
Computational Interpretation for
Natural Deduction plus cut and structural rules

The λ_{lr} -calculus

(Terms)	$t, u ::= x$	<i>variable</i>
	$\lambda x.t$	<i>abstraction</i>
	$t u$	<i>application</i>
	$t[x/u]$	<i>substitution</i>
	$W_x(t)$	<i>weakening</i>
	$C_x^{y,z}(t)$	<i>contraction</i>

We only consider *well-formed* terms :

- Linearity
- Compulsory presence
- Barendregt's convention

Typing Rules for the λ_{lr} -calculus

$$\frac{}{x : A \vdash x : A} \quad (ax) \quad \frac{\Delta \vdash u : B \quad \Gamma, x : B \vdash t : A}{\Gamma, \Delta \vdash t[x/u] : A} \quad (cut)$$

$$\frac{\Gamma \vdash t : A \rightarrow B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash (t u) : B} \quad (\rightarrow e) \quad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \rightarrow B} \quad (\rightarrow i)$$

$$\frac{\Gamma, x : A, y : A \vdash t : B}{\Gamma, z : A \vdash C_z^{x,y}(t) : B} \quad (c) \quad \frac{\Gamma \vdash t : A}{\Gamma, x : B \vdash W_x(t) : A} \quad (w)$$

Congruence I

AC of contraction :

$$C_w^{x,v}(C_x^{z,y}(t)) \equiv C_w^{x,y}(C_x^{z,v}(t)) \quad \text{if } x \neq y, v$$

$$C_x^{y,z}(t) \equiv C_x^{z,y}(t)$$

$$C_{x'}^{y',z'}(C_x^{y,z}(t)) \equiv C_x^{y,z}(C_{x'}^{y',z'}(t)) \quad \text{if } x \neq y', z' \ \& \ x' \neq y, z$$

C of weakening :

$$W_x(W_y(t)) \equiv W_y(W_x(t))$$

Congruence II

Commutativity of substitutions :

$$t[x/u][y/v] \equiv t[y/v][x/u] \text{ if } y \notin FV(u) \ \& \ x \notin FV(v)$$

Contraction and substitution have the same status :

$$C_w^{y,z}(t)[x/u] \equiv C_w^{y,z}(t[x/u]) \text{ if } x \neq w \ \& \ y, z \notin FV(u)$$

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Reduction Rules for the $\lambda 1x$ -calculus

$$(B) \quad (\lambda x.t) u \rightarrow t[x/u]$$

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SubSystem x

$$\begin{aligned} (Abs) \quad & (\lambda y.t)[x/u] \rightarrow \lambda y.t[x/u] \\ (App1) \quad & (t \ v)[x/u] \rightarrow t[x/v] \ v \quad \text{if } x \in FV(t) \\ (App2) \quad & (t \ v)[x/u] \rightarrow t \ v[x/u] \quad \text{if } x \in FV(v) \\ (Var) \quad & x[x/u] \rightarrow u \\ (Weak1) \quad & W_x(t)[x/u] \rightarrow W_{FV(u)}(t) \\ (Weak2) \quad & W_y(t)[x/u] \rightarrow W_y(t[x/u]) \quad \text{if } x \neq y \\ (Cont1) \quad & C_x^{y,z}(t)[x/u] \rightarrow C_{\Phi}^{\Delta, \Pi}(t[y/u_1][z/u_2]) \\ & \text{where } \Phi := FV(u) \\ & u_1 = R_{\Delta}^{\Phi}(u) \ \text{and} \ u_2 = R_{\Pi}^{\Phi}(u) \\ (Comp) \quad & t[y/v][x/u] \rightarrow t[y/v[x/u]] \ \text{if } x \in FV(v) \end{aligned}$$

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SubSystem t

$$\begin{aligned} (W Abs) \quad & \lambda x.W_y(t) \rightarrow W_y(\lambda x.t) \quad x \neq y \\ (W App1) \quad & W_y(u) \ v \rightarrow W_y(u \ v) \\ (W App2) \quad & u \ W_y(v) \rightarrow W_y(u \ v) \\ (W Subs) \quad & t[x/W_y(u)] \rightarrow W_y(t[x/u]) \\ (Merge) \quad & C_w^{y,z}(W_y(t)) \rightarrow R_w^z(t) \\ (Cross) \quad & C_w^{y,z}(W_x(t)) \rightarrow W_x(C_w^{y,z}(t)) \quad x \neq y, \ x \neq z \\ (CAbs) \quad & C_w^{y,z}(\lambda x.t) \rightarrow \lambda x.C_w^{y,z}(t) \\ (C App1) \quad & C_w^{y,z}(t \ u) \rightarrow C_w^{y,z}(t) \ u \quad y, z \in FV(t) \\ (C App2) \quad & C_w^{y,z}(t \ u) \rightarrow t \ C_w^{y,z}(u) \quad y, z \in FV(u) \\ (C Subs) \quad & C_w^{y,z}(t[x/u]) \rightarrow t[x/C_w^{y,z}(u)] \quad y, z \in FV(u) \end{aligned}$$

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The reduction relation λ_{lxr}

The reduction relation is generated by the previous rewriting rules and congruence axioms :

$$t \rightarrow_{\lambda_{\text{lxr}}} t' \text{ iff } \exists t_1, t_2 \ t \equiv t_1 \rightarrow_{B+\mathbf{x}+\mathbf{t}} t_2 \equiv t'$$

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Example

$$\begin{aligned} (\lambda x. W_u(C_x^{y,z}(y \ z))) \ w & \rightarrow \\ W_u(C_x^{y,z}(y \ z))[x/w] & \rightarrow \\ W_u(C_x^{y,z}(y \ z)[x/w]) & \rightarrow \\ W_u(C_w^{w_1,w_2}((y \ z)[y/w_1][z/w_2])) & \rightarrow \\ W_u(C_w^{w_1,w_2}((y[y/w_1] \ z)[z/w_2])) & \rightarrow \\ W_u(C_w^{w_1,w_2}(y[y/w_1] \ z[z/w_2])) & \rightarrow \\ W_u(C_w^{w_1,w_2}(w_1 \ z[z/w_2])) & \rightarrow \\ W_u(C_w^{w_1,w_2}(w_1 \ w_2)) & \end{aligned}$$

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Properties of the λ_{lxr} -calculus

1. **(Full composition)** $t[x/v] \rightarrow^* t\{x = v\}$
even when t contains non-evaluated substitutions
2. **(Free variables are preserved)** If $t \rightarrow_{\lambda_{\text{lxr}}} t'$, then
 $FV(t) = FV(t')$
3. **(Subject reduction)** If $\Gamma \vdash t : A$ et $t \rightarrow_{\lambda_{\text{lxr}}} t'$, then $\Gamma \vdash t' : A$.
4. **(Convergence)** $\mathbf{xt} = \mathbf{x} \cup \mathbf{t}$ is convergent.

Which is the form of a term in \mathbf{xt} -normal form ?

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Connexion with λ -calculus

$\mathcal{B}()$ hides resource control

$$t \rightarrow_{\mathbf{xt}}^* W_{\Pi}(\mathcal{A}(\mathcal{B}(t))) \quad \lambda_{\text{lxr}} \quad \begin{array}{c} \xrightarrow{\mathcal{B}()} \\ \lambda \\ \xleftarrow{\mathcal{A}()} \end{array} \quad t = \mathcal{B}(\mathcal{A}(t))$$

$\mathcal{A}()$ introduces resource operators

Example : $\lambda x. y \ y$ is represented by $\lambda x. W_x(C_y^{z,z'}(z \ z'))$

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From λlxr to λ

$$\begin{aligned}\mathcal{B}(x) &= x \\ \mathcal{B}(\lambda x.t) &= \lambda x.\mathcal{B}(t) \\ \mathcal{B}(W_x(t)) &= \mathcal{B}(t) \\ \mathcal{B}(C_x^{y,z}(t)) &= \mathcal{B}(t)\{y \leftarrow x\}\{z \leftarrow x\} \\ \mathcal{B}(t\ u) &= \mathcal{B}(t)\ \mathcal{B}(u) \\ \mathcal{B}(t[x/u]) &= \mathcal{B}(t)\{x \leftarrow \mathcal{B}(u)\}\end{aligned}$$

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Relating λlxr and λ

Lemma

1. If $M \equiv N$, then $\mathcal{B}(M) = \mathcal{B}(N)$.
2. If $M \rightarrow_B N$, then $\mathcal{B}(M) \rightarrow_{\beta}^* \mathcal{B}(N)$.
3. If $M \rightarrow_{\text{xt}} N$, then $\mathcal{B}(M) = \mathcal{B}(N)$.

Proposition [Projecting λlxr -reductions]

$M \rightarrow_{\lambda\text{lxr}} N$, then $\mathcal{B}(M) \rightarrow_{\beta}^* \mathcal{B}(N)$.

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From λ to λlxr

$$\begin{aligned}\mathcal{A}(x) &:= x \\ \mathcal{A}(\lambda x.t) &:= \lambda x.\mathcal{A}(t) && \text{if } x \in FV(t) \\ \mathcal{A}(\lambda x.t) &:= \lambda x.W_x(\mathcal{A}(t)) && \text{if } x \notin FV(t) \\ \mathcal{A}(tu) &:= C_{\Phi}^{\Delta, \Pi}(R_{\Delta}^{\Phi}(\mathcal{A}(t))) R_{\Pi}^{\Phi}(\mathcal{A}(u)) && \text{where } \Phi := FV(t) \cap FV(u)\end{aligned}$$

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Relating λ and λlxr

Lemma For all λ -terms t and u such that $x \in FV(t)$, we have

$$C_{\Phi}^{\Delta, \Pi}(R_{\Delta}^{\Phi}(\mathcal{A}(t))[x/R_{\Pi}^{\Phi}(\mathcal{A}(u))]) \rightarrow_{\text{xt}}^* \mathcal{A}(t\{x \leftarrow u\})$$

where $\Phi := (FV(t) \setminus \{x\}) \cap FV(u)$.

Proposition [Simulating β -reductions]

If $t \rightarrow_{\beta} t'$, then $\mathcal{A}(t) \rightarrow_{\lambda\text{lxr}}^+ W_{FV(t) \setminus FV(t')}(\mathcal{A}(t'))$.

Example $t = (\lambda x.y)z \rightarrow_{\beta} y = t'$ and
 $\mathcal{A}(t) = (\lambda x.W_x(y))z \rightarrow_{\lambda\text{lxr}}^+ W_z(\mathcal{A}(y))$.

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Confluence

Lemma The xt-normal form of t is $W_{FV(t) \setminus FV(\mathcal{B}(t))}(\mathcal{A}(\mathcal{B}(t)))$.

Example Let $t = C_x^{x_1, x_2}((\lambda y. x_1 (x_2 W_y(z))) W_k(w))$. Then $\text{xt}(t) = W_k((\lambda y. W_y(C_x^{x_1, x_2}(x_1 (x_2 z)))) w)$.

Theorem [Confluence modulo] The reduction relation λlxr is confluent (even on terms with meta-variables).

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More Properties of the λlxr -calculus

1. (Preservation of typing)

(a) If $\Gamma \vdash_\lambda t : A$ then $\Gamma \vdash_{\lambda\text{lxr}} W_{\Gamma \setminus FV(t)}(\mathcal{A}(t)) : A$

(b) If $\Gamma \vdash_{\lambda\text{lxr}} t : A$ then $\Gamma \vdash_\lambda \mathcal{B}(t) : A$

2. (PSN) If $M \in SN^\beta$ then $\mathcal{A}(M) \in SN^{\lambda\text{lxr}}$.

breaks Mellies' counter-example of non-termination
(with $t[y/v][x/u] \rightarrow t[y/v[x/u]]$ if $x \notin t$)

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Strong Normalization and Proof Nets

Translating types :

$$\begin{aligned} A^* &= A && \text{for atomic types} \\ (A \rightarrow B)^* &= ?((A^*)^\perp) \wp B^* && \text{otherwise} \end{aligned}$$

Translating terms :

$T(B_1, \dots, B_n \vdash t : A)$ gives a proof-net having wires labelled with $?((B_1^*)^\perp), \dots, ?((B_n^*)^\perp), A^*$.

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Simulating λlxr with Proof Nets

Lemma Let t be a λlxr -typed term.

- If $t \equiv t'$, then $T(t) \sim_E T(t')$.
- If $t \rightarrow_B t'$, then $T(t) \rightarrow^+_{R/E} T(t')$.
- If $t \rightarrow_{\text{xt}} t'$, then $T(t) \rightarrow^*_{R/E} T(t')$.

So that λlxr is **sound** w.r.t proof-nets :

If t is λlxr -typed, then $t \rightarrow_{\lambda\text{lxr}} t'$ implies $T(t) \rightarrow^*_{R/E} T(t')$.

Theorem [Strong Normalisation] The relation λlxr is strongly normalising on well-typed λlxr -terms.

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Towards completeness

- Define a congruence \approx for proof-nets.
- Define a congruence \cong for λlr -terms.
- Show that $T(t_1) \approx T(t_2)$ implies $t_1 \cong t_2$.

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Summary

The λlr -calculus is a computational interpretation of natural deduction plus cut and structural rules enjoying the following properties :

- Confluence on all the terms.
- Simulation of one-step β -reduction.
- Preservation of β -strong normalization.
- Strong normalization of well-typed terms.
- Full and safe composition.
- Sound and complete with respect to proof-nets.
- Explicit operators for implementation issues.

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