Simplicial approximations and coverings for state-spaces (very preliminary)

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INVAL, second meeting

Modelisation and Analysis of Systems in Interaction lab

CEA/Saclay and Ecole Polytechnique

Outline of the talk

- Po-spaces, components, and the problem of loops
- Streams
- Simplicial sets, classifying spaces and rewriting systems
- 2nd idea for definition of components of looping spaces

Some motivating examples from nature

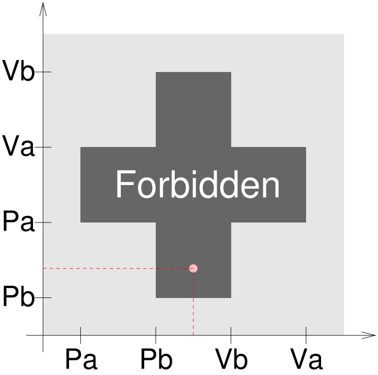
1. Topological semilattices
2. Ordered topological vector spaces
3. Time-oriented Einstein manifolds
4. Domains with their Lawson topologies
5. "Topological" directed graphs

Basic definitions

A partially ordered space $(X, \leqslant_X)$ is a space X with a partial order $\leqslant_X$ on its points.

Example: Swiss flag

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



Some reasonable axioms

A subset A of a partially ordered space $(X, \leqslant_X)$ is *convex* if $y \in A$ whenever $x \leqslant_X y \leqslant_X z$ and $x, z \in A$.

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A partially ordered space $(X, \leqslant_X)$ is *locally convex* if every point has a neighborhood basis of convex subsets, i.e. $(X, \leqslant_X)$ is "locally determined."

Some reasonable axioms

In a partially ordered space $(X, \leqslant_X)$, call a set

$$[a, b] = \{x \mid a \leqslant_X x \leqslant_X b\}$$

a *bounded interval*.

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Typically, state spaces are locally convex and their bounded intervals have connected closure.

Example of a looping space...

```
A=Pa.(Pb.Vb.Pc.Vc)*.Va
B=Pc.Pb.Vc.Pa.Va.Vb
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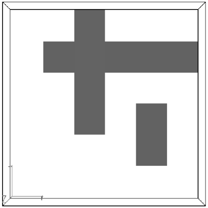

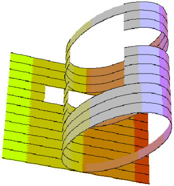

Time travel: quotients of partially ordered spaces

Given a partially ordered space $(X, \leqslant_X)$, want to define

$$"(X, \leqslant_X)/\sim\ "$$

to be $X/\sim$ plus something more...

Locally preordered spaces: streams

A *stream* $(X, \leqslant)$ is a space X with a function $\leqslant$ assigning a preorder $\leqslant_U$ on each open subset U such that $\leqslant$ sends unions to transitive unions.

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Define *stream maps* to preserve all structure in sight, giving us a category $\mathscr{S}$ of streams.

Categorical goodness

Restricting to a category $\mathscr{S}'$ of quotients of locally compact Hausdorff streams by closed equivalences:

Theorem

$\mathscr{S}'$ is complete, cocomplete, Cartesian closed.

The category generalizes reasonable things

Let $\mathscr{K}$ be category of compact Hausdorff, locally convex preordered spaces whose bounded intervals have connected closure.

Some examples in $\mathscr{K}$:

1. Connected, compact Hausdorff topological lattices.
2. Connected chains whose open intervals generate the topology.

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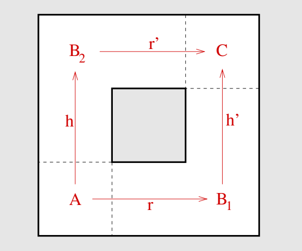
Theorem

$\mathscr{K}$ fully and faithfully embeds into $\mathscr{S}'$.

How to retract?

The fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ of a pospace $\overrightarrow{X}$

- Starting with a variation on the Poincaré groupoid, $\pi_1(X)$ defined as the category:
  - objects: points of X,
  - morphisms: classes of dipaths up to dihomotopy: a morphism from x to y is a dihomotopy class $[\alpha]$ of a dipath α going from x to y.
- We see that in most interesting (to static analysis) case, it is "essentially" finite



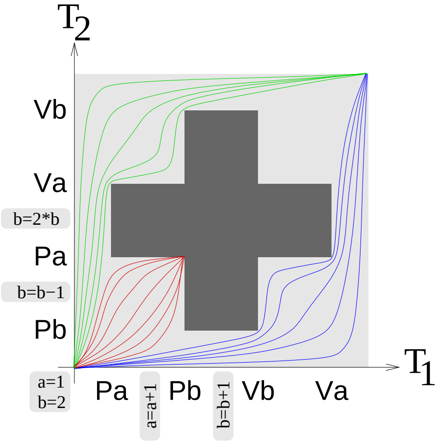
Classes of equivalent dipaths

up to dihomotopy

T1 gets a and b before T2 => a=2 and b=4

T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1

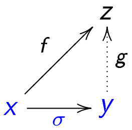


Yoneda morphism

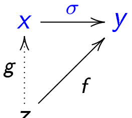
axiomatizing the preservation of the future and the past (1)

Let $\mathcal{C}$ be a small category. A *Yoneda* morphism σ is an element of $\mathcal{C}[x,y]$ such that for all object z of $\mathcal{C}$,

future if $\mathcal{C}[y,z] \neq \emptyset$ then for all $f \in \mathcal{C}[x,z]$, there is a unique $g \in \mathcal{C}[y,z]$ such that



past if $\mathcal{C}[z,x] \neq \emptyset$ then for all $f \in \mathcal{C}[z,y]$, there is a unique $g \in \mathcal{C}[z,x]$ such that

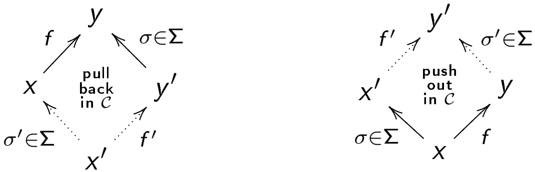


Yoneda system of a small category $\mathcal{C}$

axiomatizing the preservation of the future and the past (2)

A collection Σ of morphisms of $\mathcal{C}$ such that:

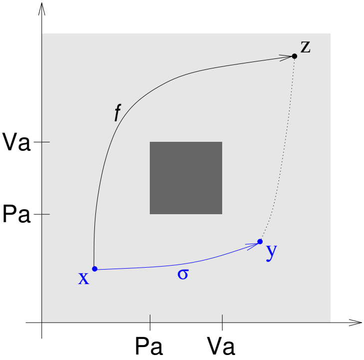
1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.

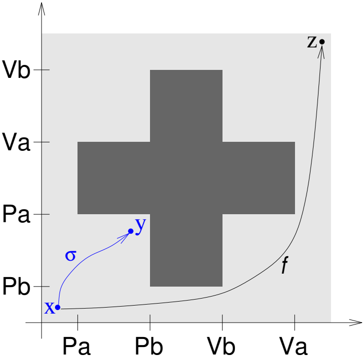


Define now the component category to be $\overrightarrow{\pi_0}(X)$ equal to the category of fractions of $\pi_1(X)$ by the maximal Yoneda System (equivalently as we shall see, as the quotient category of the same two categories).

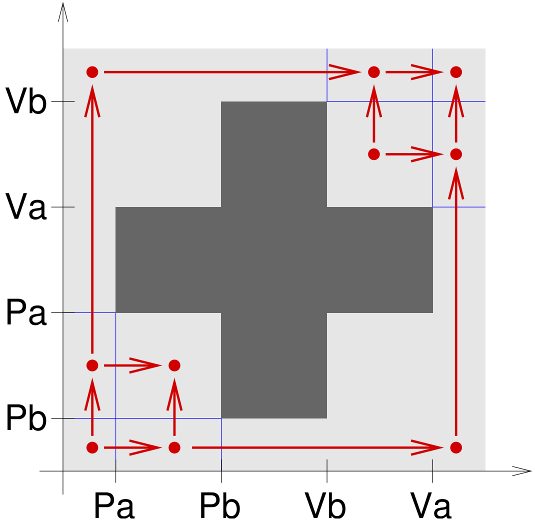
Examples

of morphisms which do not belong to a *Yoneda* system

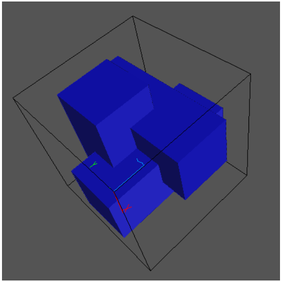


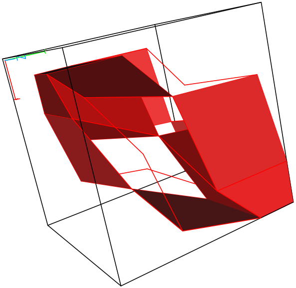


The category of components of the Swiss flag



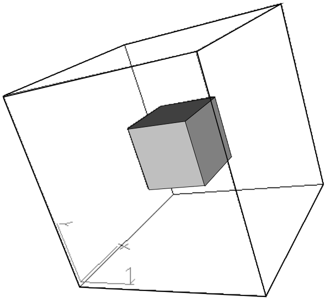
The components category of the 3 philosophers

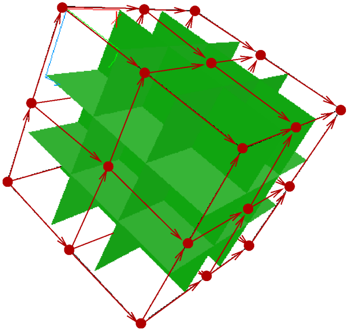




the pospace | its category of components

The components category of a 2-semaphore





the pospace its category of components

Notice: a certain amount of the classical π_2 is apparent; and $\overrightarrow{\pi_1}$ has no "cancellation" property in general

Main interesting properties of components

fractions *vs* quotients

Let $\mathcal{C}$ be a small loop-free category and Σ a *Yoneda* system of $\mathcal{C}$:

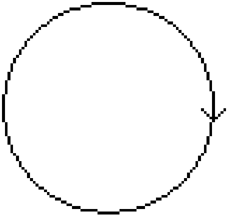
1. the collection Σ is pure in $\mathcal{C}$,
2. the small category $\mathcal{C}/_{\Sigma}$ is loop-free,
3. the small categories $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}/_{\Sigma}$ are equivalent and
4. the category $\mathcal{C}[\Sigma^{-1}]$ is fibered over $\mathcal{C}/_{\Sigma}$.
5. Seifert/van Kampen on component categories
6. if $\overrightarrow{K}$ is a compact pospace, then any component of $\overrightarrow{\pi_1}(\overrightarrow{K})$ has both a greatest lower bound and an least upper bound in $(|K|, \sqsubseteq)$.

see *Ph. D. Thesis* of E. Haucourt

see *Components of the Fundamental Category* - APCS 04, L. Fajstrup, E. Goubault, E. Haucourt, M. Raussen

see also *Components of the Fundamental Category II* - APCS 07, E. Goubault, E. Haucourt

The problem with loops



- only identities are Yoneda invertible!
- we would like something like $\mathbb{N}$

A conjectural dictionary

homotopy	dihomotopy
set of path components	component category
minimal Kan complexes	”weakly minimal” weak Kan complexes

conjecture: Under this dictionary, nerve of component category coincides with weakly minimal Kan complexes in dimensions 0,1,2 except degenerate cases?

Monoids as looping processes

Definition

The nerve $N_*(M)$ of a monoid M is the simplicial set defined by

$$N_k(M) = M^k$$

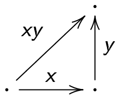
where degeneracy maps are defined by

$$d_i(m_1, \ldots, m_k) = \begin{cases} (m_2, \ldots, m_k) & i = 0 \\ (m_1, \ldots, m_{i-1}, m_i m_{i+1}, m_{i+2}, \ldots, m_k) & 0 < i < k \\ (m_1, \ldots, m_{n-1}) & i = k \end{cases}$$

and face maps are defined by

$$s_i(m_1, \ldots, m_{n+1}) = (m_1, \ldots, m_{i-1}, 1, m_i, \ldots, m_k).$$

The nerve of a monoid: a very pretty picture



Classifying spaces and rewriting

Let BM be the geometric realization $|N_*(M)|$.

Theorem (K. Brown, 89)

BM is homotopy equivalent to a CW complex of finite type if it is presented by a finite, complete rewriting system.

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proof: complete rewriting system specifies how to collapse cells.

Classifying streams and rewriting

We can define a *classifying stream* $\vec{B}M$. The same proof shows:

Theorem

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$\vec{B}M$ is homotopy equivalent to a CW stream of finite type if it is presented by a finite, complete, rewriting system.

proof: complete rewriting system specifies "monotone" collapses

Stream realization of simplicial sets

For each $n > 0$, define $\nabla(n)$ to be partially ordered space

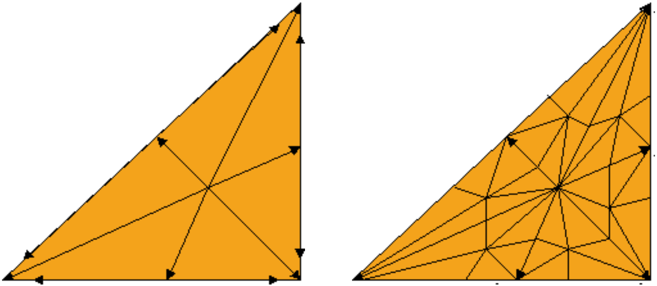
$$\{(t_0, \ldots, t_n) \mid 0 = t_0 \leq \ldots \leq t_n \leq 1\}.$$

and redefine, in the category $\mathscr{S}'$,

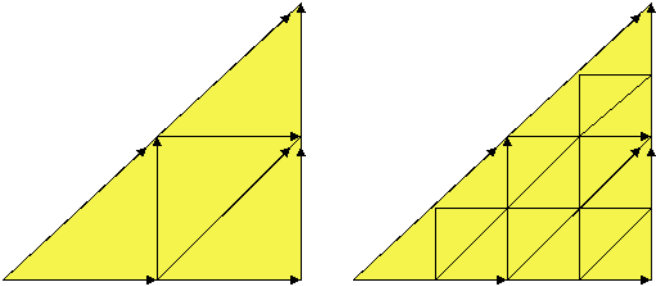
$$|X_*| = \coprod_{n=0}^{\infty} X_n \times \nabla(n)/\sim$$

$\vec{B}M = |N_*(M)|$.

Barycentric subdivision



Cartesian subdivision



Simplicial Approximation Result

Theorem

Every stream map of the form

$$f : |X_*| \to |Y_*|$$

for X_* finite is homotopic, after replacing X_* with sufficiently many Cartesian subdivisions of X_*, to a map of the form $|\alpha : X_* \to Y_*|$.

Future work

For an Abelian monoid A, let $N_*^p(A) = N_*(\ldots(N_*(A))\ldots)$,

$$\vec{B}^p(A) = |N^p(A)|, \qquad H^p(M;A) = [\vec{B}M, \vec{B}^n(A)].$$

To express $H^p(M;A)$ in terms of cochains, need approximation result for $X_* = N_*(M)$ and $Y_* = N_*^n(M)$.

Another potential solution

We want to study the looping space (for instance a stream) B:

- Consider pointed stream maps $p : (E, e_0) \to (B, b_0)$ s.t.
- p is a (topological) covering map:
  - for all $b \in B$, there exists a neighborhood V of b in B, a non-empty discrete space F and a homeomorphism $\Phi : p^{-1}(V) \to V \times F$ s.t.
  - $p_1 \circ \Phi = p$ where $p_1 : V \times F \to V$ is the first projection

Relationship with recent work by Lisbeth Fajstrup.

Should imply...

- lifting of (di-) paths to upper space E
- lifting of (di-) homotopies to upper space E
- existence of universal (di-) covering (which is a [non-compact] po-space)

Definition of $\overrightarrow{\pi_1}(\overrightarrow{B})$

- Morally: σ is invertible in B if and only if $p^{-1}(\sigma)$ (thanks to the lifting of paths for coverings) is invertible for all coverings p
- Definition: Σ "Yoneda system" of B is the image of any Σ_∞ Yoneda system of E_∞ by p_∞
- $\overrightarrow{\pi_0}(\overrightarrow{B}) =_{def} \overrightarrow{\pi_1}(\overrightarrow{B})[\Sigma^{-1}]$

Claim

- For streams with finitely generated (ordinary) π_1, we need only consider a compact sub-space of E_∞ of the universal (di-) covering $p_\infty : E_\infty \to B$ to get the component category of B
- This amounts to saying that one has only to examine the effect of the glueing of some compact (di-) covering space of B on the components

Sketch of proof. By a careful study of the action of $\pi_1(B)$ on E_∞ and its components.

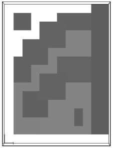
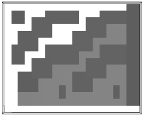
Case of "unsafe regions"

Part of this was known already (Lisbeth Fajstrup, in particular MSCS 2000) for one particular component: the unsafe region.

- Consider PV terms:

$$T1 = P_D.(P_A.V_D.P_B.V_A.P_D.V_B.P_E.V_E)^*.P_F.V_D.V_F$$

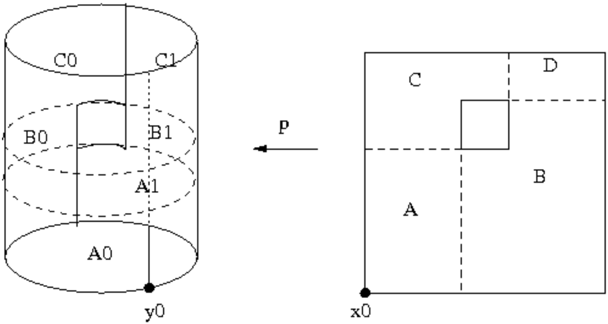
$$T2 = P_F.P_E.P_A.V_E.P_B.V_A.P_D.V_B.P_A.V_D.P_B.V_A.P_D.V_B.V_D.V_F$$

-  

Properties

- We want to show that we have a lifting property as in the case of components for po-spaces: we read off the properties of $\vec{\pi_1}(\vec{B})$ in $\vec{\pi_0}(\vec{B})$
  Sketch of proof. This should be a direct consequence of the lifting property for component categories of po-spaces E' for any p-cover $p' : (E', x_0) \longrightarrow (B, y_0)$ and because of the "di-covering like" definition of p'.
- We want to show that $\vec{\pi_1}(\vec{p})[\Sigma^{-1}]$ is equivalent to $\vec{\pi_1}(\vec{p})/\Sigma$.

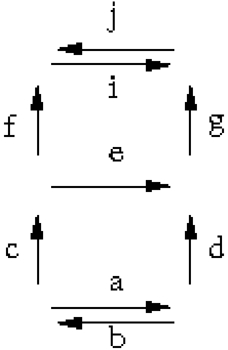
Examples



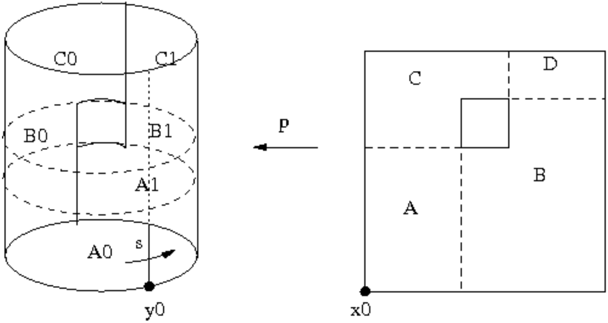
shows a simple folded po-space (a square hole cut onto a cylinder). It is realized as the folding of E, the pospace which is a square minus an inner square, onto two of its parallel boundary edges.

Resulting component category

- Its component category is made up of regions $A1$, $A2$, $B1$, $B2$ and $C1$, $C2$. We have extra relations:
  - $d \circ a = e \circ c$
  - $g \circ e = i \circ f$
-

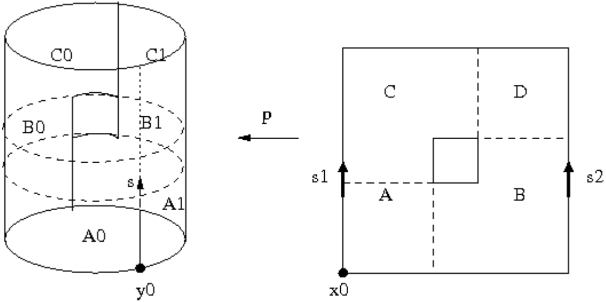


A non-invertible morphism



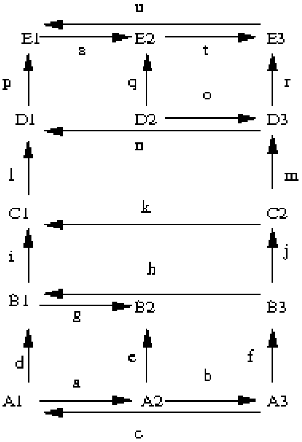
shows an example of an element of $\overrightarrow{\pi_1}(\overrightarrow{E})(s)$ which does not have any inverse by p, since it crosses the vertical line coming up from the based point y_0, which is thus the image of the two parallel edges identified by p. This element s then cannot be Yoneda invertible: this distingues regions $A1$ from $A2$ here.

Another non-invertible morphism



This time, it is non invertible since $s1$ is non Yoneda invertible in E ($s2$ is invertible though). By the pullback stability, we get a distinction between $A2$ and $B2$ here.

A folded Swiss flag



with the extra relations:

- $e \circ a = g \circ d$
- $h \circ f = d \circ c$
- $k \circ j = i \circ h$
- $n \circ m = l \circ k$
- $t \circ q = r \circ o$
- $u \circ r = p \circ n$

A torus-knot example (TCS 2006)

Consider

$$PE.PA.(PB.VA.PC.VB.PA.VC.PB.VA.PC.VB.PA.VC)^*.$$

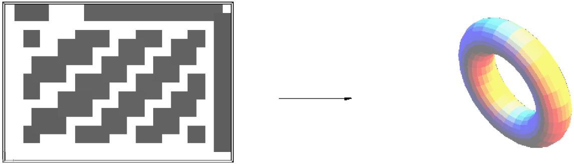
$$VA.PD.VE.VD$$

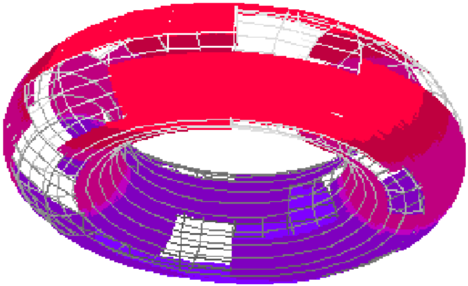
in parallel with

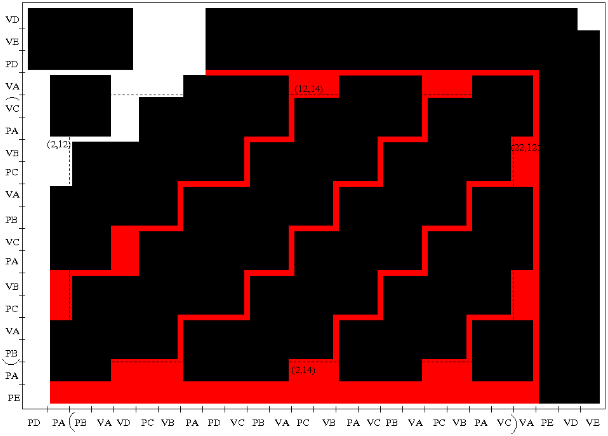
$$PD.PA.(PB.VA.VD.PC.VB.PA.PD.VC.PB.VA.PC.VB.$$

$$PA.VC.PB.VA.PC.VB.PA.VC)^*.VA.PE.VD.VE$$

The semantics of these terms is






VD
VE
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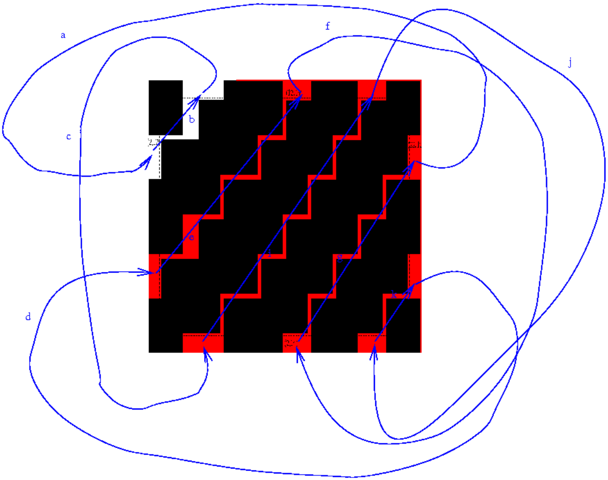
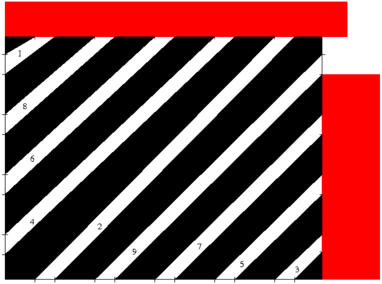
PD PA (PB VA VD PC VB PA PD VC PB VA PC VB PA VC PB VA PC VB PA VC) VA PE VD VE

Its component category



depict the morphisms in the corresponding component category

Continued...



Continued

The relation between the morphisms of the component category is geometrically depicted above:
it consists of an simplified view of the p-cover found by unfolding two times in the y axis, and three times in the x axis. This corresponds to the only loop in $\overrightarrow{\pi_1}(\overrightarrow{B})$, generated by:

$$a \circ g \circ f \circ d \circ k \circ j \circ i \circ c \circ b$$

The loop goes three times on the x-axis and two times on the y axis: just follow bands 1 to 9 on this unfolding.

Algorithmics

Given looping PV terms, we construct easily a folded po-space (E is obtained by unfolding exactly once all "syntactic" loops). We get "enough" p-covers by all syntactic unfoldings of the loops.

- For a p-cover in a set of "suitable" (syntactic) unfoldings, compute the components $\overrightarrow{\pi_0}(\overrightarrow{E'})$ of the po-space E' using one of the classical algorithms (this can be done inductively for all p'-covers etc.)
- Look for stability of $p(\overrightarrow{\pi_0}(\overrightarrow{E'}))$?? Or impose extra non invertibility conditions, as given by the folded relations?