From λ-terms to MELL Proof-Nets

General Picture

λ-calculus $\Rightarrow$ intermediate language $\Rightarrow$ MELL Proof-Nets

The intermediate language : terms + reduction rules

Alternatives :

- λs-calculus.
- λlxr-calculus.
- λj-calculus.

The λ-calculus

$$t, u ::= x \mid \lambda x.t \mid t\ u$$

Typing Rules for λ-calculus

$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i}\ (Ax)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ (\to i) \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}\ (\to e)$$

We denote by $\Gamma \vdash_\lambda t : A$ or by $\Gamma \vdash_{nd} t : A$ the derivability relation.

Reduction Rules for λ-calculus

$$(\lambda x.t)\ u \rightarrow_{\beta}\ t\{x \leftarrow u\}$$

Properties of λ-calculus

Theorem [Confluence] If $t \rightarrow^*_{\beta}\ u$ and $t \rightarrow^*_{\beta}\ v$, then there is t' such that $u \rightarrow^*_{\beta}\ t'$ and $v \rightarrow^*_{\beta}\ t'$.

Theorem [Subject Reduction] If $\Gamma \vdash_{nd} t : A$ and $t \rightarrow_{\beta}\ t'$, then $\Gamma \vdash_{nd} t' : A$.

Theorem [Strong Normalization] If $\Gamma \vdash_{nd} t : A$, then $t \in SN(\beta)$, i.e., there is no infinite β-reduction sequence starting at t (every β-reduction sequence starting at t terminates).

The λs-calculus

$$t, u ::= x \mid \lambda x.t \mid t\ u \mid t[x/u]$$

- Free and bound variables.
- Alpha-conversion.
- Pure terms.

Reduction rules for the λs-calculus

$$t[x/v][y/u] \equiv t[y/u][x/v] \text{ if } x \notin FV(u)\ \&\ y \notin FV(v)$$

$(\lambda x.t)\ u$	$\to_B$	$t[x/u]$	
$(t\ u)[x/v]$	$\to$	$(t[x/v]\ u)$	if $x \in FV(t)\ \&\ x \notin FV(u)$
$(t\ u)[x/v]$	$\to$	$(t\ u[x/v])$	if $x \notin FV(t)\ \&\ x \in FV(u)$
$(t\ u)[x/v]$	$\to$	$(t[x/v]\ u[x/v])$	if $x \in FV(t)\ \&\ x \in FV(u)$
$(\lambda y.t)[x/v]$	$\to$	$\lambda y.t[x/v]$	if $x \neq y\ \&\ y \notin FV(v)$
$t[x/v][y/u]$	$\to$	$t[y/u][x/v[y/u]]$	if $y \in FV(v)\ \&\ y \in FV(t)$
$t[x/v][y/u]$	$\to$	$t[x/v[y/u]]$	if $y \in FV(v)\ \&\ y \notin FV(t)$
$t[x/v]$	$\to$	t	if $x \notin FV(t)$
$x[x/v]$	$\to$	v	

We define $\mathsf{s} = \lambda\mathsf{s} \setminus \{B\}$.

Example

$(\lambda w.(\lambda x.xxy)(zw))w'$	$\to_B$
$(\lambda w.(xxy)[x/zw])w'$	$\to_B$
$(xxy)[x/zw][w/w']$	$\to$
$(xxy)[x/(zw)[w/w']]$	$\to$
$(xxy)[x/zw[w/w']]$	$\to$
$(xxy)[x/zw']$	$\to$
$(xx)[x/zw']y$	$\to$
$x[x/zw']x[x/zw']y$	$\to$
$(zw')x[x/zw']y$	$\to$
$(zw')(zw')y$	

Computing s-normal forms

Lemma The system s is confluent and terminating and thus every term has a unique s-normal form, denoted $\mathsf{s}\text{-}nf(t)$.

Exercice : Define an inductive function s such that $\mathsf{s}(t) = \mathsf{s}\text{-}nf(t)$.

Purification

Lemma If t is a pure term, then $\mathsf{s}(t[y/u]) = t\{y \leftarrow \mathsf{s}(u)\}$.

Lemma The s-normal forms of λs-terms are pure terms.

Corollary For every term t, $\mathsf{s}(t[y/u]) = \mathsf{s}(t)\{y \leftarrow \mathsf{s}(u)\}$.

Full Composition

Exercice : Which is the minimal system $\mathcal{R} \subseteq \lambda\mathsf{s}$ such that $t[x/u] \rightarrow^*_{\mathcal{R}} t\{x \leftarrow u\}$.

Confluence

Lemma If $t \rightarrow_{\lambda\mathsf{s}} t'$, then $\mathsf{s}(t) \rightarrow^*_{\beta} \mathsf{s}(t')$.

Theorem [Simulation] If t is a pure term and $t \rightarrow_{\beta} t'$, then $t \rightarrow^*_{\lambda\mathsf{s}} t'$.

Theorem [Confluence] If $t \rightarrow^*_{\lambda\mathsf{s}} t_1$ and $t \rightarrow^*_{\lambda\mathsf{s}} t_2$, then there is t_3 s.t. $t_1 \rightarrow^*_{\lambda\mathsf{s}} t_3$ and $t_2 \rightarrow^*_{\lambda\mathsf{s}} t_3$.

Normalization Properties

Définition [PSN]

Let $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Let R_1 be a relation on $\mathcal{O}_1$ and R_2 be a relation on $\mathcal{O}_2$. We say that R_2 preserves $SN(R_1)$ iff $t \in SN(R_1)$ implies $t \in SN(R_2)$.

Proposition [PSN for $\lambda\mathsf{s}$] If $t \in SN(\beta)$, then $t \in SN(\lambda\mathsf{s})$.

How PSN can be lost

Let us consider the λx-calculus :

$$
\begin{array}{llll}
(\lambda x.t)\ u & \rightarrow_B & t[x/u] & \\
(t\ u)[x/v] & \rightarrow & (t[x/v]\ u[x/v]) & \\
(\lambda y.t)[x/v] & \rightarrow & \lambda y.t[x/v] & \text{if } x \neq y \,\&\, y \notin FV(v) \\
t[x/v] & \rightarrow & t & \text{if } x \notin FV(t) \\
x[x/v] & \rightarrow & v &
\end{array}
$$

Define $\lambda\mathrm{xc} = \lambda\mathrm{x} \cup \{Comp\}$, where

$$(Comp) \quad t[x/u][y/v] \rightarrow t[x/u[y/v]] \text{ if } y \notin FV(t)$$

- Define $\alpha_0 = [x/(\lambda y.u)u]$ and $\alpha_{m+1} = [x/u\alpha_m]$
- Show that $u\alpha_0\alpha_{m+1} \to^* \ \ldots u\alpha_{m+1}\alpha_{m+2}\ldots$
- Show that $u\alpha_{m+1}\alpha_{n+1} \to^* \ \ldots u\alpha_m\alpha_{n+1}\ldots$
- Show that there is a λx-typable term that admits an infinite reduction sequence in the new system λxc

$\lambda u.(\lambda x.(\lambda y.u)u)((\lambda x.u)u)$	$\to_B$
$\lambda u.((\lambda y.u)u)[x/(\lambda x.u)u]$	$\to_{\mathtt{s}}^*$
$\lambda u.((\lambda y.u[x/(\lambda x.u)u])u[x/(\lambda x.u)u])$	$=$
$\lambda u.((\lambda y.u\alpha_0)u\alpha_0)$	$\to_B$
$\lambda u.(u\alpha_0)[y/u\alpha_0]$	$=$
$\lambda u.(u\alpha_0\alpha_1)$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_1\alpha_2\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_0\alpha_2\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_2\alpha_3\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_1\alpha_3\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_0\alpha_3\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_3\alpha_4\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_2\alpha_4\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_1\alpha_4\ldots$	$\to_{\lambda\mathtt{x}c}^*$
$\ldots u\alpha_0\alpha_4\ldots$	$\ldots$

A Special Strategy for λs-Reduction...

The *strategy* $\rightsquigarrow$ on terms is given by an inductive definition.

$$\frac{\overline{u_n} \in \mathtt{NF}_{\lambda\mathtt{s}} \quad t \rightsquigarrow t'}{x\overline{u_n}t\overline{v_m} \rightsquigarrow x\overline{u_n}t'\overline{v_m}} \qquad \frac{t \rightsquigarrow t'}{\lambda x.t \rightsquigarrow \lambda x.t'} \qquad \frac{}{(\lambda x.t)u\overline{u_n} \rightsquigarrow t[x/u]\overline{u_n}}$$

$$\frac{u \in SN(\lambda\mathtt{s})}{t[x/u]\overline{v_n} \rightsquigarrow t\{x \leftarrow u\}\overline{v_n}} \qquad \frac{u \notin SN(\lambda\mathtt{s}) \quad u \rightsquigarrow u'}{t[x/u]\overline{v_n} \rightsquigarrow t[x/u']\overline{v_n}}$$

... which is Perpetual

Theorem [Perpetuality] If $t \rightsquigarrow t'$ and $t' \in SN(\lambda\mathsf{s})$, then $t \in SN(\lambda\mathsf{s})$ ($t \notin SN(\lambda\mathsf{s})$ implies $t' \notin SN(\lambda\mathsf{s})$).

Characterisation of $SN(\lambda\mathsf{s})$

$$\frac{t_1, \ldots, t_n \in \mathtt{ISN} \quad n \geq 0}{x t_1 \ldots t_n \in \mathtt{ISN}} \qquad \frac{u[x/v] t_1 \ldots t_n \in \mathtt{ISN} \quad n \geq 0}{(\lambda x.u) v t_1 \ldots t_n \in \mathtt{ISN}}$$

$$\frac{u\{x \leftarrow v\} t_1 \ldots t_n \in \mathtt{ISN} \quad v \in \mathtt{ISN} \quad n \geq 0}{u[x/v] t_1 \ldots t_n \in \mathtt{ISN}} \qquad \frac{u \in \mathtt{ISN}}{\lambda x.u \in \mathtt{ISN}}$$

Lemma $SN(\lambda\mathsf{s}) = \mathtt{ISN}$.

Proof : Uses the Perpetuality Theorem.

Typing Rules for the $\lambda\mathsf{s}$-calculus

$$\frac{}{x : A \vdash x : A} \ (ax) \qquad \frac{\Gamma \vdash t : B}{\Gamma \setminus \{x : A\} \vdash \lambda x.t : A \to B} \ (\to i)$$

$$\frac{\Gamma \vdash t : B \to A \quad \Delta \vdash u : B}{\Gamma \uplus \Delta \vdash t\ u : A} \ (\to e)$$

$$\frac{\Gamma \vdash u : B \quad \Delta \vdash t : A}{\Gamma \uplus (\Delta \setminus \{x : B\}) \vdash t[x/u] : A} \ (cut)$$

We denote by $\Gamma \vdash_{\lambda\mathsf{s}} t : A$ the derivability relation. We write t^A if there exists Γ s.t. $\Gamma \vdash_{\lambda\mathsf{s}} t : A$.

Reduction Preserves Types

Theorem [Subject Reduction] If $\Gamma \vdash_{\lambda\mathsf{s}} t : A$ and $t \to_{\lambda\mathsf{s}} t'$, then $\Gamma' \vdash_{\lambda\mathsf{s}} t' : A$ for some $\Gamma' \subseteq \Gamma$.

Strong Normalisation

Lemma If $t^A, u^B \in SN(\lambda\mathtt{s})$, then $t\{x^B \leftarrow u\} \in SN(\lambda\mathtt{s})$.

Proof : Induction on the lexicographic 3-tuple $\langle B, \eta_{\lambda\mathtt{s}}(t), t\rangle$.

Theorem [SN for λs] If $\Gamma \vdash_{\lambda\mathtt{s}} t : A$, then $t \in SN(\lambda\mathtt{s})$.

Proof : Induction on t and previous lemma using `ISN`.

Summary

The λs-calculus is a computational interpretation of Natural Deduction plus cut enjoying the following properties :

- Full composition.
- Confluence on terms.
- Simulation of one-step β-reduction.
- Preservation of β-strong normalization.
- Strong normalization of well-typed terms.
- Admits a translation into MELL Proof-Nets.