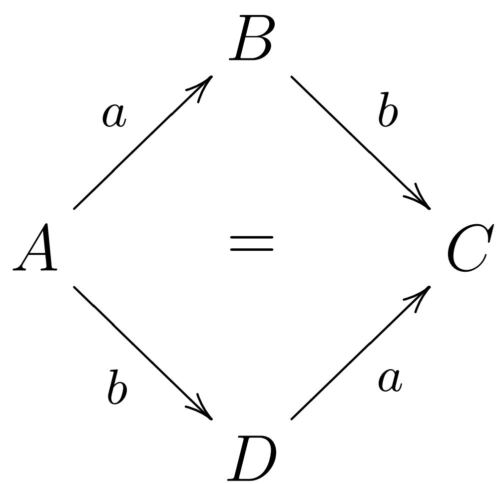


- K precubical set

- $$|K| := \varinjlim_{\square[n] \to K} \{\widehat{0} < \widehat{1}\}^n$$

Parallel composition (local situation)

- Construction of $|\square[m]| \otimes_\sigma |\square[n]|$:
  - Operational semantics: $(\square[m] \otimes_\sigma \square[n])_{\leqslant 1}$
  - Realization as labelled 1-dimensional flow
  - Adding one algebraic relation for each pair of commuting labels (a, b):



- $$\boxed{|\square[m]| \otimes_\sigma |\square[n]| \simeq |\square[m] \otimes_\sigma \square[n]|}$$
- Coskeleton construction removed !

Parallel composition (global situation)

- Two labelled precubical sets K and L
- $$\boxed{|K \otimes_\sigma L| \simeq \varinjlim_{\square[m]\to K}{}^{\!\!\text{holim}} \ \varinjlim_{\square[n]\to L}{}^{\!\!\text{holim}} |\square[m]| \otimes_\sigma |\square[n]| := |K| \otimes_\sigma |L|}$$
- Formula above false if $\underrightarrow{\text{holim}}$ replaced by $\varinjlim$
- Problem: how to define $X \otimes_\sigma Y$ for any labelled flow X and Y ?