

Reduction Rules and Equations for the λ_S -calculus

Equations :			
$t[x/u][y/v]$	$=_c$	$t[y/v][x/u]$	if $y \notin \text{fv}(u)$ & $x \notin \text{fv}(v)$
Reduction Rules :			
$(\lambda x.t) u$	\rightarrow_B	$t[x/u]$	
$x[x/u]$	\rightarrow_{Var}	u	
$t[x/u]$	\rightarrow_{Gc}	t	if $x \notin \text{fv}(t)$
$(t u)[x/v]$	$\rightarrow_{\text{App}_1}$	$(t[x/v] u[x/v])$	if $x \in \text{fv}(t)$ & $x \in \text{fv}(u)$
$(t u)[x/v]$	$\rightarrow_{\text{App}_2}$	$(t u[x/v])$	if $x \notin \text{fv}(t)$ & $x \in \text{fv}(u)$
$(t u)[x/v]$	$\rightarrow_{\text{App}_3}$	$(t[x/v] u)$	if $x \in \text{fv}(t)$ & $x \notin \text{fv}(u)$
$(\lambda y.t)[x/v]$	$\rightarrow_{\text{Lamb}}$	$\lambda y.t[x/v]$	if $y \notin \text{fv}(v)$
$t[x/u][y/v]$	$\rightarrow_{\text{Comp}_1}$	$t[y/v][x/u[y/v]]$	if $y \in \text{fv}(u)$ & $y \in \text{fv}(t)$
$t[x/u][y/v]$	$\rightarrow_{\text{Comp}_2}$	$t[x/u][y/v]$	if $y \in \text{fv}(u)$ & $y \notin \text{fv}(t)$

Let $s = \{\text{Var}, \text{Gc}, \text{App}_1, \text{App}_2, \text{App}_3, \text{Lamb}, \text{Comp}_1, \text{Comp}_2\}$.

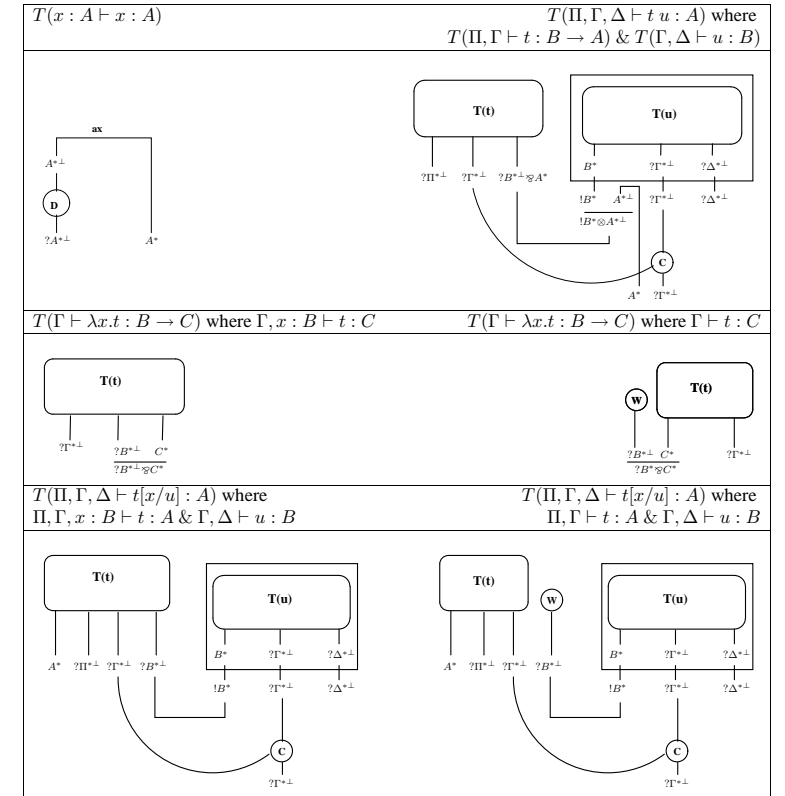
From λ_S -calculus to MELL Proof-Nets

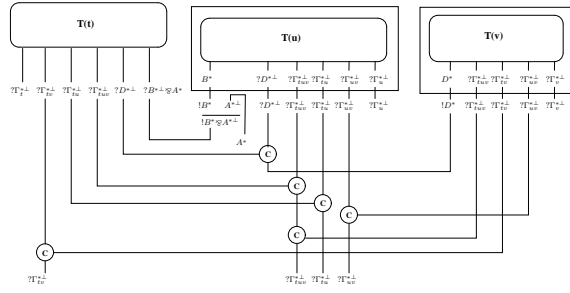
Translating types

$$A^* := A \quad \text{if } A \text{ is atomic}$$

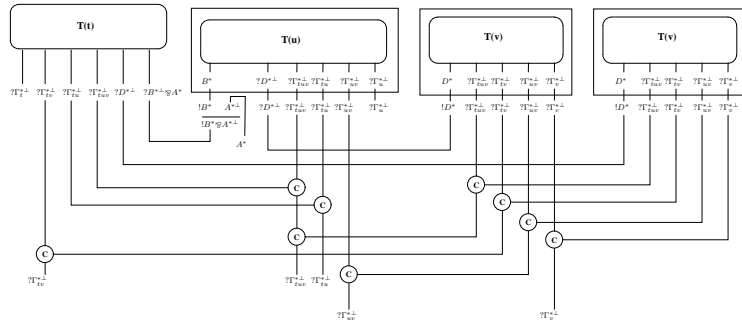
$$(A \rightarrow B)^* := ?((A^*)^\perp) \wp B^*$$

Translating terms

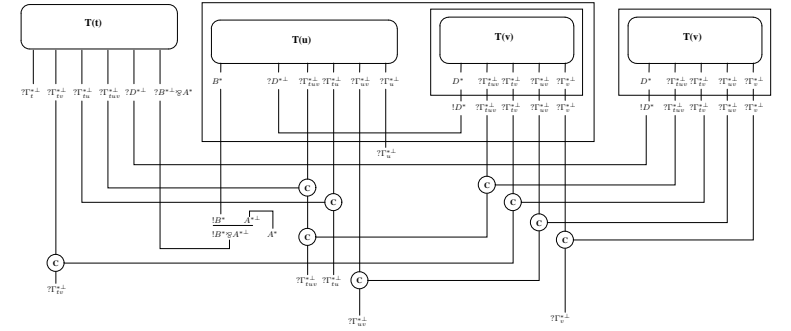




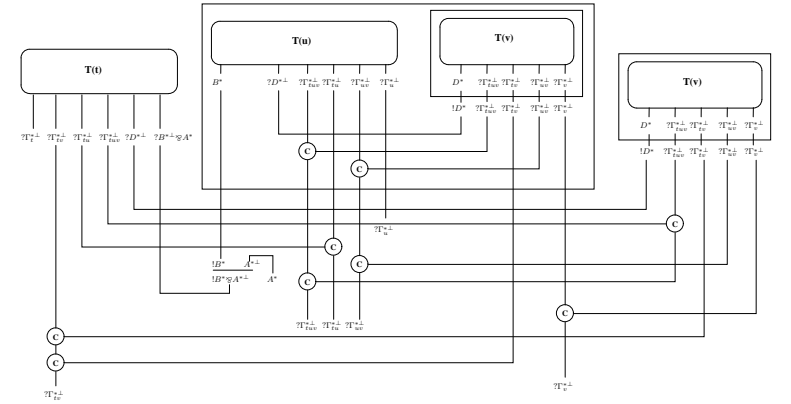
which reduces by \rightarrow_{c-b} to the proof-net



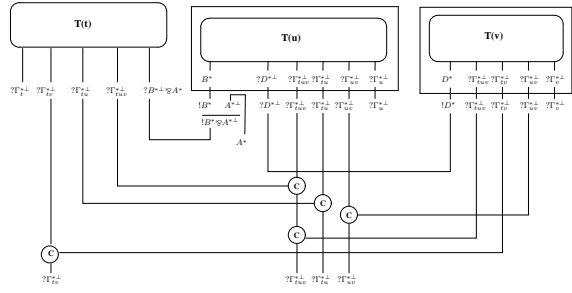
which reduces by \rightarrow_{b-b} to the proof-net



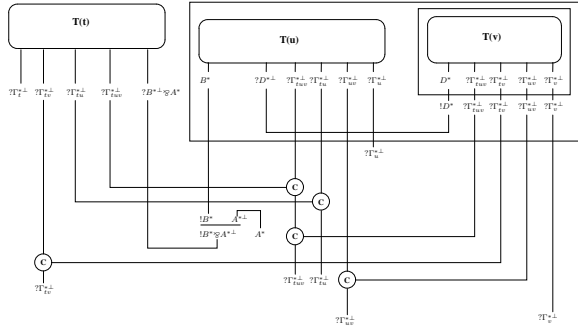
which is equivalent via \sim_E to the proof-net $T(s')$



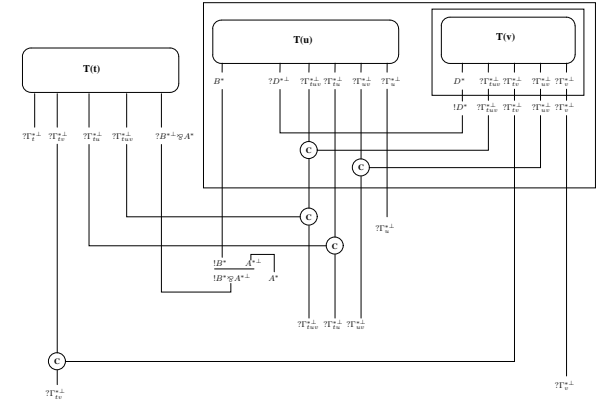
- For $s = (t u)[x/v] \rightarrow_{\text{App}_2} (t u[x/v]) = s'$, with $x \notin \text{fv}(t)$ & $x \in \text{fv}(u)$, coming from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t \vdash t : B \rightarrow A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, x : D \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \text{fv}(t) \cap \text{fv}(u) \cap \text{fv}(v)$, $\Gamma_{tu} := \text{fv}(t) \cap \text{fv}(u) \setminus x \setminus \text{fv}(v)$, $\Gamma_{tv} := \text{fv}(t) \cap \text{fv}(v) \setminus \text{fv}(u)$, $\Gamma_{uv} := \text{fv}(u) \cap \text{fv}(v) \setminus \text{fv}(t)$, $\Gamma_t := \text{fv}(t) \setminus \text{fv}(u) \setminus \text{fv}(v)$, $\Gamma_u := \text{fv}(u) \setminus \text{fv}(t) \setminus \text{fv}(v)$ and $\Gamma_v := \text{fv}(v) \setminus \text{fv}(t) \setminus \text{fv}(u)$. The proof-net $T(s)$ is given by



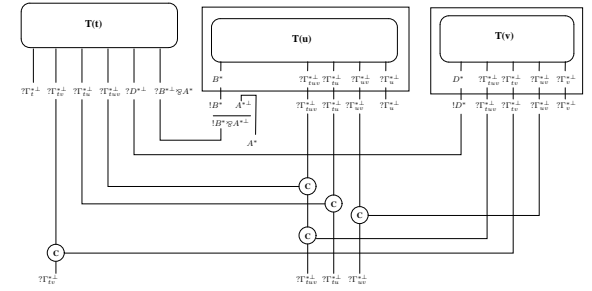
which reduces by \rightarrow_{b-1} to the proof-net



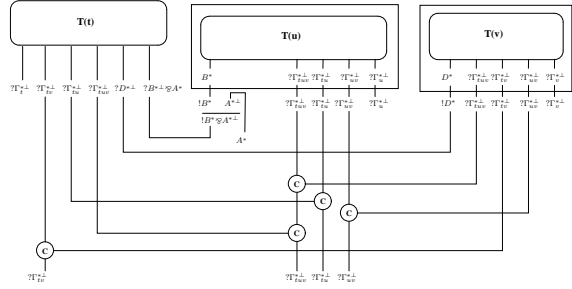
which is equivalent via \sim_E to the proof-net $T(s')$



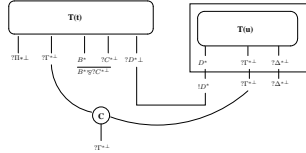
- For $s = (t u)[x/v] \rightarrow_{app_3} (t[x/v] u) = s'$, with $x \in \text{fv}(t)$ & $x \notin \text{fv}(u)$, coming from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : D \vdash t : B \rightarrow A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_u \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_u, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \text{fv}(t) \cap \text{fv}(u) \cap \text{fv}(v)$, $\Gamma_{tu} := \text{fv}(t) \cap \text{fv}(u) \setminus x \setminus \text{fv}(v)$, $\Gamma_{tv} := \text{fv}(t) \cap \text{fv}(v) \setminus \text{fv}(u)$, $\Gamma_{uv} := \text{fv}(u) \cap \text{fv}(v) \setminus \text{fv}(t)$, $\Gamma_t := \text{fv}(t) \setminus \text{fv}(u) \setminus \text{fv}(v)$, $\Gamma_u := \text{fv}(u) \setminus \text{fv}(t) \setminus \text{fv}(v)$ and $\Gamma_v := \text{fv}(v) \setminus \text{fv}(t) \setminus \text{fv}(u)$. The proof-net $T(s)$ is given by



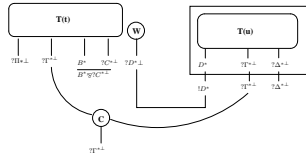
which is equivalent via \sim_E to the proof-net $T(s')$



- For $s = (\lambda y.t)[x/u] \rightarrow_{\text{Lamb}} \lambda y.t[x/u] = s'$, with $x \in \text{fv}(\lambda y.t)$, coming from $\Pi, \Gamma, x : D \vdash \lambda y.t : B \rightarrow C$ and $\Gamma, \Delta \vdash u : D$ where $\Gamma := \text{fv}(\lambda y.t) \cap \text{fv}(u)$ and $\Pi := \text{fv}(\lambda y.t) \setminus x \setminus \text{fv}(u)$ and $\Delta := \text{fv}(u) \setminus \text{fv}(\lambda y.t)$. We show here the case $y \in \text{fv}(t)$, the case $y \notin \text{fv}(t)$ being similar. We have exactly the same interpretation $T(_)$ for both terms s and s' which is given by the proof-net:



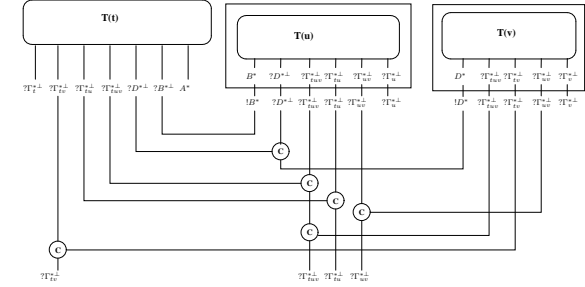
- For $s = (\lambda y.t)[x/u] \rightarrow_{\text{Lamb}} \lambda y.t[x/u] = s'$, where $x \notin \text{fv}(\lambda y.t)$, coming from $\Pi, \Gamma \vdash \lambda y.t : B \rightarrow C$ and $\Gamma, \Delta \vdash u : D$ where $\Gamma := \text{fv}(\lambda y.t) \cap \text{fv}(u)$ and $\Pi := \text{fv}(\lambda y.t) \setminus \text{fv}(u)$ and $\Delta := \text{fv}(u) \setminus \text{fv}(\lambda y.t)$. We show here the case $y \in \text{fv}(t)$, the case $y \notin \text{fv}(t)$ being similar. We have exactly the same interpretation $T(_)$ for both terms s and s' which is given by the following proof-net.



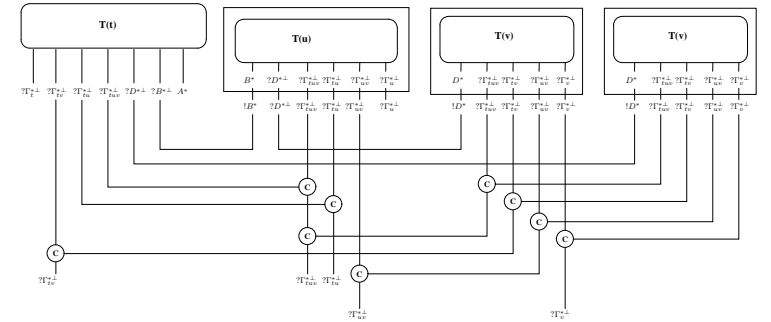
- For $s = t[x/u][y/v] \rightarrow_{\text{Comp}_1} t[y/v][x/u][y/v] = s'$, with $y \in \text{fv}(t)$ & $y \in \text{fv}(u)$. We show here the case $x \in \text{fv}(t)$, the case $x \notin \text{fv}(t)$ being similar. Thus, $\Gamma \vdash s : A$ comes from $\Gamma_{tu}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : B, y : D \vdash t : A$ and

$\Gamma_{tu}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, y : D \vdash u : B$ and $\Gamma_{tu}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tu} := \text{fv}(t) \cap \text{fv}(u) \cap \text{fv}(u)$, $\Gamma_{tu} := \text{fv}(t) \cap \text{fv}(u) \setminus y \setminus \text{fv}(v)$, $\Gamma_{tv} := \text{fv}(t) \cap \text{fv}(v) \setminus \text{fv}(u)$, $\Gamma_{uv} := \text{fv}(u) \cap \text{fv}(v) \setminus \text{fv}(t)$, $\Gamma_t := \text{fv}(t) \setminus \text{fv}(u) \setminus \text{fv}(v)$, $\Gamma_u := \text{fv}(u) \setminus \text{fv}(v) \setminus \text{fv}(t)$ and $\Gamma_v := \text{fv}(v) \setminus \text{fv}(t) \setminus \text{fv}(u)$.

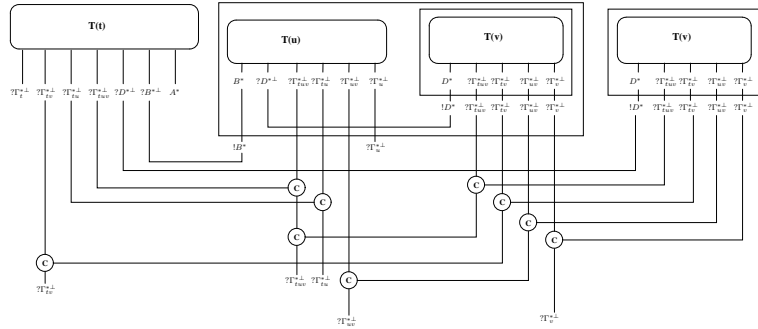
This case is similar to App₁. The proof-net $T(s)$ is given by



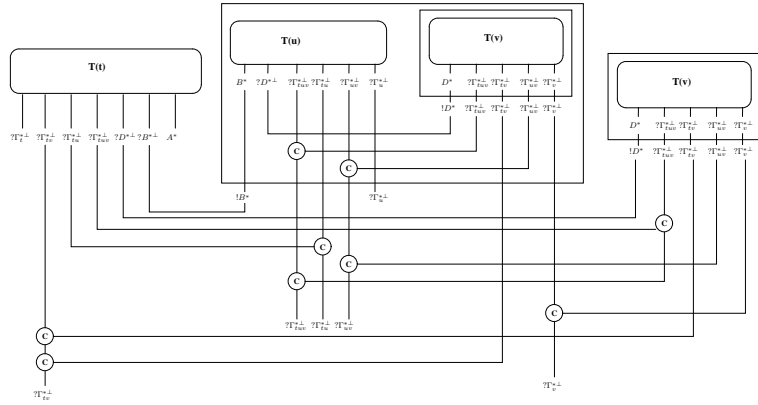
which reduces by \rightarrow_{c-b} to the proof-net



which reduces by \rightarrow_{b-b} to the proof-net

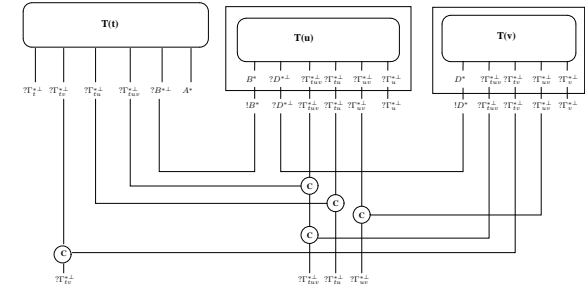


which is equivalent via \sim_E to the proof-net $T(s')$

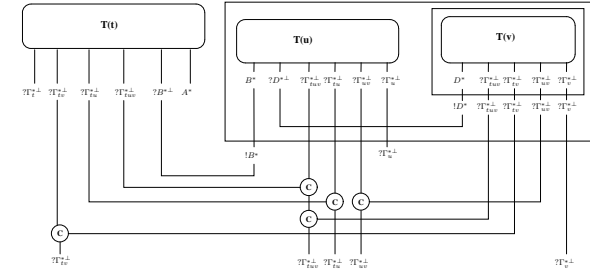


- $s = t[x/u][y/v] \rightarrow_{\text{Comp}_2} t[x/u][y/v] = s'$, with $y \notin \text{fv}(t)$ & $y \in \text{fv}(u)$. We show here the case $x \in \text{fv}(t)$, the case $x \notin \text{fv}(t)$ begin similar. Thus, $\Gamma \vdash s : A$ comes from $\Gamma_{tw}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : B \vdash t : A$ and $\Gamma_{tw}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, y : D \vdash u : B$ and $\Gamma_{tw}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tw} := \text{fv}(t) \cap \text{fv}(u) \cap \text{fv}(u)$, $\Gamma_{tu} := \text{fv}(t) \cap \text{fv}(u) \setminus y \setminus \text{fv}(v)$, $\Gamma_{tv} := \text{fv}(t) \cap \text{fv}(v) \setminus \text{fv}(u)$, $\Gamma_{uv} := \text{fv}(u) \cap \text{fv}(v) \setminus \text{fv}(t)$, $\Gamma_t := \text{fv}(t) \setminus \text{fv}(u) \setminus \text{fv}(v)$, $\Gamma_u := \text{fv}(u) \setminus \text{fv}(t) \setminus \text{fv}(v)$ and $\Gamma_v := \text{fv}(v) \setminus \text{fv}(t) \setminus \text{fv}(u)$.

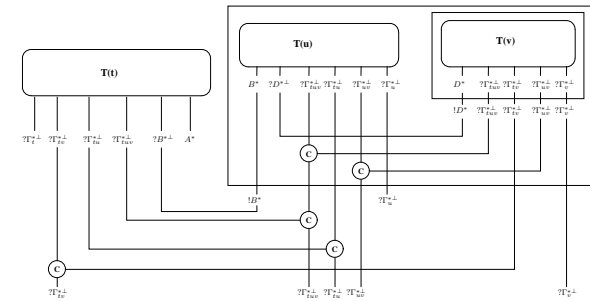
This case is similar to App₂. The proof-net $T(s)$ is given by



which reduces by \rightarrow_{b-b} to the proof-net



which is equivalent via \sim_E to the proof-net $T(s')$



We now consider the cases where $s \rightarrow_{\lambda_S} s'$ is an internal reduction step.

- If $s =_C s'$ or $s \rightarrow_{\text{App}_3, \text{Lamb}} s'$ then the property trivially holds since \sim_E is a congruence.
- If $s \rightarrow_{\text{Bs} \setminus \{\text{App}_3, \text{Lamb}\}} s'$ is $\lambda x.t \rightarrow \lambda x.t'$ or $t u \rightarrow t' u$ or $t[x/u] \rightarrow t'[x/u]$ coming from $t \rightarrow t'$, then we obtain $T(t) \rightarrow_{R/E}^+ C[T(t')]$ by i.h. and the property holds by the fact that the context $C[\]$ of weakening wires surrounding $T(t')$ can also be considered as a context of weakening wires surrounding $T(s')$.
- If $s \rightarrow_{\text{Bs} \setminus \{\text{App}_3, \text{Lamb}\}} s'$ is $u t \rightarrow u t'$ or $u[x/t] \rightarrow u[x/t']$ coming from $t \rightarrow t'$, then we obtain $T(t) \rightarrow_{R/E}^+ C[T(t')]$ by i.h. and the property holds by the fact that the context $C[\]$ of weakening wires surrounding $T(t')$ can be pushed outside the box containing $T(t')$ by using the rule \rightarrow_V in order to obtain a context of weakening wires surrounding $T(s')$.

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Remark that the only case where we get a non empty context in Lemma 0.1 is when simulating the rule Gc. This is because Gc is the only rule which looses free variables, all the other ones preserve the same set of free variables.

Corollary 0.2 (SN for λ_S -typed terms) *If $\Gamma \vdash_{\lambda_S} t : A$, then $t \in \mathcal{SN}_{\lambda_S}$.*

Proof. We can apply the abstract theorem 0.3 : \mathcal{E} is C, \mathcal{R}_1 is the relation $\rightarrow_{\text{App}_3, \text{Lamb}}$ (for which we can trivially show that $\rightarrow_{\text{App}_3, \text{Lamb}} / =_{\mathcal{E}}$ is well-founded), \mathcal{R}_2 is the relation $\rightarrow_{\text{es} \setminus \{\text{App}_3, \text{Lamb}\}}$, K is the relation given by the translation $T(\cdot)$, \mathcal{S} is the reduction relation R/E on MELL proof-nets (which is well-founded Polonovski), and properties (ES), (WS), (SS) hold by Lemma 0.1.

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An abstract theorem

Theorem 0.3 *Let \mathcal{O} and \mathcal{P} be two sets. Let $\mathcal{R}_1, \mathcal{R}_2$ be two relations on $\mathcal{O} \times \mathcal{O}$, \mathcal{S} be a relation on $\mathcal{P} \times \mathcal{P}$, \mathbb{K} a relation $\subseteq \mathcal{O} \times \mathcal{P}$ and \mathcal{E} an equivalence relation on \mathcal{O} such that $\mathcal{R}_1/\mathcal{E}$ is well-founded. Suppose also*

(ES) $t \mathcal{E} t'$ and $t \mathbb{K} T$ implies $t' \mathbb{K} T$

(WS) $t \mathcal{R}_1 t'$ and $t \mathbb{K} T$ implies there is T' such that $t' \mathbb{K} T'$ and $T \mathcal{S}^* T'$

(SS) $t \mathcal{R}_2 t'$ and $t \mathbb{K} T$ implies there is T' such that $t' \mathbb{K} T'$ and $T \mathcal{S}^+ T'$

Then, if $t \mathbb{K} T$ and \mathcal{S} is a well-founded relation on T , then $(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{E}$ is well-founded on t .

Proof. Suppose $(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{E}$ is not well-founded on t . Since $\mathcal{R}_1/\mathcal{E}$ is well-founded by hypothesis, there is an infinite sequence on \mathcal{O} where $\mathcal{R}_2/\mathcal{E}$ occurs infinitely many times so it is of the form

$$t \dots (\mathcal{R}_2/\mathcal{E}) t_1 \dots (\mathcal{R}_2/\mathcal{E}) t_2 \dots (\mathcal{R}_2/\mathcal{E}) t_i \dots$$

that is,

$$t (\mathcal{R}_1/\mathcal{E})^* \mathcal{E} \mathcal{R}_2 \mathcal{E} t_1 (\mathcal{R}_1/\mathcal{E})^* \mathcal{E} \mathcal{R}_2 \mathcal{E} t_2 \dots (\mathcal{R}_1/\mathcal{E})^* \mathcal{E} \mathcal{R}_2 \mathcal{E} t_i \dots$$

But $t_j \mathbb{K} T_j$ and $t_j (\mathcal{R}_1/\mathcal{E})^* \mathcal{E} \mathcal{R}_2 \mathcal{E} t_{j+1}$ imply, by (ES), (WS) and (SS), that there is T_{j+1} s.t. $t_{j+1} \mathbb{K} T_{j+1}$ and $T_j \mathcal{S}^+ T_{j+1}$. Thus, there are $T_1, T_2, \dots, T_i, \dots \in \mathcal{P}$ such that $t_1 \mathbb{K} T_1, t_2 \mathbb{K} T_2, \dots, t_i \mathbb{K} T_i, \dots$ and the following infinite \mathcal{S} -reduction sequence exists

$$T \mathcal{S}^+ T_1 \mathcal{S}^+ T_2 \mathcal{S}^+ \dots \mathcal{S}^+ T_i \dots$$

This leads to a contradiction with the fact that \mathcal{S} is well-founded on T .

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