

Realizability algebras II : new models of ZF + DC

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Introduction

The technology of *classical realizability* was developed in [15, 18] in order to extend the proof-program correspondence (also known as *Curry-Howard correspondence*) from pure intuitionistic logic to the whole of mathematical proofs, with excluded middle, axioms of ZF, dependent choice, existence of a well ordering on $\mathcal{P}(\mathbb{N})$, ...

We show here that this technology is also a new method in order to build models of ZF and to obtain relative consistency results.

The main tools are :

- The notion of *realizability algebra* [18], which comes from combinatory logic [2] and plays a role similar to a set of forcing conditions. The extension from intuitionistic to classical logic was made possible by Griffin's discovery [7] of the relation between the law of Peirce and the instruction `call-with-current-continuation` of the programming language SCHEME.

In this paper, we only use the simplest case of realizability algebra, which I call *standard realizability algebra*; somewhat like the *binary tree* in the case of forcing.

- The theory ZF_ε [13] which is a conservative extension of ZF, with a notion of *strong membership*, denoted as ε .

The theory ZF_ε is essentially ZF without the extensionality axiom. We note an analogy with the Fraenkel-Mostowski models with “urelements” : we obtain a non well orderable set, which is a Boolean algebra denoted $\mathfrak{J}2$, all elements of which (except 1) are empty. But we also notice two important differences :

- The final model of $ZF + \neg AC$ is obtained directly, without taking a suitable submodel.
- There exists an injection from the “pathological set” $\mathfrak{J}2$ into \mathbb{R} , and therefore \mathbb{R} is *also not well orderable*.

We show the consistency, relatively to the consistency of ZF, of the theory $ZF + DC$ (dependent choice) with the following properties :

there exists a sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$ of infinite subsets of \mathbb{R} , the “cardinals” of which are strictly increasing (this means that there is an injection but no surjection from \mathcal{X}_n to \mathcal{X}_{n+1}), and such that $\mathcal{X}_m \times \mathcal{X}_n$ is equipotent with \mathcal{X}_{mn} for $m, n \geq 2$;

there exists a sequence of infinite subsets of \mathbb{R} , the “cardinals” of which are strictly decreasing.

More detailed properties of \mathbb{R} in this model are given in theorems 35 and 39.

As far as I know, these consistency results are new, and it seems they cannot be obtained by forcing. But, in any case, the fact that the simplest non trivial realizability model (which I call the *model of threads*) has a real line with such unusual properties, is of interest in itself. Another aspect of these results, which is interesting from the point of view of computer science, is the following : in [18], we introduce *read* and *write* instructions in a global memory, in order to realize a weak form of the axiom of choice (well ordering of \mathbb{R}). Therefore, what we show here, is that these instructions are *indispensable* : without them, we can build a realizability model in which \mathbb{R} is not well ordered.

Standard realizability algebras

The structure of *realizability algebra*, and the particular case of *standard realizability algebra* are defined in [18]. They are variants of the usual notion of *combinatory algebra*. Here, we only need the *standard* realizability algebras, the definition of which we recall below :

We have a countable set Π_0 which is the set of *stack constants*.

We define recursively two sets : Λ (the set of *terms*) and Π (the set of *stacks*). Terms and stacks are finite sequences of elements of the set :

$$\Pi_0 \cup \{B, C, E, I, K, W, cc, \zeta, k, (,), [,], \bullet\}$$

which are obtained by the following rules :

- $B, C, E, I, K, W, cc, \zeta$ are terms (*elementary combinators*) ;
- each element of Π_0 is a stack (*empty stacks*) ;
- if ξ, η are terms, then $(\xi)\eta$ is a term (this operation is called *application*) ;
- if ξ is a term and π a stack, then $\xi \bullet \pi$ is a stack (this operation is called *push*) ;
- if π is a stack, then $k[\pi]$ is a term.

A term of the form $k[\pi]$ is called a *continuation*. From now on, it will be denoted as k_π .

A term which does not contain any continuation (i.e. in which the symbol k does not appear) is called *proof-like*.

Every stack has the form $\pi = \xi_1 \bullet \dots \bullet \xi_n \bullet \pi_0$, where $\xi_1, \dots, \xi_n \in \Lambda$ and $\pi_0 \in \Pi_0$, i.e. π_0 is a stack constant.

If $\xi \in \Lambda$ and $\pi \in \Pi$, the ordered pair (ξ, π) is called a *process* and denoted as $\xi \star \pi$;

ξ and π are called respectively the *head* and the *stack* of the process $\xi \star \pi$.

The set of processes $\Lambda \times \Pi$ will also be written $\Lambda \star \Pi$.

Notation.

For sake of brevity, the term $(\dots((\xi)\eta_1)\eta_2)\dots\eta_n$ will be also denoted as $(\xi)\eta_1\eta_2\dots\eta_n$ or $\xi\eta_1\eta_2\dots\eta_n$, if the meaning is clear. For example : $\xi\eta\zeta = (\xi)\eta\zeta = (\xi\eta)\zeta = ((\xi)\eta)\zeta$.

We now choose a recursive bijection from Λ onto \mathbb{N} , which is written $\xi \mapsto n_\xi$.

We put $\sigma = (BW)(B)B$ (the characteristic property of σ is given below).

For each $n \in \mathbb{N}$, we define $\underline{n} \in \Lambda$ recursively, by putting : $\underline{0} = KI$; $\underline{n+1} = (\sigma)\underline{n}$;

\underline{n} is the *n-th integer* and σ is the *successor* in combinatory logic.

We define a preorder relation $>$ on $\Lambda \star \Pi$. It is the least reflexive and transitive relation such that, for all $\xi, \eta, \zeta \in \Lambda$ and $\pi, \omega \in \Pi$, we have :

$(\xi)\eta \star \pi > \xi \star \eta \bullet \pi.$
 $I \star \xi \bullet \pi > \xi \star \pi.$
 $K \star \xi \bullet \eta \bullet \pi > \xi \star \pi.$
 $E \star \xi \bullet \eta \bullet \pi > (\xi)\eta \star \pi.$
 $W \star \xi \bullet \eta \bullet \pi > \xi \star \eta \bullet \eta \bullet \pi.$
 $C \star \xi \bullet \eta \bullet \zeta \bullet \pi > \xi \star \zeta \bullet \eta \bullet \pi.$
 $B \star \xi \bullet \eta \bullet \zeta \bullet \pi > (\xi)(\eta)\zeta \star \pi.$
 $cc \star \xi \bullet \pi > \xi \star k_\pi \bullet \pi.$
 $k_\pi \star \xi \bullet \omega > \xi \star \pi.$
 $\varsigma \star \xi \bullet \eta \bullet \pi > \xi \star \underline{\eta} \bullet \pi.$

For instance, with the definition of $\underline{0}$ and σ given above, we have :

$\underline{0} \star \xi \bullet \eta \bullet \pi > \eta \star \pi ; \sigma \star \xi \bullet \eta \bullet \zeta \bullet \pi > (\xi\eta)(\eta)\zeta \star \pi.$

Finally, we have a subset \perp of $\Lambda \star \Pi$ which is a final segment for this preorder, which means that : $\xi \star \pi \in \perp, \xi' \star \pi' > \xi \star \pi \Rightarrow \xi' \star \pi' \in \perp.$

In other words, we ask that \perp has the following properties :

$(\xi)\eta \star \pi \notin \perp \Rightarrow \xi \star \eta \bullet \pi \notin \perp.$
 $I \star \xi \bullet \pi \notin \perp \Rightarrow \xi \star \pi \notin \perp.$
 $K \star \xi \bullet \eta \bullet \pi \notin \perp \Rightarrow \xi \star \pi \notin \perp.$
 $E \star \xi \bullet \eta \bullet \pi \notin \perp \Rightarrow (\xi)\eta \star \pi \notin \perp.$
 $W \star \xi \bullet \eta \bullet \pi \notin \perp \Rightarrow \xi \star \eta \bullet \eta \bullet \pi \notin \perp.$
 $C \star \xi \bullet \eta \bullet \zeta \bullet \pi \notin \perp \Rightarrow \xi \star \zeta \bullet \eta \bullet \pi \notin \perp.$
 $B \star \xi \bullet \eta \bullet \zeta \bullet \pi \notin \perp \Rightarrow (\xi)(\eta)\zeta \star \pi \notin \perp.$
 $cc \star \xi \bullet \pi \notin \perp \Rightarrow \xi \star k_\pi \bullet \pi \notin \perp.$
 $k_\pi \star \xi \bullet \omega \notin \perp \Rightarrow \xi \star \pi \notin \perp.$
 $\varsigma \star \xi \bullet \eta \bullet \pi \notin \perp \Rightarrow \xi \star \underline{\eta} \bullet \pi \notin \perp.$

Remark. Thus, the only arbitrary elements in a standard realizability algebra are the set Π_0 of stack constants and the set \perp of processes.

c-terms and λ -terms

We call *c-term* a term which is built with variables, the elementary combinators $B, C, E, I, K, W, cc, \varsigma$ and the application (binary function). A closed c-term is exactly what we have called a proof-like term.

Given a c-term t and a variable x , we define inductively on t , a new c-term denoted by $\lambda x t$, which does not contain x . To this aim, we apply the first possible case in the following list :

1. $\lambda x t = (K)t$ if t does not contain x .
2. $\lambda x x = I.$
3. $\lambda x tu = (C)\lambda x(E)t u$ if u does not contain x .
4. $\lambda x tx = (E)t$ if t does not contain x .
5. $\lambda x tx = (W)\lambda x(E)t$ (if t contains x).
6. $\lambda x(t)(u)v = \lambda x(B)tuv$ (if uv contains x).

In [18], it is shown that this definition is correct. This allows us to translate every λ -term into a c-term. In the following, almost every c-term will be written as a λ -term.

The fundamental property of this translation is given by theorem 1, which is proved in [18] :

Theorem 1. Let t be a c -term with the only variables x_1, \dots, x_n ; let $\xi_1, \dots, \xi_n \in \Lambda$ and $\pi \in \Pi$. Then $\lambda x_1 \dots \lambda x_n t \star \xi_1 \bullet \dots \bullet \xi_n \bullet \pi > t[\xi_1/x_1, \dots, \xi_n/x_n] \star \pi$.

Remark. The property we need for the term σ (the *successor*) is $\sigma \star \xi \bullet \eta \bullet \zeta \bullet \pi > (\xi\eta)(\eta)\zeta \star \pi$ (to prove theorem 18). Therefore, by theorem 1, we could define $\sigma = \lambda n \lambda f \lambda x (nf)(f)x$. The definition we chose is much simpler.

The formal system

We write formulas and proofs in the language of first order logic. This formal language consists of :

- *individual variables* x, y, \dots ;
- *function symbols* f, g, \dots ; each one has an arity, which is an integer; function symbols of arity 0 are called *constant symbols*.
- *relation symbols*; each one has an arity; relation symbols of arity 0 are called *propositional constants*. We have two particular propositional constants \top, \perp and three particular binary relation symbols $\mathcal{E}, \notin, \subseteq$.

The *terms* are built in the usual way with individual variables and function symbols.

Remark. We use the word “term” with two different meanings: here as a term in a first order language, and previously as an element of the set Λ of a realizability algebra. I think that, with the help of the context, no confusion is possible.

The *atomic formulas* are the expressions $R(t_1, \dots, t_n)$, where R is a n -ary relation symbol, and t_1, \dots, t_n are terms.

Formulas are built as usual, from atomic formulas, *with the only logical symbols* \rightarrow, \forall :

- each atomic formula is a formula;
- if A, B are formulas, then $A \rightarrow B$ is a formula;
- if A is a formula and x an individual variable, then $\forall x A$ is a formula.

Notations. The formula $A_1 \rightarrow (A_2 \rightarrow (\dots (A_n \rightarrow B) \dots))$ will be written $A_1, A_2, \dots, A_n \rightarrow B$.

The usual logical symbols are defined as follows :

$$\neg A \equiv A \rightarrow \perp; A \vee B \equiv (A \rightarrow \perp), (B \rightarrow \perp) \rightarrow \perp; A \wedge B \equiv (A, B \rightarrow \perp) \rightarrow \perp; \exists x F \equiv \forall x (F \rightarrow \perp) \rightarrow \perp.$$

More generally, we shall write $\exists x\{F_1, \dots, F_k\}$ for $\forall x(F_1, \dots, F_k \rightarrow \perp) \rightarrow \perp$.

We shall sometimes write \vec{F} for a finite sequence of formulas F_1, \dots, F_k ;

Then, we shall also write $\vec{F} \rightarrow G$ for $F_1, \dots, F_k \rightarrow G$ and $\exists x\{\vec{F}\}$ for $\forall x(\vec{F} \rightarrow \perp) \rightarrow \perp$.

$A \leftrightarrow B$ is the pair of formulas $\{A \rightarrow B, B \rightarrow A\}$.

The rules of natural deduction are the following (the A_i 's are formulas, the x_i 's are variables of c -term, t, u are c -terms, written as λ -terms) :

1. $x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i$.
2. $x_1 : A_1, \dots, x_n : A_n \vdash t : A \rightarrow B, x_1 : A_1, \dots, x_n : A_n \vdash u : A \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash tu : B$.
3. $x_1 : A_1, \dots, x_n : A_n, x : A \vdash t : B \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash \lambda x t : A \rightarrow B$.
4. $x_1 : A_1, \dots, x_n : A_n \vdash t : A \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash t : \forall x A$ where x is an individual variable which does not appear in A_1, \dots, A_n .
5. $x_1 : A_1, \dots, x_n : A_n \vdash t : \forall x A \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash t : A[\tau/x]$ where x is an individual variable and τ is a term.

6. $x_1 : A_1, \dots, x_n : A_n \vdash \text{cc} : ((A \rightarrow B) \rightarrow A) \rightarrow A$ (law of Peirce).
 7. $x_1 : A_1, \dots, x_n : A_n \vdash t : \perp \Rightarrow x_1 : A_1, \dots, x_n : A_n \vdash t : A$ for every formula A .

The theory ZF_ε

We write below a set of axioms for a theory called ZF_ε . Then :

- We show that ZF_ε is a conservative extension of ZF
- We define the *realizability models* and we show that each axiom of ZF_ε is realized by a proof-like c-term, in every realizability model.

It follows that the axioms of ZF are also realized by proof-like c-terms in every realizability model.

We write the axioms of ZF_ε with the three binary relation symbols $\not\in, \notin, \subseteq$. Of course, $x \varepsilon y$ and $x \in y$ are the formulas $x \not\in y \rightarrow \perp$ and $x \notin y \rightarrow \perp$.

The notation $x \simeq y \rightarrow F$ means $x \subseteq y, y \subseteq x \rightarrow F$. Thus $x \simeq y$, which represents the usual (extensional) equality of sets, is the pair of formulas $\{x \subseteq y, y \subseteq x\}$.

We use the notations $(\forall x \varepsilon a)F(x)$ for $\forall x(\neg F(x) \rightarrow x \not\in a)$ and $(\exists x \varepsilon a)\vec{F}(x)$ for $\neg \forall x(\vec{F}(x) \rightarrow x \not\in a)$.

For instance, $(\exists x \varepsilon y) t \simeq u$ is the formula $\neg \forall x(t \subseteq u, u \subseteq t \rightarrow x \not\in y)$.

The axioms of ZF_ε are the following :

0. Extensionality axioms.

$$\forall x \forall y [x \in y \leftrightarrow (\exists z \varepsilon y) x \simeq z] ; \forall x \forall y [x \subseteq y \leftrightarrow (\forall z \varepsilon x) z \in y].$$

1. Foundation scheme.

$$\forall x_1 \dots \forall x_n \forall a (\forall x ((\forall y \varepsilon x) F[y, x_1, \dots, x_n] \rightarrow F[x, x_1, \dots, x_n]) \rightarrow F[a, x_1, \dots, x_n])$$

for every formula $F[x, x_1, \dots, x_n]$.

The intuitive meaning of axioms 0 and 1 is that ε is a well founded relation, and that the relation \in is obtained by “collapsing” ε into an extensional binary relation.

The following axioms essentially express that the relation ε satisfies the axioms of Zermelo-Fraenkel *except extensionality*.

2. Comprehension scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b \forall x (x \varepsilon b \leftrightarrow (x \varepsilon a \wedge F[x, x_1, \dots, x_n]))$$

for every formula $F[x, x_1, \dots, x_n]$.

3. Pairing axiom.

$$\forall a \forall b \exists x \{a \varepsilon x, b \varepsilon x\}.$$

4. Union axiom.

$$\forall a \exists b (\forall x \varepsilon a) (\forall y \varepsilon x) y \varepsilon b.$$

5. Power set axiom.

$$\forall a \exists b \forall x (\exists y \varepsilon b) \forall z (z \varepsilon y \leftrightarrow (z \varepsilon a \wedge z \varepsilon x)).$$

6. Collection scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b (\forall x \varepsilon a) (\exists y F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \varepsilon b) F[x, y, x_1, \dots, x_n])$$

for every formula $F[x, y, x_1, \dots, x_n]$.

7. Infinity scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b \{a \varepsilon b, (\forall x \varepsilon b) (\exists y F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \varepsilon b) F[x, y, x_1, \dots, x_n])\}$$

for every formula $F[x, y, x_1, \dots, x_n]$.

The usual Zermelo-Fraenkel set theory is obtained from ZF_ε by identifying the predicate symbols ε and \in . Thus, the axioms of ZF are written as follows, with the predicate symbols \in, \subseteq (recall that $x \simeq y$ is the conjunction of $x \subseteq y$ and $y \subseteq x$) :

0. Equality and extensionality axioms.

$$\forall x \forall y [x \in y \leftrightarrow (\exists z \in y) x \simeq z] ; \forall x \forall y [x \subseteq y \leftrightarrow (\forall z \in x) z \in y].$$

1. Foundation scheme.

$$\forall x_1 \dots \forall x_n \forall a (\forall x ((\forall y \in x) F[y, x_1, \dots, x_n] \rightarrow F[x, x_1, \dots, x_n]) \rightarrow F[a, x_1, \dots, x_n])$$

for every formula $F[x, x_1, \dots, x_n]$ written with the only relation symbols \in, \subseteq .

2. Comprehension scheme.

$$\forall a \exists b \forall x (x \in b \leftrightarrow (x \in a \wedge F[x, x_1, \dots, x_n]))$$

for every formula $F[x, x_1, \dots, x_n]$ written with the only relation symbols \in, \subseteq .

3. Pairing axiom.

$$\forall a \forall b \exists x \{a \in x, b \in x\}.$$

4. Union axiom.

$$\forall a \exists b (\forall x \in a) (\forall y \in x) y \in b.$$

5. Power set axiom.

$$\forall a \exists b \forall x (\exists y \in b) \forall z (z \in y \leftrightarrow (z \in a \wedge z \in x)).$$

6. Collection scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b (\forall x \in a) (\exists y \in b) (F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \in b) F[x, y, x_1, \dots, x_n])$$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \in, \subseteq .

7. Infinity scheme.

$$\forall x_1 \dots \forall x_n \forall a \exists b \{a \in b, (\forall x \in b) (\exists y \in b) (F[x, y, x_1, \dots, x_n] \rightarrow (\exists y \in b) F[x, y, x_1, \dots, x_n])\}$$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \in, \subseteq .

Remark. The usual statement of the axiom of infinity is the particular case of this scheme, where a is \emptyset , and $F(x, y)$ is the formula $y \simeq x \cup \{x\}$.

Let us show that ZF_ε is a conservative extension of ZF. First, it is clear that, if $ZF_\varepsilon \vdash F$, where F is a formula of ZF (i.e. written only with \in and \subseteq), then $ZF \vdash F$; indeed, it is sufficient to replace ε with \in in any proof of $ZF_\varepsilon \vdash F$.

Conversely, we must show that each axiom of ZF is a consequence of ZF_ε .

Theorem 2.

i) $ZF_\varepsilon \vdash \forall a (a \subseteq a)$ (and thus $a \simeq a$).

ii) $ZF_\varepsilon \vdash \forall a \forall x (x \varepsilon a \rightarrow x \in a)$.

i) Using the foundation axiom, we assume $\forall x (x \varepsilon a \rightarrow x \subseteq x)$, and we must show $a \subseteq a$; therefore, we add the hypothesis $x \varepsilon a$. It follows that $x \subseteq x$, then $x \simeq x$, and therefore :

$\exists y \{x \simeq y, y \varepsilon a\}$, that is to say $x \in a$. Thus, we have $\forall x (x \varepsilon a \rightarrow x \in a)$, and therefore $a \subseteq a$.

ii) Just shown.

Q.E.D.

Corollary 3. $ZF_\varepsilon \vdash \forall x (x \varepsilon a \rightarrow x \in b) \rightarrow a \subseteq b$.

We must show $x \varepsilon a \rightarrow x \in b$, which follows from $x \in a \rightarrow x \in b$ and $x \varepsilon a \rightarrow x \in a$ (theorem 2(ii)).

Q.E.D.

Lemma 4. $ZF_\varepsilon \vdash a \subseteq b, \forall x(x \in b \rightarrow x \in c) \rightarrow a \subseteq c$.

We must show $x \varepsilon a \rightarrow x \in c$, which follows from $x \varepsilon a \rightarrow x \in b$ and $x \in b \rightarrow x \in c$.

Q.E.D.

Theorem 5. $ZF_\varepsilon \vdash \forall y \forall z (y \simeq a, a \in z \rightarrow y \in z)$; $ZF_\varepsilon \vdash \forall y \forall z (a \subseteq y, z \in a \rightarrow z \in y)$.

Call $F(a), F'(a)$ these two formulas. We show $F(a)$ by foundation :

thus, we suppose $(\forall x \varepsilon a) F(x)$ and we first show $F'(a)$: by hypothesis, we have $a \subseteq y, z \in a$; thus, there exists a' such that $z \simeq a'$ and $a' \varepsilon a$, and thus $F(a')$. From $a' \varepsilon a$ and $a \subseteq y$, we deduce $a' \in y$. From $z \simeq a'$ and $a' \in y$, we deduce $z \in y$ by $F(a')$.

Then, we show $F(a)$: by hypothesis, we have $y \simeq a, a \in z$, thus $a \simeq y'$ and $y' \varepsilon z$ for some y' . In order to show $y \in z$, it is sufficient to show $y \simeq y'$.

Now, we have $y \simeq a, a \simeq y'$, and thus $y' \subseteq a, a \subseteq y$. From $F'(a)$, we get $\forall z(z \in a \rightarrow z \in y)$; from $y' \subseteq a$, we deduce $y' \subseteq y$ by lemma 4.

We have also $y \subseteq a, a \subseteq y'$. From $F'(a)$, we get $\forall z(z \in a \rightarrow z \in y')$; from $y \subseteq a$, we deduce $y \subseteq y'$ by lemma 4.

Q.E.D.

With corollary 3, we obtain :

Corollary 6. $ZF_\varepsilon \vdash b \subseteq c \leftrightarrow \forall x(x \in b \rightarrow x \in c)$.

It is now easy to deduce the equality and extensionality axioms of ZF :

$\forall x(x \simeq x)$; $\forall x \forall y(x \simeq y \rightarrow y \simeq x)$; $\forall x \forall y \forall z(x \simeq y, y \simeq z \rightarrow x \simeq z)$;
 $\forall x \forall x' \forall y \forall y'(x \simeq x', y \simeq y', x \notin y \rightarrow x' \notin y')$; $\forall x \forall y(\forall z(z \notin x \leftrightarrow z \notin y) \rightarrow x \simeq y)$;
 $\forall x \forall y(x \subseteq y \leftrightarrow \forall z(z \notin y \rightarrow z \notin x))$.

Remark. This shows that \simeq is an equivalence relation which is compatible with the relations \in and \subseteq ; but, in general, it is *not compatible with ε* . This is the equality relation for ZF ; it will be called *extensional equivalence*.

Notation. The formula $\forall z(z \notin y \rightarrow z \notin x)$ will be written $x \subset y$. The ordered pair of formulas $x \subset y, y \subset x$ will be written $x \sim y$.

By theorem 2, we get $ZF_\varepsilon \vdash \forall x \forall y(x \subset y \rightarrow x \subseteq y)$. Thus \subset will be called *strong inclusion*, and \sim will be called *strong extensional equivalence*.

• Foundation scheme.

Let $F[x]$ be written with only \notin, \subseteq and let $G[x]$ be the formula $\forall y(y \simeq x \rightarrow F[y])$. Clearly, $\forall x G[x]$ is equivalent to $\forall x F[x]$. Therefore, from axiom scheme 1 of ZF_ε , it is sufficient to show : $\forall b(\forall x(x \in b \rightarrow F[x]) \rightarrow F[b]) \rightarrow (\forall x(x \varepsilon a \rightarrow G[x]) \rightarrow G[a])$, i.e. :

$\forall b(\forall x(x \in b \rightarrow F[x]) \rightarrow F[b]), \forall x \forall y(x \varepsilon a, y \simeq x \rightarrow F[y]), a \simeq b \rightarrow F[b]$.

Therefore, it is sufficient to prove : $\forall x \forall y(x \varepsilon a, y \simeq x \rightarrow F[y]), a \simeq b \rightarrow \forall x(x \in b \rightarrow F[x])$.

From $x \in b, a \simeq b$, we deduce $x \in a$ and therefore (by axiom 0), $x' \varepsilon a$ for some $x' \simeq x$. Finally, we get $F[x]$ from $\forall x \forall y(x \varepsilon a, y \simeq x \rightarrow F[y])$.

- Comprehension scheme : $\forall a \exists b \forall x (x \in b \leftrightarrow (x \in a \wedge F[x]))$

for every formula $F[x, x_1, \dots, x_n]$ written with \notin, \subseteq .

From the axiom scheme 2 of ZF_ε , we get $\forall x (x \varepsilon b \leftrightarrow (x \varepsilon a \wedge F[x]))$. If $x \in b$, then $x \simeq x'$, $x' \varepsilon b$ for some x' . Thus $x' \varepsilon a$ and $F[x']$. From $x \simeq x'$ and $x' \varepsilon a$, we deduce $x \in a$. Since \subseteq and ε are compatible with \simeq , it is the same for F ; thus, we obtain $F[x]$.

Conversely, if we have $F[x]$ and $x \in a$, we have $x \simeq x'$ and $x' \varepsilon a$ for some x' . Since F is compatible with \simeq , we get $F[x']$, thus $x' \varepsilon b$ and $x \in b$.

- Pairing axiom : $\forall x \forall y \exists z \{x \in z, y \in z\}$.

Trivial consequence of axiom 3 of ZF_ε , and theorem 2(ii).

- Union axiom : $\forall a \exists b \forall x \forall y (x \in a, y \in x \rightarrow y \in b)$.

From $x \in a$ we have $x \simeq x'$ and $x' \varepsilon a$ for some x' ; we have $y \in x$, therefore $y \in x'$, thus $y \simeq y'$ and $y' \varepsilon x'$ for some y' . From axiom 4 of ZF_ε , $x' \varepsilon a$ and $y' \varepsilon x'$, we get $y' \varepsilon b$; therefore $y \in b$, by $y \simeq y'$.

- Power set axiom : $\forall a \exists b \forall x \exists y \{y \in b, \forall z (z \in y \leftrightarrow (z \in a \wedge z \in x))\}$

Given a , we obtain b by axiom 5 of ZF_ε ; given x , we define x' by the condition :

$\forall z (z \varepsilon x' \leftrightarrow (z \varepsilon a \wedge z \in x))$ (comprehension scheme of ZF_ε). By definition of b , there exists $y \varepsilon b$ such that $\forall z (z \varepsilon y \leftrightarrow z \varepsilon a \wedge z \in x')$, and therefore $\forall z (z \varepsilon y \leftrightarrow z \varepsilon a \wedge z \in x)$. It follows easily that $\forall z (z \in y \leftrightarrow z \in a \wedge z \in x)$.

- Collection scheme : $\forall a \exists b (\forall x \in a) (\exists y F[x, y] \rightarrow (\exists y \in b) F[x, y])$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

From $x \in a$ and $\exists y F[x, y]$, we get $x \simeq x'$, $x' \varepsilon a$ for some x' , and thus $\exists y F[x', y]$ since F is compatible with \simeq . From axiom scheme 6 of ZF_ε , we get $(\exists y \varepsilon b) F[x', y]$, and therefore : $(\exists y \in b) F[x, y]$, by theorem 2(ii), again because F is compatible with \simeq .

- Infinity scheme : $\forall a \exists b \{a \in b, (\forall x \in b) (\exists y F[x, y] \rightarrow (\exists y \in b) F[x, y])\}$

for every formula $F[x, y, x_1, \dots, x_n]$ written with the only relation symbols \notin, \subseteq .

Same proof.

Q.E.D.

Realizability models of ZF_ε

As usual in relative consistency proofs, we start with a model \mathcal{M} of ZFC, called *the ground model* or *the standard model*. In particular, the integers of \mathcal{M} are called *the standard integers*. The elements of \mathcal{M} will be called *individuals*.

In the sequel, the model \mathcal{M} will be our universe, which means that every notion we consider is defined in \mathcal{M} . In particular, the realizability algebra (Λ, Π, \perp) is an individual of \mathcal{M} .

We define a *realizability model* \mathcal{N} , with the same set of individuals as \mathcal{M} . But \mathcal{N} is not a model in the usual sense, because its truth values are subsets of Π instead of being 0 or 1. Therefore, although \mathcal{M} and \mathcal{N} have the same domain (the quantifier $\forall x$ describes the same domain for both), the model \mathcal{N} may (and will, in all non trivial cases) have much more individuals than \mathcal{M} , because it has individuals which are *not named*. In particular, it will have *non standard integers*.

Remark. This is a great difference between *realizability* and *forcing* models of ZF. In a forcing model, each individual is named in the ground model ; it follows that integers, and even ordinals, are not changed.

For each closed formula F with parameters in \mathcal{M} , we define two truth values :

$\|F\| \subseteq \Pi$ and $|F| \subseteq \Lambda$.

$|F|$ is defined immediately from $\|F\|$ as follows :

$$\xi \in |F| \Leftrightarrow (\forall \pi \in \|F\|) \xi \star \pi \in \perp.$$

Notation. We shall write $\xi \Vdash F$ (read “ ξ realizes F ”) for $\xi \in |F|$.

$\|F\|$ is now defined by recurrence on the length of F :

- F is atomic ;

then F has one of the forms $\top, \perp, a \neq b, a \subseteq b, a \notin b$ where a, b are parameters in \mathcal{M} . We set :

$$\|\top\| = \emptyset ; \|\perp\| = \Pi ; \|a \neq b\| = \{\pi \in \Pi ; (a, \pi) \in b\}.$$

$\|a \subseteq b\|, \|a \notin b\|$ are defined simultaneously by induction on $(\text{rk}(a) \cup \text{rk}(b), \text{rk}(a) \cap \text{rk}(b))$ ($\text{rk}(a)$ being the rank of a).

$$\|a \subseteq b\| = \bigcup_c \{\xi \cdot \pi ; \xi \in \Lambda, \pi \in \Pi, (c, \pi) \in a, \xi \Vdash c \notin b\} ;$$

$$\|a \notin b\| = \bigcup_c \{\xi \cdot \xi' \cdot \pi ; \xi, \xi' \in \Lambda, \pi \in \Pi, (c, \pi) \in b, \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a\}.$$

- $F \equiv A \rightarrow B$; then $\|F\| = \{\xi \cdot \pi ; \xi \Vdash A, \pi \in \|B\|\}$.
- $F \equiv \forall x A$: then $\|F\| = \bigcup_a \|A[a/x]\|$.

The following theorem is an essential tool :

Theorem 7 (Adequacy lemma).

Let A_1, \dots, A_n, A be closed formulas of ZF_ε , and suppose that $x_1 : A_1, \dots, x_n : A_n \vdash t : A$.

If $\xi_1 \Vdash A_1, \dots, \xi_n \Vdash A_n$ then $t[\xi_1/x_1, \dots, \xi_n/x_n] \Vdash A$. In particular, if $\vdash t : A$, then $t \Vdash A$.

We need to prove a (seemingly) more general result, that we state as a lemma :

Lemma 8. Let $A_1[\vec{z}], \dots, A_n[\vec{z}], A[\vec{z}]$ be formulas of ZF_ε , with $\vec{z} = (z_1, \dots, z_k)$ as free variables, and suppose that $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : A[\vec{z}]$.

If $\xi_1 \Vdash A_1[\vec{a}], \dots, \xi_n \Vdash A_n[\vec{a}]$ for some parameters (i.e. individuals in \mathcal{M}) $\vec{a} = (a_1, \dots, a_k)$, then $t[\xi_1/x_1, \dots, \xi_n/x_n] \Vdash A[\vec{a}]$.

Proof by recurrence on the length of the derivation of $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : A[\vec{z}]$.

We consider the last used rule.

1. $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash x_i : A_i[\vec{z}]$. This case is trivial.

2. We have the hypotheses :

$$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash u : B[\vec{z}] \rightarrow A[\vec{z}] ; x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash v : B[\vec{z}] ; t = uv.$$

By the induction hypothesis, we have $u[\vec{\xi}/\vec{x}] \Vdash B[\vec{a}/\vec{z}] \rightarrow A[\vec{a}/\vec{z}]$ and $v[\vec{\xi}/\vec{x}] \Vdash B[\vec{a}/\vec{z}]$.

Therefore $(uv)[\vec{\xi}/\vec{x}] \Vdash A[\vec{a}/\vec{z}]$ which is the desired result.

3. We have the hypotheses :

$$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}], y : B[\vec{z}] \vdash u : C[\vec{z}] ; A[\vec{z}] \equiv B[\vec{z}] \rightarrow C[\vec{z}] ; t = \lambda y u.$$

We want to show that $(\lambda y u)[\vec{\xi}/\vec{x}] \Vdash B[\vec{a}/\vec{z}] \rightarrow C[\vec{a}/\vec{z}]$. Thus, let :

$\eta \Vdash B[\vec{a}/\vec{z}]$ and $\pi \in \|C[\vec{a}/\vec{z}]\|$. We must show :

$(\lambda y u)[\vec{\xi}/\vec{x}] \star \eta \bullet \pi \in \perp$ or else $u[\vec{\xi}/\vec{x}, \eta/y] \star \pi \in \perp$.

Now, by the induction hypothesis, we have $u[\vec{\xi}/\vec{x}, \eta/y] \Vdash C[\vec{a}/\vec{z}]$, which gives the result.

4. We have the hypotheses :

$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : B[\vec{z}]$; $A[\vec{z}] \equiv \forall z_1 B[\vec{z}]$; $\xi_i \Vdash A_i[a_1/z_1, a_2/z_2, \dots, a_k/z_k]$; the variable z_1 is not free in $A_1[\vec{z}], \dots, A_n[\vec{z}]$.

We have to show that $t[\vec{\xi}/\vec{x}] \Vdash \forall z_1 B[\vec{a}/\vec{z}]$ i.e. $t[\vec{\xi}/\vec{x}] \Vdash \forall z_1 B[a_2/z_2, \dots, a_k/z_k]$. Thus, we take an arbitrary set b in \mathcal{M} and we show $t[\vec{\xi}/\vec{x}] \Vdash B[b/z_1, a_2/z_2, \dots, a_k/z_k]$.

By the induction hypothesis, it is sufficient to show that $\xi_i \Vdash A_i[b/z_1, a_2/z_2, \dots, a_k/z_k]$.

But this follows from the hypothesis on ξ_i , because z_1 is not free in the formulas A_i .

5. We have the hypotheses :

$x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : \forall y B[y, \vec{z}]$; $A[\vec{z}] \equiv B[\tau[\vec{z}]/y, \vec{z}]$; $\xi_i \Vdash A_i[\vec{a}]$.

By the induction hypothesis, we have $t[\vec{\xi}/\vec{x}] \Vdash \forall y B[y, \vec{a}/\vec{z}]$; therefore $t[\vec{\xi}/\vec{x}] \Vdash B[b/y, \vec{a}/\vec{z}]$ for every parameter b . We get the desired result by taking $b = \tau[\vec{a}]$.

6. The result follows from the following :

Theorem 9. For every formulas A, B , we have $cc \Vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$.

Let $\xi \Vdash (A \rightarrow B) \rightarrow A$ and $\pi \in \|A\|$. Then $cc \star \xi \bullet \pi > \xi \star k_\pi \bullet \pi$ which is in \perp , because $k_\pi \Vdash A \rightarrow B$ by lemma 10.

Q.E.D.

Lemma 10. If $\pi \in \|A\|$, then $k_\pi \Vdash A \rightarrow B$.

Indeed, let $\xi \Vdash A$; then $k_\pi \star \xi \bullet \pi' > \xi \star \pi \in \perp$ for every stack $\pi' \in \|B\|$.

Q.E.D.

7. We have the hypothesis $x_1 : A_1[\vec{z}], \dots, x_n : A_n[\vec{z}] \vdash t : \perp$.

By the induction hypothesis, we have $t[\vec{\xi}/\vec{x}] \Vdash \perp$. Since $\|\perp\| = \Pi$, we have $t[\vec{\xi}/\vec{x}] \star \pi \in \perp$ for every $\pi \in \|A[\vec{a}/\vec{z}]\|$, and therefore $t[\vec{\xi}/\vec{x}] \Vdash A[\vec{a}/\vec{z}]$ which is the desired result.

This completes the proof of lemma 8 and theorem 7.

Q.E.D.

Realized formulas and coherent models

In the ground model \mathcal{M} , we interpret the formulas of the *language of ZF* : this language consists of \notin, \subseteq ; we add some function symbols, but these functions are always defined, in \mathcal{M} , by some formulas written with \notin, \subseteq . We suppose that this ground model satisfies ZFC. The value, in \mathcal{M} , of a closed formula F of the language of ZF, with parameters in \mathcal{M} , is of course 1 or 0. In the first case, we say that \mathcal{M} *satisfies* F , and we write $\mathcal{M} \models F$.

In the realizability model \mathcal{N} , we interpret the formulas of the *language of ZF_ε*, which consists of $\notin, \notin, \subseteq$ and the same function symbols as in the language of ZF. The domain of \mathcal{N} and the interpretation of the function symbols are the same as for the model \mathcal{M} .

The value, in \mathcal{N} , of a closed formula F of ZF_ε with parameters (in \mathcal{M} or in \mathcal{N} , which is the same thing) is an element of $\mathcal{P}(\Pi)$ which is denoted as $\|F\|$, the definition of which has been given above.

Thus, we can no longer say that \mathcal{N} satisfies (or not) a given closed formula F . But we shall

say that \mathcal{N} realizes F (and we shall write $\mathcal{N} \Vdash F$), if there exists a proof-like term θ such that $\theta \Vdash F$. We say that two closed formulas F, G are *interchangeable* if $\mathcal{N} \Vdash F \leftrightarrow G$. Notice that, if $\|F\| = \|G\|$, then F, G are interchangeable (indeed $I \Vdash F \rightarrow G$), but the converse is far from being true.

The model \mathcal{N} allows us to make relative consistency proofs, since it is clear, from the adequacy lemma (theorem 7), that the class of formulas which are realized in \mathcal{N} is closed by deduction in classical logic. Nevertheless, we must check that the realizability model \mathcal{N} is *coherent*, i.e. that it does not realize the formula \perp . We can express this condition in the following form :

For every proof-like term θ , there exists a stack $\pi \in \Pi$ such that $\theta \star \pi \notin \perp$.

When the model \mathcal{N} is coherent, it is not *complete*, except in trivial cases. This means that there exist closed formulas F of ZF_ε such that $\mathcal{N} \not\Vdash F$ and $\mathcal{N} \not\Vdash \neg F$.

The axioms of ZF_ε are realized in \mathcal{N}

- Extensionality axioms.

We have $\|\forall z(z \notin b \rightarrow z \notin a)\| = \bigcup_c \{\xi \cdot \pi; \xi \Vdash c \notin b, \pi \in \|c \notin a\|\}$

by definition of the value of $\|\forall z(z \notin b \rightarrow z \notin a)\|$;

and $\|a \subseteq b\| = \bigcup_c \{\xi \cdot \pi; (c, \pi) \in a, \xi \Vdash c \notin b\}$ by definition of $\|a \subseteq b\|$.

Therefore, we have $\|a \subseteq b\| = \|\forall z(z \notin b \rightarrow z \notin a)\|$, so that :

$I \Vdash \forall x \forall y (x \subseteq y \rightarrow \forall z (z \notin y \rightarrow z \notin x))$ and $I \Vdash \forall x \forall y (\forall z (z \notin y \rightarrow z \notin x) \rightarrow x \subseteq y)$.

In the same way, we have :

$\|\forall z(a \subseteq z, z \subseteq a \rightarrow z \notin b)\| = \bigcup_c \{\xi \cdot \xi' \cdot \pi; \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a; \pi \in \|c \notin b\|\}$

by definition of the value of $\|\forall z(a \subseteq z, z \subseteq a \rightarrow z \notin b)\|$;

and $\|a \notin b\| = \bigcup_c \{\xi \cdot \xi' \cdot \pi; (c, \pi) \in b, \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a\}$ by definition of $\|a \notin b\|$.

Therefore, we have $\|a \notin b\| = \|\forall z(a \subseteq z, z \subseteq a \rightarrow z \notin b)\|$, so that :

$I \Vdash \forall x \forall y (x \notin y \rightarrow \forall z (x \subseteq z, z \subseteq x \rightarrow z \notin y))$; $I \Vdash \forall x \forall y (\forall z (x \subseteq z, z \subseteq x \rightarrow z \notin y) \rightarrow x \notin y)$.

Notation. We shall write $\vec{\xi}$ for a finite sequence (ξ_1, \dots, ξ_n) of terms. Therefore, we shall write $\vec{\xi} \Vdash \vec{A}$ for $\xi_i \Vdash A_i$ ($i = 1, \dots, n$).

In particular, the notation $\vec{\xi} \Vdash a \simeq b$ means $\xi_1 \Vdash a \subseteq b, \xi_2 \Vdash b \subseteq a$;

the notation $\vec{\xi} \Vdash A \leftrightarrow B$ means $\xi_1 \Vdash A \rightarrow B, \xi_2 \Vdash B \rightarrow A$.

- Foundation scheme.

Theorem 11. *For every finite sequence $\vec{F}[x, x_1, \dots, x_n]$ of formulas, we have :*

$Y \Vdash \forall x (\forall y (\vec{F}[y] \rightarrow y \notin x), \vec{F}[x] \rightarrow \perp) \rightarrow \forall x (\vec{F}[x] \rightarrow \perp)$

with $Y = AA$ and $A = \lambda a \lambda f (f)(a)af$ (Turing fixed point combinator).

Let $\xi \Vdash \forall x (\forall y (\vec{F}[y] \rightarrow y \notin x), \vec{F}[x] \rightarrow \perp)$. We show, by induction on the rank of a , that :

$Y \star \xi \cdot \vec{\eta} \cdot \pi \in \perp$, for every $\pi \in \Pi$ and $\vec{\eta} \Vdash \vec{F}[a]$.

Since $Y \star \xi \cdot \vec{\eta} \cdot \pi \succ \xi \star Y\xi \cdot \vec{\eta} \cdot \pi$, it suffices to show $\xi \star Y\xi \cdot \vec{\eta} \cdot \pi \in \perp$.

Now, $\xi \Vdash \forall y (\vec{F}[y] \rightarrow y \notin a), \vec{F}[a] \rightarrow \perp$, so that it suffices to show $Y\xi \Vdash \forall y (\vec{F}[y] \rightarrow y \notin a)$, in other words $Y\xi \Vdash \vec{F}[b] \rightarrow b \notin a$ for every b . Let $\vec{\zeta} \Vdash \vec{F}[b]$ and $\omega \in \|b \notin a\|$. Thus, we have

$(b, \omega) \in a$, therefore $\text{rk}(b) < \text{rk}(a)$ so that $Y \star \xi \cdot \vec{\zeta} \cdot \omega \in \perp$ by induction hypothesis. It follows that $Y \xi \star \vec{\zeta} \cdot \omega \in \perp$, which is the desired result.

Q.E.D.

It follows from theorem 11 that the axiom scheme 1 of ZF_e (foundation) is realized.

- Comprehension scheme.

Let a be a set, and $F[x]$ a formula with parameters. We put $b = \{(x, \xi \cdot \pi); (x, \pi) \in a, \xi \Vdash F[x]\}$; then, we have trivially $\|y \not\in b\| = \|F(x) \rightarrow x \not\in a\|$.

Therefore $I \Vdash \forall x(x \not\in b \rightarrow (F(x) \rightarrow x \not\in a))$ and $I \Vdash \forall x((F(x) \rightarrow x \not\in a) \rightarrow x \not\in b)$.

- Pairing axiom.

We consider two sets a and b , and we put $c = \{a, b\} \times \Pi$. We have $\|a \not\in c\| = \|b \not\in c\| = \|\perp\|$, thus $I \Vdash a \in c$ and $I \Vdash b \in c$.

Remark. Except in trivial cases, c has many other elements than a and b , which have no name in \mathcal{M} .

- Union axiom.

Given a set a , let $b = \text{Cl}(a)$ (the transitive closure of a , i.e. the least transitive set which contains a). We show $\|y \not\in b \rightarrow x \not\in a\| \subseteq \|y \not\in x \rightarrow x \not\in a\|$: indeed, let $\xi \cdot \pi \in \|y \not\in b \rightarrow x \not\in a\|$, i.e. $\xi \Vdash y \not\in b$ and $(x, \pi) \in a$. Therefore, $x \subseteq \text{Cl}(a)$, i.e. $x \subseteq b$ and thus $\|y \not\in b\| \supseteq \|y \not\in x\|$.

Thus, we have $\xi \Vdash y \not\in x$, which gives the result.

It follows that $I \Vdash \forall x \forall y ((y \not\in x \rightarrow x \not\in a) \rightarrow (y \not\in b \rightarrow x \not\in a))$.

- Power set axiom.

Given a set a , let $b = \mathcal{P}(\text{Cl}(a) \times \Pi) \times \Pi$. For every set x , we put :

$y = \{(z, \xi \cdot \pi); \xi \Vdash z \in x, (z, \pi) \in a\}$. We have $y = \{(z, \xi \cdot \pi); \xi \Vdash z \in x, \pi \in \|z \not\in a\|\}$, and therefore : $\|z \not\in y\| = \|z \in x \rightarrow z \not\in a\|$. Thus :

$I \Vdash \forall z(z \not\in y \rightarrow (z \in x \rightarrow z \not\in a))$ and $I \Vdash \forall z((z \in x \rightarrow z \not\in a) \rightarrow z \not\in y)$.

Now, it is obvious that $y \in \mathcal{P}(\text{Cl}(a) \times \Pi)$, and therefore $(y, \pi) \in b$ for every $\pi \in \Pi$.

Thus, we have $\|y \not\in b\| = \Pi = \|\perp\|$. It follows that :

$\lambda f(f) I I \Vdash \forall x(\forall y(\forall z(z \not\in y \rightarrow (z \in x \rightarrow z \not\in a)), \forall z((z \in x \rightarrow z \not\in a) \rightarrow z \not\in y) \rightarrow y \not\in b) \rightarrow \perp)$.

- Collection scheme.

Given a set a , and a formula $F[x, y]$ with parameters, let :

$b = \bigcup \{\Phi(x, \xi) \times \text{Cl}(a); x \in \text{Cl}(a), \xi \in \Lambda\}$ with

$\Phi(x, \xi) = \{y \text{ of minimum rank}; \xi \Vdash F[x, y]\}$ or $\Phi(x, \xi) = \emptyset$ if there is no such y .

We show that $\|\forall y(F[x, y] \rightarrow x \not\in a)\| \subseteq \|\forall y(F[x, y] \rightarrow y \not\in b)\|$:

Suppose indeed that $\xi \cdot \pi \in \|\forall y(F[x, y] \rightarrow x \not\in a)\|$, i.e. $(x, \pi) \in a$ and $\xi \Vdash F[x, y]$ for some y .

By definition of $\Phi(x, \xi)$, there exists $y' \in \Phi(x, \xi)$. Moreover, we have $x \in \text{Cl}(a)$, $\pi \in \text{Cl}(a)$, and therefore $(y', \pi) \in b$; it follows that $\pi \in \|y' \not\in b\|$. But, since $y' \in \Phi(x, \xi)$, we have $\xi \Vdash F[x, y']$ and thus $\xi \cdot \pi \in \|F[x, y'] \rightarrow y' \not\in b\|$, which gives the result.

We have proved that $I \Vdash \forall x(\forall y(F[x, y] \rightarrow y \not\in b) \rightarrow \forall y(F[x, y] \rightarrow x \not\in a))$.

- Infinity scheme.

Given a set a , we define b as the least set such that :

$$\{a\} \times \Pi \subseteq b \text{ and } \forall x(\forall \pi \in \Pi)(\forall \xi \in \Lambda)((x, \pi) \in b \Rightarrow \Phi(x, \xi) \times \{\pi\} \subseteq b)$$

where $\Phi(x, \xi)$ is defined as above.

We have $\{a\} \times \Pi \subseteq b$, thus $\|a \not\in b\| = \|\perp\|$, and therefore $I \Vdash a \in b$.

We now show that $\|\forall y(F[x, y] \rightarrow x \not\in b)\| \subseteq \|\forall y(F[x, y] \rightarrow y \not\in b)\|$:

Suppose indeed that $\xi \cdot \pi \in \|\forall y(F[x, y] \rightarrow x \not\in b)\|$, i.e. $(x, \pi) \in b$ and $\xi \Vdash F[x, y]$ for some y .

By definition of $\Phi(x, \xi)$, there exists $y' \in \Phi(x, \xi)$. By definition of b , we have $(y', \pi) \in b$, i.e.

$\pi \in \llbracket y' \neq b \rrbracket$. Now, since $y' \in \Phi(x, \xi)$, we have $\xi \Vdash F[x, y']$ and thus :

$\xi \bullet \pi \in \llbracket F[x, y'] \rightarrow y' \neq b \rrbracket$, which gives the result.

We have proved that $I \Vdash a \varepsilon b$ and $I \Vdash \forall x(\forall y(F[x, y] \rightarrow y \neq b) \rightarrow \forall y(F[x, y] \rightarrow x \neq b))$.

Function symbols and equality

According to our needs, we shall add to the language of ZF_ε , some *function symbols* f, g, \dots of any arity. A k -ary function symbol f will be interpreted, in the realizability model \mathcal{N} , by a *functional relation*, which is defined *in the ground model* \mathcal{M} by a formula $F[x_1, \dots, x_k, y]$ of ZF. Thus, we assume that $\mathcal{M} \models \forall x_1 \dots \forall x_k \exists! y F[x_1, \dots, x_k, y]$

($\exists! y F[y]$ is the conjunction of $\forall y \forall y' (F[y], F[y'] \rightarrow y = y')$ and $\exists y F[y]$).

The axiom schemes of ZF_ε , written in the extended language, are still realized in the model \mathcal{N} , because the above proofs remain valid.

On the other hand, in order to make sure that the axiom schemes of ZF, which use a k -ary function symbol f , are still realized, one must check that this symbol is *compatible with* \simeq , i.e. that the following formula is realized in \mathcal{N} :

$\forall x_1 \dots \forall x_k (x_1 \simeq y_1, \dots, x_k \simeq y_k \rightarrow f x_1 \dots x_k \simeq f y_1 \dots y_k)$.

We now add a new rule to build formulas of ZF_ε :

If t, u are two terms and F is a formula of ZF_ε , then $t = u \leftrightarrow F$ is a formula of ZF_ε .

The formula $t = u \leftrightarrow \perp$ is denoted $t \neq u$.

The formula $t \neq u \rightarrow \perp$, i.e. $(t = u \leftrightarrow \perp) \rightarrow \perp$ is denoted $t = u$.

The truth value of these new formulas is defined as follows, assuming that t, u, F are closed, with parameters in \mathcal{N} :

$\llbracket t = u \leftrightarrow F \rrbracket = \emptyset$ if $t \neq u$; $\llbracket t = u \leftrightarrow F \rrbracket = \llbracket F \rrbracket$ if $t = u$.

It follows that :

$\llbracket t \neq u \rrbracket = \emptyset = \llbracket \top \rrbracket$ if $t \neq u$; $\llbracket t \neq u \rrbracket = \Pi = \llbracket \perp \rrbracket$ if $t = u$;

$\llbracket t = u \rrbracket = \llbracket \top \rightarrow \perp \rrbracket$ if $t \neq u$; $\llbracket t = u \rrbracket = \llbracket \perp \rightarrow \perp \rrbracket$ if $t = u$.

Proposition 12 shows that $t = u \leftrightarrow F$ and $t = u \rightarrow F$ are interchangeable.

Proposition 12.

i) $\lambda x(x)I \Vdash (t = u \rightarrow F) \rightarrow (t = u \leftrightarrow F)$;

ii) $\lambda x \lambda y (\text{cc}) \lambda k(y) (k)x \Vdash (t = u \leftrightarrow F), t = u \rightarrow F$.

i) Let $\xi \Vdash t = u \rightarrow F$ and $\pi \in \llbracket t = u \leftrightarrow F \rrbracket$. Thus, we have $t = u$ and $\pi \in \llbracket F \rrbracket$.

We must show $\lambda x(x)I \star \xi \bullet \pi \in \perp$, that is $\xi \star I \bullet \pi \in \perp$. This is immediate, by hypothesis on ξ , since $I \Vdash t = u$.

ii) Let $\xi \Vdash t = u \leftrightarrow F$, $\eta \Vdash t = u$ and $\pi \in \llbracket F \rrbracket$. We must show that :

$\lambda x \lambda y (\text{cc}) \lambda k(y) (k)x \star \xi \bullet \eta \bullet \pi \in \perp$, soit $\eta \star k_\pi \xi \bullet \pi \in \perp$.

If $t \neq u$, then $\eta \Vdash \top \rightarrow \perp$, hence the result.

If $t = u$, then $\xi \Vdash F$, thus $\xi \star \pi \in \perp$, therefore $k_\pi \xi \Vdash \perp$.

But we have $\eta \Vdash \perp \rightarrow \perp$, and therefore $\eta \star k_\pi \xi \bullet \pi \in \perp$.

Q.E.D.

Proposition 13 shows that the formulas $t = u$ and $\forall x (u \neq x \rightarrow t \neq x)$ (*Leibniz equality*) are interchangeable.

In particular, $a \notin \mathbb{J}E$ is identical with $a \in \mathbb{J}E \leftrightarrow \perp$ that is $1_E(a) \neq 1$.

We shall write $\forall x^{\mathbb{J}E} A[x]$ for $\forall x(x \in \mathbb{J}E \leftrightarrow A[x])$.

Proposition 12 shows that $x \in \mathbb{J}E \leftrightarrow A$ and $x \in \mathbb{J}E \rightarrow A$ are interchangeable.

Therefore $\forall x^{\mathbb{J}E} A[x]$ and $\forall x(x \in \mathbb{J}E \rightarrow A[x])$ are also interchangeable. We have :

$$\|\forall x^{\mathbb{J}E} A[x]\| = \bigcup_{a \in E} \|A[a/x]\| \quad \text{and} \quad |\forall x^{\mathbb{J}E} A[x]| = \bigcap_{a \in E} |A[a/x]|.$$

As already said, we shall add to the language of ZF_ε , some function symbols of any arity, which will be interpreted in the ground model \mathcal{M} by some functional relations. Then every formula of the form $\forall \vec{x}(t_1[\vec{x}] = u_1[\vec{x}], \dots, t_k[\vec{x}] = u_k[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$ which is satisfied in the model \mathcal{M} , is *realized* in the model \mathcal{N} ($t_1, u_1, \dots, t_k, u_k, t, u$ are terms of the language).

Indeed, we verify immediately that :

$$I \Vdash \forall \vec{x}(t_1[\vec{x}] = u_1[\vec{x}] \leftrightarrow (\dots \leftrightarrow (t_k[\vec{x}] = u_k[\vec{x}] \leftrightarrow t[\vec{x}] = u[\vec{x}])) \dots).$$

It follows that if, for instance, $t[x_0, x_1]$ sends $E_0 \times E_1$ into D in the model \mathcal{M} , then it sends $\mathbb{J}E_0 \times \mathbb{J}E_1$ into $\mathbb{J}D$ in the model \mathcal{N} . Indeed, we have then :

$\mathcal{M} \models \forall x_0 \forall x_1 (1_{E_0}(x_0) = 1, 1_{E_1}(x_1) = 1 \rightarrow 1_D(t[x_0, x_1]) = 1)$ and therefore, we have :

$$I \Vdash \forall x_0 \forall x_1 (1_{E_0}(x_0) = 1 \leftrightarrow (1_{E_1}(x_1) = 1 \leftrightarrow 1_D(t[x_0, x_1]) = 1)), \text{ in other words :}$$

$$I \Vdash \forall x_0^{\mathbb{J}E_0} \forall x_1^{\mathbb{J}E_1} (t[x_0, x_1] \varepsilon \mathbb{J}D).$$

Notice, in particular, that the characteristic function 1_E , which takes its values in the set $\mathbf{2} = \{0, 1\}$ in the model \mathcal{M} , sends $\mathbb{J}E$ into $\mathbb{J}\mathbf{2}$ in the realizability model \mathcal{N} .

We shall denote \wedge, \vee, \neg the (trivial) Boolean algebra operations in $\{0, 1\}$ (they should not be confused with the logical connectives \wedge, \vee, \neg). In this way, we have defined three function symbols of the language of ZF_ε ; thus, in the realizability model \mathcal{N} , they define a *Boolean algebra structure* on the set $\mathbb{J}\mathbf{2}$.

Remarks.

i) A set of the form $\mathbb{J}E$ behaves somewhat like a *type*, in the sense of computer science, because any function of the model \mathcal{M} with domain (resp. range) $E_1 \times \dots \times E_k$ becomes a function of the model \mathcal{N} with domain (resp. range) $\mathbb{J}E_1 \times \dots \times \mathbb{J}E_k$.

ii) The Boolean algebra $\mathbb{J}\mathbf{2}$ is, in general, non trivial i.e. it has ε -elements $\neq 0, 1$. Notice that they are all empty : indeed, it is easy to check that $I \Vdash \forall x^{\mathbb{J}\mathbf{2}} \forall y (x \neq 1 \rightarrow y \notin x)$.

The set $\tilde{\mathbb{N}}$ of integers in \mathcal{N}

We add to the language of ZF_ε a constant symbol 0 and a unary function symbol s . Their interpretation in the model \mathcal{M} is as follows :

0 is \emptyset ; $s(a)$ is $\{a\} \times \Pi$ for every set a , in other words $s(a) = \mathbb{J}(\{a\})$.

In the realizability model \mathcal{N} , $s(a)$ is the singleton of a . Indeed, we have trivially :

$\|b \notin s(a)\| = \|b \neq a\|$ (i.e. \emptyset if $a \neq b$ and Π if $a = b$) and it follows that :

$$I \Vdash \forall x \forall y (y \notin sx \rightarrow x \neq y) ; I \Vdash \forall x \forall y (x \neq y \rightarrow y \notin sx).$$

For each $n \in \mathbb{N}$, the term $s^n 0$ will also be written n .

Remark. In the definition of the set of integers in the realizability model \mathcal{N} , we prefer to use the singleton as the successor function s , instead of the usual one $x \mapsto x \cup \{x\}$, which is more complicated to define. It would give : $s(a) = \{(a, K \bullet \pi) ; \pi \in \Pi\} \cup \{(x, \underline{0} \bullet \pi) ; (x, \pi) \in a\}$.

Theorem 16. *The following formulas are realized in \mathcal{N} :*

- i) $\forall x \forall y (sx = sy \leftrightarrow x = y)$;
- ii) $\forall x (sx \neq 0)$;
- iii) $\forall x \forall y (x \simeq y \rightarrow sx \simeq sy)$;
- iv) $\forall x \forall y (sx \simeq sy \rightarrow x \simeq y)$.

This shows, in particular, that the function s is *compatible with the extensional equivalence* \simeq .

i) We check that $I \Vdash sa = sb \leftrightarrow a = b$. We may suppose $sa = sb$, because

$\|sa = sb \leftrightarrow a = b\| = \emptyset$ if $sa \neq sb$. But, in this case, we have $a = b$, by definition of sa, sb .

ii) We have $\|a \notin 0\| = \|\forall x (x \simeq a \rightarrow x \notin 0)\| = \emptyset$, since $\|x \notin 0\| = \emptyset$. Now $\|a \notin sa\| = \Pi$ and therefore we have, for any $\xi \in \Lambda$, $\lambda x (x) \xi \Vdash (a \notin \emptyset \rightarrow a \notin sa) \rightarrow \perp$; thus :

$\lambda x (x) \xi \Vdash \forall x (x \notin \emptyset \rightarrow x \notin sa) \rightarrow \perp$. But this means exactly that $\lambda x (x) \xi \Vdash sa \subseteq 0 \rightarrow \perp$, and therefore $\lambda x \lambda y (x) \xi \Vdash sa \simeq 0 \rightarrow \perp$.

iii) We show that the formula $a \simeq b \rightarrow sa \simeq sb$ is realized ; it suffices to realize the formula $a \simeq b \rightarrow sa \subseteq sb$. We prove it by means of already realized sentences.

We need to prove $a \simeq b, x \notin sb \rightarrow x \notin sa$. But $x \notin sa$ has the same truth value as $x \neq a$. Thus, we simply have to prove $a \simeq b \rightarrow a \in sb$. But $a \in sb$ follows from $b \in sb$ and $a \simeq b$.

iv) In the same way, we prove the formula $sa \simeq sb \rightarrow a \simeq b$ and, in fact $sa \subseteq sb \rightarrow a \simeq b$.

The formula $sa \subseteq sb$ is $\forall x (x \notin sb \rightarrow x \notin sa)$; but $x \notin sa$ is the same as $x \neq a$. Thus, from $sa \subseteq sb$ we obtain $a \in sb$, i.e. $(\exists x \varepsilon sb) x \simeq a$. But $x \varepsilon sb$ is the same as $x = b$, so that we obtain $a \simeq b$.

Q.E.D.

The individuals $s^n 0$ are obviously distinct, for $n \in \mathbb{N}$. Therefore, we can define :

$$\tilde{\mathbb{N}} = \{(s^n 0, \underline{n} \bullet \pi) ; n \in \mathbb{N}, \pi \in \Pi\}$$

and we have :

$\|a \notin \tilde{\mathbb{N}}\| = \emptyset$ if a is not of the form $s^n 0$, with $n \in \mathbb{N}$;

$\|s^n 0 \notin \tilde{\mathbb{N}}\| = \{\underline{n} \bullet \pi ; \pi \in \Pi\}$.

The formula $x \varepsilon \tilde{\mathbb{N}}$ will also be written $\text{ent}(x)$.

In the sequel, we shall use the restricted quantifier $\forall x^{\tilde{\mathbb{N}}}$, which we also write $\forall x^{\text{ent}}$, with the following meaning :

$\|\forall x^{\text{ent}} F[x]\| = \|\forall x^{\tilde{\mathbb{N}}} F[x]\| = \{\underline{n} \bullet \pi ; n \in \mathbb{N}, \pi \in \|F[s^n 0]\|\}$.

The restricted existential quantifier $\exists x^{\tilde{\mathbb{N}}}$ or $\exists x^{\text{ent}}$ is defined as :

$\exists x^{\text{ent}} F[x] \equiv \exists x^{\tilde{\mathbb{N}}} F[x] \equiv \neg \forall x^{\text{ent}} \neg F[x]$.

Proposition 17 shows that these quantifiers have indeed the intended meaning : the formulas $\forall x^{\text{ent}} F[x]$ and $\forall x (x \varepsilon \tilde{\mathbb{N}} \rightarrow F[x])$ are interchangeable.

Proposition 17.

i) $\lambda x \lambda y \lambda z (y)(x)z \Vdash \forall x^{\text{ent}} F[x] \rightarrow \forall x (\neg F[x] \rightarrow x \notin \tilde{\mathbb{N}})$;

ii) $\lambda x \lambda y (\text{cc}) \lambda k (x) k y \Vdash \forall x (\neg F[x] \rightarrow x \notin \tilde{\mathbb{N}}) \rightarrow \forall x^{\text{ent}} F[x]$.

i) Let $\xi \Vdash \forall x^{\text{ent}} F[x]$, $\eta \Vdash \neg F[a]$ and $\omega \in \|a \notin \tilde{\mathbb{N}}\|$. Thus, we have $a = s^n 0$ for some $n \in \mathbb{N}$ (since $\|a \notin \tilde{\mathbb{N}}\| \neq \emptyset$) and $\omega = \underline{n} \bullet \pi$. We must show that $\eta \star \xi \underline{n} \bullet \pi \in \perp$.

Now, by hypothesis on ξ , we have $\xi \star \underline{n} \bullet \rho \in \perp$ for any $\rho \in \|F[s^n 0]\|$; that is $\xi \underline{n} \Vdash F[s^n 0]$.

Since $\eta \Vdash \neg F[s^n 0]$, we have $\eta \star \xi \underline{n} \bullet \pi \in \perp$, which is the desired result.

ii) Let $\xi \Vdash \forall x(\neg F[x] \rightarrow x \notin \tilde{\mathbb{N}})$ and $\underline{n} \cdot \pi \in \|\forall x^{\text{ent}} F[x]\|$, with $n \in \mathbb{N}$ and $\pi \in \|F[s^n 0]\|$.
We have : $\lambda x \lambda y (\text{cc}) \lambda k(x) k y \star \xi \cdot \underline{n} \cdot \pi > \xi \star k_\pi \cdot \underline{n} \cdot \pi$.

Now, we have $k_\pi \Vdash \neg F[s^n 0]$ and $\underline{n} \cdot \pi \in \|s^n 0 \notin \tilde{\mathbb{N}}\|$. Therefore $\xi \star k_\pi \cdot \underline{n} \cdot \pi \in \perp$.

Q.E.D.

Theorem 18 (Recurrence scheme). *For every formula $F[\vec{x}, y]$:*

i) $I \Vdash \forall \vec{x} \forall n^{\tilde{\mathbb{N}}} (\forall y (F[\vec{x}, sy] \rightarrow F[\vec{x}, y]), F[\vec{x}, n] \rightarrow F[\vec{x}, 0])$.

ii) $I \Vdash \forall \vec{x} \forall n^{\tilde{\mathbb{N}}} (\forall y (F[\vec{x}, y] \rightarrow F[\vec{x}, sy]), F[\vec{x}, 0] \rightarrow F[\vec{x}, n])$.

i) Let $n \in \mathbb{N}$, \vec{a} a sequence of individuals, $\xi \Vdash \forall y (F[\vec{a}, sy] \rightarrow F[\vec{a}, y])$, $\pi \in \|F[\vec{a}, 0]\|$.

We must show that, for every $\alpha \Vdash F[\vec{a}, n]$, we have $I \star \underline{n} \cdot \xi \cdot \alpha \cdot \pi \in \perp$.

In fact, we show, by recurrence on n , that $\underline{n} \star \xi \cdot \alpha \cdot \pi \in \perp$.

This is immediate if $n = 0$. In order to go from n to $n + 1$, we suppose now $\alpha \Vdash F[\vec{a}, sn]$;

we have $\underline{n+1} \star \xi \cdot \alpha \cdot \pi > \sigma \underline{n} \star \xi \cdot \alpha \cdot \pi > \sigma \star \underline{n} \cdot \xi \cdot \alpha \cdot \pi > \underline{n} \star \xi \cdot \alpha \cdot \pi$.

But, by hypothesis on ξ , we have $\xi \Vdash F[\vec{a}, sn] \rightarrow F[\vec{a}, n]$; thus $\xi \alpha \Vdash F[\vec{a}, n]$.

Hence the result, by the recurrence hypothesis.

ii) Let $n \in \mathbb{N}$, \vec{a} a sequence of individuals, $\xi \Vdash \forall y (F[\vec{a}, y] \rightarrow F[\vec{a}, sy])$, $\alpha \Vdash F[\vec{a}, 0]$ and $\pi \in \|F[\vec{a}, 0]\|$. We must show that $I \star \underline{n} \cdot \xi \cdot \alpha \cdot \pi \in \perp$; this follows from lemma 19, with $k = 0$.

Q.E.D.

Lemma 19. *Let $n, k \in \mathbb{N}$, $\xi \Vdash \forall y (F[y] \rightarrow F[sy])$, $\alpha \Vdash F[s^k 0]$ and $\pi \in \|F[s^k n]\|$.*

Then $\underline{n} \star \xi \cdot \alpha \cdot \pi \in \perp$.

The proof is done for all integers k , by recurrence on n . This is immediate if $n = 0$.

In order to go from n to $n + 1$, we suppose now $\pi \in \|F[s^k(n+1)]\|$, i.e. $\pi \in \|F[s^{k+1} n]\|$.

We have $\underline{n+1} \star \xi \cdot \alpha \cdot \pi > \sigma \underline{n} \star \xi \cdot \alpha \cdot \pi > \sigma \star \underline{n} \cdot \xi \cdot \alpha \cdot \pi > \underline{n} \star \xi \cdot \alpha \cdot \pi$.

But, by hypothesis on ξ , we have $\xi \Vdash F[s^k 0] \rightarrow F[s^{k+1} 0]$; thus $\xi \alpha \Vdash F[s^{k+1} 0]$.

Hence the result, by the recurrence hypothesis.

Q.E.D.

Definition. We denote by $\text{int}(n)$ the formula $\forall x (\forall y (sy \notin x \rightarrow y \notin x), n \notin x \rightarrow 0 \notin x)$.

Theorem 21 shows that the formulas $\text{int}(n)$ and $n \varepsilon \tilde{\mathbb{N}}$ are interchangeable, i.e. the formula $\forall n (\text{int}(n) \leftrightarrow n \varepsilon \tilde{\mathbb{N}})$ is realized by a proof-like term : this is the *storage theorem for integers*.

Lemma 20. $\lambda g \lambda x (g)(\sigma)x \Vdash \forall y (sy \notin \tilde{\mathbb{N}} \rightarrow y \notin \tilde{\mathbb{N}})$.

We show that $\lambda g \lambda x (g)(\sigma)x \Vdash sb \notin \tilde{\mathbb{N}} \rightarrow b \notin \tilde{\mathbb{N}}$ for every individual b .

This is obvious if b is not of the form $s^n 0$, since then $\|b \notin \tilde{\mathbb{N}}\| = \emptyset$. Thus, it remains to show :

$\lambda g \lambda x (g)(\sigma)x \Vdash s^{n+1} 0 \notin \tilde{\mathbb{N}} \rightarrow s^n 0 \notin \tilde{\mathbb{N}}$. Thus, let $\xi \Vdash s^{n+1} 0 \notin \tilde{\mathbb{N}}$; we must show :

$\lambda g \lambda x (g)(\sigma)x \star \xi \cdot \underline{n} \cdot \pi \in \perp$, i.e. $\xi \star \sigma \underline{n} \cdot \pi \in \perp$, which is clear, since $\sigma \underline{n} = \underline{n+1}$.

Q.E.D.

Theorem 21 (Storage theorem).

i) $I \Vdash \forall x^{\tilde{\mathbb{N}}} \text{int}(x)$.

ii) $T \Vdash \forall x (\text{int}(x), x \notin \tilde{\mathbb{N}} \rightarrow \perp)$ with $T = \lambda n \lambda f ((n) \lambda g \lambda x (g)(\sigma)x) f 0$.

i) It is theorem 18(i), if we take for $F[x, y]$ the formula $y \neq x$.

ii) Let $v \Vdash \text{int}(a)$, $\phi \Vdash a \neq \tilde{\mathbb{N}}$ and $\pi \in \Pi$. We must show $T \star v \bullet \phi \bullet \pi \in \perp$, that is :
 $v \star \lambda g \lambda x (g)(\sigma) x \bullet \phi \bullet \underline{0} \bullet \pi \in \perp$.

By hypothesis, we have $v \Vdash \forall y (s y \neq \tilde{\mathbb{N}} \rightarrow y \neq \tilde{\mathbb{N}})$, $a \neq \tilde{\mathbb{N}} \rightarrow 0 \neq \tilde{\mathbb{N}}$.

But we have $\underline{0} \bullet \pi \in \parallel 0 \neq \tilde{\mathbb{N}} \parallel$ by definition of $\tilde{\mathbb{N}}$ and, by lemma 20 :
 $\lambda g \lambda x (g)(\sigma) x \Vdash \forall y (s y \neq \tilde{\mathbb{N}} \rightarrow y \neq \tilde{\mathbb{N}})$. Hence the result.

Q.E.D.

From theorem 18(ii), it follows immediately that the *recurrence scheme of ZF* is realized in \mathcal{N} ; it is the scheme :

$\forall \vec{x} (\forall y (F[\vec{x}, y] \rightarrow F[\vec{x}, s y]), F[\vec{x}, 0] \rightarrow (\forall n \in \tilde{\mathbb{N}}) F[\vec{x}, n])$ for every formula $F[\vec{x}, y]$ of ZF (i.e. written with $\neq, \subseteq, 0, s$).

Then, indeed, the formula F is compatible with the extensional equivalence \simeq .

Since the function s is compatible with \simeq , we deduce from lemma 20 that the formula :
 $\forall y (y \in \tilde{\mathbb{N}} \rightarrow s y \in \tilde{\mathbb{N}})$ is realized in \mathcal{N} ; the formula $0 \in \tilde{\mathbb{N}}$ is also obviously realized.

From the recurrence scheme just proved, we deduce that :

$\tilde{\mathbb{N}}$ is the set of integers of the model \mathcal{N} , considered as a model of ZF.

Theorem 22.

i) Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a recursive function. Then, the formula :

$\forall x_1^{\tilde{\mathbb{N}}} \dots \forall x_k^{\tilde{\mathbb{N}}} (f(x_1, \dots, x_k) \in \tilde{\mathbb{N}})$ is realized in \mathcal{N} .

ii) Let $g : \mathbb{N}^k \rightarrow 2$ be a recursive function. Then, the formula :

$\forall x_1^{\tilde{\mathbb{N}}} \dots \forall x_k^{\tilde{\mathbb{N}}} (g(x_1, \dots, x_k) = 1 \vee g(x_1, \dots, x_k) = 0)$ is realized in \mathcal{N} .

i) This can be written $\forall x_1^{\text{ent}} \dots \forall x_k^{\text{ent}} \text{ent}(f(x_1, \dots, x_k))$. The proof is done in [18, 15].

ii) We have $\mathcal{N} \Vdash (\forall x_1 \in \mathbb{J}\mathbb{N}) \dots (\forall x_k \in \mathbb{J}\mathbb{N}) g(x_1, \dots, x_k) \in \mathbb{J}2$.

Now, since g is recursive, we have, by (i) :

$\mathcal{N} \Vdash (\forall x_1 \in \tilde{\mathbb{N}}) \dots (\forall x_k \in \tilde{\mathbb{N}}) g(x_1, \dots, x_k) \in \tilde{\mathbb{N}}$.

Hence the result, by lemma 23.

Q.E.D.

Lemma 23. $\lambda x \lambda y \lambda f (f) x y \Vdash \forall x^{\mathbb{J}2} (x \neq 1, x \neq 0 \rightarrow x \neq \tilde{\mathbb{N}})$.

We have to show :

$\lambda x \lambda y \lambda f (f) x y \Vdash \top, \perp \rightarrow 0 \neq \tilde{\mathbb{N}}$ and $\lambda x \lambda y \lambda f (f) f x y \Vdash \perp, \top \rightarrow 1 \neq \tilde{\mathbb{N}}$.

Thus let $\xi \Vdash \top$ (i.e. $\xi \in \Lambda$ arbitrary) and $\eta \Vdash \perp$. We have to show :

$\lambda x \lambda y \lambda f (f) x y \star \xi \bullet \eta \bullet \underline{0} \bullet \pi \in \perp$ and $\lambda x \lambda y \lambda f (f) x y \star \eta \bullet \xi \bullet \underline{1} \bullet \pi \in \perp$

which is trivial.

Q.E.D.

Remarks. i) In the present paper, theorem 22 is used only in trivial particular cases.

ii) Let us recall the difference between $\mathbb{J}\mathbb{N}$ and $\tilde{\mathbb{N}}$ (the set of integers in the model \mathcal{N}) ; we have :

$\xi \Vdash \forall x^{\mathbb{J}\mathbb{N}} F[x]$ iff $(\forall n \in \mathbb{N}) (\forall \pi \in \parallel F[s^n 0] \parallel) \xi \star \pi \in \perp$.

$\xi \Vdash \forall x^{\tilde{\mathbb{N}}} F[x]$ iff $(\forall n \in \mathbb{N}) (\forall \pi \in \parallel F[s^n 0] \parallel) \xi \star \underline{n} \bullet \pi \in \perp$.

Notice that we have $K \Vdash \forall x (x \neq \mathbb{J}\mathbb{N} \rightarrow x \neq \tilde{\mathbb{N}})$, in other words $K \Vdash \tilde{\mathbb{N}} \subset \mathbb{J}\mathbb{N}$. This means that, in \mathcal{N} , the set $\tilde{\mathbb{N}}$ of integers is strongly included in $\mathbb{J}\mathbb{N}$. In the particular realizability model considered below (and, in fact, in every non trivial realizability model), the formula $\mathbb{J}\mathbb{N} \not\subseteq \tilde{\mathbb{N}}$ is realized.

Non extensional and dependent choice

For each formula $F(x, y_1, \dots, y_m)$ of ZF_ε , we add a function symbol f_F of arity $m + 1$, with the

axiom : $\forall \vec{y} (\forall k \overset{\sim}{\mathbb{N}} F[f_F(k, \vec{y}), \vec{y}] \rightarrow \forall x F[x, \vec{y}])$

or else : $\forall \vec{y} (\forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}] \rightarrow \forall x F[x, \vec{y}])$.

It is the *axiom scheme of non extensional choice*, in abbreviated form NEAC.

Remarks. i) The axiom scheme NEAC does not imply the axiom of choice in ZF, because we do not suppose that the symbol f_F is compatible with the extensional equivalence \approx . It is the reason why we speak about *non extensional* axiom of choice. On the other hand, as we show below, it implies DC (the axiom of dependent choice).

ii) It seems that we could take for f_F a m -ary function symbol and use the following simpler (and logically equivalent) axiom scheme NEAC' : $\forall \vec{y} (F[f_F(\vec{y}), \vec{y}] \rightarrow \forall x F[x, \vec{y}])$.

But this axiom scheme cannot be realized, even though the axiom scheme NEAC is realized by a very simple proof-like term (theorem 24), *provided the instruction ζ is present*.

More precisely, we can define a function f_F in \mathcal{M} , such that NEAC is realized in \mathcal{N} , but this is impossible for NEAC'.

Theorem 24 (NEAC).

For each closed formula $\forall x \forall \vec{y} F$, we can define a $(m + 1)$ -ary function symbol f_F such that : $\lambda x(\zeta)xx \Vdash \forall \vec{y} (\forall k^{\text{ent}} F[f_F(k, \vec{y})/x, \vec{y}] \rightarrow \forall x F[x, \vec{y}])$.

For each $k \in \mathbb{N}$ we put $P_k = \{\pi \in \Pi; \xi \star \underline{k} \bullet \pi \notin \perp, k = n_\xi\}$.

For each individual x , we have : $\|\forall x F[x, \vec{y}]\| = \bigcup_a \|F[a, \vec{y}]\|$.

Thus, there exists a function f_F such that, given $k \in \mathbb{N}$ and \vec{y} such that $P_k \cap \|\forall x F[x, \vec{y}]\| \neq \emptyset$, we have $P_k \cap \|F[f_F(k, \vec{y}), \vec{y}]\| \neq \emptyset$.

Now, we want to show $\lambda x(\zeta)xx \Vdash \forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}] \rightarrow F[x, \vec{y}]$, for every individuals x, \vec{y} .

Thus, let $\xi \Vdash \forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}]$ and $\pi \in \|F[a, \vec{y}]\|$; we must show $\lambda x(\zeta)xx \star \xi \bullet \pi \in \perp$.

If this is false, we have $\zeta \star \xi \bullet \xi \bullet \pi \notin \perp$ and therefore $\xi \star \underline{j} \bullet \pi \notin \perp$ with $j = n_\xi$.

It follows that $\pi \in P_j \cap \|F[a, \vec{y}]\|$; thus, there exists $\pi' \in \bar{P}_j \cap \|F[f_F(j, \vec{y}), \vec{y}]\|$.

Now, we have $\underline{j} \bullet \pi' \in \|\forall k^{\text{ent}} F[f_F(k, \vec{y}), \vec{y}]\|$, and therefore, by hypothesis on ξ , we have :

$\xi \star \underline{j} \bullet \pi' \in \perp$. This is in contradiction with $\pi' \in P_j$.

$\overline{\text{Q.E.D.}}$

NEAC implies DC

Let us call DCS (*dependent choice scheme*) the following axiom scheme :

$\forall \vec{z} (\forall x \exists y F[x, y, \vec{z}] \rightarrow \forall n^{\text{ent}} \exists ! y S_F[n, y, \vec{z}] \wedge \forall n^{\text{ent}} \exists y \exists y' \{S_F[n, y, \vec{z}], S_F[sn, y', \vec{z}], F[y, y', \vec{z}]\})$.

where F is a formula of ZF_ε with free variables x, y, \vec{z} ; the formula S_F is written below.

In the following, we omit the variables \vec{z} (the parameters), for sake of simplicity.

The usual axiom of dependent choice DC is obtained by taking for $F[x, y, z_0, z_1]$ the formula $y \varepsilon z_0 \wedge (x \varepsilon z_0 \rightarrow \langle x, y \rangle \varepsilon z_1)$.

We now show how to define the formula S_F , so that $ZF_\varepsilon, \text{NEAC} \vdash \text{DCS}$; we shall conclude that DC is realized.

So, let us assume $\forall x \exists y F[x, y]$. By NEAC, there is a function symbol f such that :

$\forall x \exists k^{\text{ent}} F[x, f(k, x)]$. We define the formula $R_F[x, y]$ as follows :

$R_F[x, y] \equiv \exists k^{\text{ent}} \{F[x, f(k, x)], \forall i^{\text{ent}} (i < k \rightarrow \neg F[x, f(i, x)]), y = f(k, x)\}$.

This means : “ $y = f(k, x)$ for the first integer k such that $F[x, f(k, x)]$ ”.

Therefore, R_F is functional, i.e. we have $\forall x \exists! y R_F(x, y)$.

S_F is defined so as to represent a sequence obtained by iteration of the function given by R_F , beginning (arbitrarily) at 0 :

$S_F(n, x) \equiv \forall z [\forall m \forall y \forall y' (< m, y > \varepsilon z, R_F(y, y') \rightarrow < sm, y' > \varepsilon z), < 0, 0 > \varepsilon z \rightarrow < n, x > \varepsilon z]$.

It should be clear that, with this definition of S_F , we obtain :

$\forall n^{\text{ent}} \exists! y S_F[n, y]$ and $\forall n^{\text{ent}} \exists y \exists y' \{S_F[n, y], S_F[sn, y'], F[y, y']\}$.

Thus, DCS is provable from ZF_ε and NEAC.

Remark. We have used the binary function symbol $\langle x, y \rangle$ which is defined, in the ground model \mathcal{M} , in the usual way : $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$. The formulas $\forall x \forall x' \forall y \forall y' (\langle x, y \rangle = \langle x', y' \rangle \leftrightarrow x = x')$, $\forall x \forall x' \forall y \forall y' (\langle x, y \rangle = \langle x', y' \rangle \leftrightarrow y = y')$, are trivially realized by I .

Properties of the Boolean algebra \mathfrak{J}

Let $(x < y)$ be the binary recursive function defined as follows in \mathcal{M} :

$(m < n) = 1$ if $m, n \in \mathbb{N}$, $m < n$; else $(m < n) = 0$.

Theorem 25. For every choice of \perp , the relation $(x < y) = 1$ is, in \mathcal{N} , a strict well founded partial order, which is the usual order on integers (i.e. on $\tilde{\mathbb{N}}$).

Indeed, the formulas : $\forall x ((x < x) \neq 1)$ and $\forall x \forall y \forall z ((x < y) = 1 \leftrightarrow ((y < z) = 1 \leftrightarrow (x < z) = 1))$ are trivially realized.

Moreover, since the relation $(x < y) = 1$ is well founded, we have (theorem 15) :

$\mathcal{Y} \Vdash \forall x (\forall y ((y < x) = 1 \leftrightarrow F[y]) \rightarrow F[x]) \rightarrow \forall x F[x]$

for every formula $F[x]$ with parameters and one free variable.

By theorem 22(ii), the binary recursive function $(x < y)$ sends $\tilde{\mathbb{N}}^2$ into $\{0, 1\}$, in the model \mathcal{N} .

Therefore, it suffices to check that the following formulas are realized in \mathcal{N} :

$\forall x^{\tilde{\mathbb{N}}} \forall y^{\tilde{\mathbb{N}}} (y \leq x \rightarrow (x < y) \neq 1)$; $\forall x^{\tilde{\mathbb{N}}} \forall y^{\tilde{\mathbb{N}}} (x < y \rightarrow (x < y) = 1)$.

Now the following formulas are trivially realized :

$\forall x^{\mathfrak{J}\mathbb{N}} \forall y^{\mathfrak{J}\mathbb{N}} \forall z^{\mathfrak{J}\mathbb{N}} (x = y + z \rightarrow (x < y) \neq 1)$; $\forall x^{\mathfrak{J}\mathbb{N}} \forall y^{\mathfrak{J}\mathbb{N}} \forall z^{\mathfrak{J}\mathbb{N}} (y = x + z + 1 \rightarrow (x < y) = 1)$.

Q.E.D.

In the ground model \mathcal{M} , we put, for each integer n :

$$\mathbf{n} = \{0, 1, \dots, n-1\} = \{0, s0, \dots, s^{n-1}0\}.$$

The functions $n \mapsto \mathbf{n}$ and $n \mapsto \mathfrak{J}\mathbf{n}$ are defined in the realizability model \mathcal{N} , with domain $\mathfrak{J}\mathbb{N}$.

Theorem 26.

The following formulas are realized in \mathcal{N} :

- i) $\forall x^{\mathfrak{J}\mathbb{N}} \forall m^{\mathfrak{J}\mathbb{N}} ((x < m) = 1 \leftrightarrow x \varepsilon \mathfrak{J}\mathbf{m})$;
- ii) $\forall m^{\mathfrak{J}\mathbb{N}} \forall n^{\mathfrak{J}\mathbb{N}} ((m < n) = 1 \rightarrow \mathfrak{J}\mathbf{m} \subset \mathfrak{J}\mathbf{n})$;
- iii) $\forall x^{\mathfrak{J}\mathbb{N}} \forall m^{\mathfrak{J}\mathbb{N}} ((x < m) = 1 \leftrightarrow \exists y^{\mathfrak{J}\mathbb{N}} (m = x + y + 1))$.

Remember that $x < y$ is the formula $\forall z (z \notin y \rightarrow z \notin x)$.

- i) We have trivially $\|(a < m) \neq 1\| = \|a \notin \mathbb{I}m\|$ for every $a, m \in \mathbb{N}$.
- ii) By transitivity of the relation $(m < n) = 1$ (theorem 25).
- iii) We observe that $\|(a < m) \neq 1\| = \|(\forall y \varepsilon \mathbb{I}N)(m \neq a + y + 1)\|$ for every $a, m \in \mathbb{N}$.

Q.E.D.

For each $n \varepsilon \mathbb{I}N$ (and, in particular, for each $n \varepsilon \tilde{\mathbb{N}}$, i.e. for each integer of \mathcal{N}), the set defined, in \mathcal{N} , by $(x < n) = 1$ (the strict initial segment defined by n) is therefore extensionally equivalent to $\mathbb{I}n$.

Theorem 27. *In \mathcal{N} , the application $(x, y) \mapsto my + x$ is a bijection from $\mathbb{I}m \times \mathbb{I}n$ onto $\mathbb{I}(mn)$. Indeed, the following formulas are realized in \mathcal{N} by I :*

- i) $\forall m \mathbb{I}N \forall n \mathbb{I}N \forall x \mathbb{I}m \forall y \mathbb{I}n ((my + x) \varepsilon \mathbb{I}mn)$;
- ii) $\forall m \mathbb{I}N \forall n \mathbb{I}N \forall x \mathbb{I}m \forall x' \mathbb{I}m \forall y \mathbb{I}n \forall y' \mathbb{I}n (my + x = my' + x' \leftrightarrow x = x')$;
 $\forall m \mathbb{I}N \forall n \mathbb{I}N \forall x \mathbb{I}m \forall x' \mathbb{I}m \forall y \mathbb{I}n \forall y' \mathbb{I}n (my + x = my' + x' \leftrightarrow y = y')$;
- iii) $\forall m \mathbb{I}N \forall n \mathbb{I}N \forall z \mathbb{I}mn \exists x \mathbb{I}m \exists y \mathbb{I}n (z = my + x)$.

i) and ii) We simply have to replace $\forall m \mathbb{I}N$ and $\forall x \mathbb{I}m$ with their definitions, which are :
 $\forall m \mathbb{I}N F \equiv \forall m (1_N(m) = 1 \leftrightarrow F)$; $\forall x \mathbb{I}m F \equiv \forall x ((x < m) = 1 \leftrightarrow F)$.

We see immediately that these two formulas are realized by I .

iii) We show that :

$$I \Vdash \forall m \mathbb{I}N \forall n \mathbb{I}N \forall z \mathbb{I}mn (\forall x \mathbb{I}m \forall y \mathbb{I}n ((x < m) = 1 \leftrightarrow ((y < n) = 1 \leftrightarrow z \neq my + x)) \rightarrow (z < mn) \neq 1).$$

Thus, we consider :

$$m, n, z_0 \in \mathbb{N} ; \xi \in \Lambda, \xi \Vdash \forall x \mathbb{I}m \forall y \mathbb{I}n ((x < m) = 1 \leftrightarrow ((y < n) = 1 \leftrightarrow z \neq my + x))$$

and $\pi \in \|(z_0 < mn) \neq 1\|$. We must show $I \star \xi \bullet \pi \in \perp$, that is $\xi \star \pi \in \perp$.

We have $\|(z_0 < mn) \neq 1\| \neq \emptyset$, therefore $z_0 < mn$. Thus, there exist $x_0, y_0 \in \mathbb{N}, x_0 < m, y_0 < n$ such that $z_0 = mx_0 + y_0$. Now, by hypothesis on ξ , we have :

$$\xi \Vdash (x_0 < m) = 1 \leftrightarrow ((y_0 < n) = 1 \leftrightarrow z_0 \neq my_0 + x_0), \text{ in other words } \xi \Vdash \perp.$$

Q.E.D.

Injection of $\mathbb{I}n$ into $\mathcal{P}(\tilde{\mathbb{N}})$

Remember that we have fixed a recursive bijection : $\xi \mapsto n_\xi$ from Λ onto \mathbb{N} . The inverse bijection will be denoted $n \mapsto \xi_n$.

This bijection is used in the execution rule of the instruction ζ , which is as follows :

$$\zeta \star \xi \bullet \eta \bullet \pi > \xi \star \underline{n}_\eta \bullet \pi.$$

We define, in \mathcal{M} , a function $\Delta : \mathbb{N} \rightarrow 2$ by putting $\Delta(n) = 0 \Leftrightarrow \xi_n \Vdash \perp$.

In this way, we have defined a function symbol Δ , in the language of ZF_ε . In the realizability model \mathcal{N} , the symbol Δ represents a function from $\mathbb{I}N$ into $\mathbb{I}2$. In particular, the function Δ sends the set $\tilde{\mathbb{N}}$ of integers of the model \mathcal{N} into the Boolean algebra $\mathbb{I}2$.

Theorem 28. *Let us put $\theta = \lambda x \lambda y (\zeta) yxx$; then, we have :*

$$\theta \Vdash \forall x \mathbb{I}2 (x \neq 0 \rightarrow \exists n^{ent} \{\Delta(n) \neq 0, \Delta(n) \leq x\})$$

where \leq is the order relation of the Boolean algebra $\mathbb{I}2$: $y \leq x$ is the formula $x = (y \vee x)$.

We must show $\theta \Vdash \forall x \mathbb{I}2 (x \neq 0, \forall n^{ent} (\Delta(n) \neq 0 \rightarrow x \neq \Delta(n) \vee x) \rightarrow \perp)$.

Thus, let $a \in \{0, 1\}$, $\xi \Vdash a \neq 0$, $\eta \Vdash \forall n^{ent} (\Delta(n) \neq 0 \rightarrow a \neq \Delta(n) \vee a)$ and $\pi \in \Pi$.

We must show $\theta \star \xi \bullet \eta \bullet \pi \in \perp$ that is $\zeta \star \eta \bullet \xi \bullet \xi \bullet \pi \in \perp$, or else $\eta \star \underline{n}_\xi \bullet \xi \bullet \pi \in \perp$.

By hypothesis on η , it suffices to show $\underline{n}_\xi \cdot \xi \cdot \pi \in \|\forall n^{\text{ent}}(\Delta(n) \neq 0 \rightarrow a \neq \Delta(n) \vee a)\|$, that is, by definition of the quantifier $\forall n^{\text{ent}}$: $\xi \cdot \pi \in \|\Delta(n_\xi) \neq 0 \rightarrow a \neq \Delta(n_\xi) \vee a\|$.

This amounts to show $\xi \Vdash \Delta(n_\xi) \neq 0$ and $a = \Delta(n_\xi) \vee a$.

- Proof of $\xi \Vdash \Delta(n_\xi) \neq 0$: if $\Delta(n_\xi) = 1$, this is trivial, because $\|\Delta(n_\xi) \neq 0\| = \emptyset$; if $\Delta(n_\xi) = 0$, then $\xi \Vdash \perp$, by definition of Δ .

- Proof of $a = \Delta(n_\xi) \vee a$: this is obvious if $a = 1$; if $a = 0$, then $\xi \Vdash \perp$, by hypothesis on ξ . Therefore $\Delta(n_\xi) = 0$ by definition of Δ , hence the result.

Q.E.D.

By theorem 28, the set $\{\Delta(n); n \in \tilde{\mathbb{N}}, \Delta(n) \neq 0\}$ is, in the realizability model \mathcal{N} , a countable dense subset of the Boolean algebra $\mathbb{J}2$: this means that each element $\neq 0$ of this Boolean algebra has a lower bound of the form $\Delta(n)$, with $n \in \tilde{\mathbb{N}}$ and $\Delta(n) \neq 0$.

It follows that the application of $\mathbb{J}2$ into $\mathcal{P}(\tilde{\mathbb{N}})$ given by:

$$x \longmapsto \{n \in \tilde{\mathbb{N}}; \Delta(n) \leq x, \Delta(n) \neq 0\}$$

is one to one: indeed, if $a, b \in \mathbb{J}2$ with $a \neq b$, then $a + b \neq 0$; thus, there exists an integer $n \in \tilde{\mathbb{N}}$ such that $\Delta(n) \neq 0$ and $\Delta(n) \leq a + b$. Therefore, we have $\Delta(n) \leq a$ iff $(b \wedge \Delta(n)) = 0$.

But, since $\Delta(n) \neq 0$, we get: $\Delta(n) \leq a$ iff $\Delta(n) \not\leq b$.

We have shown:

Theorem 29.

The formula: “there exists an injection of $\mathbb{J}2$ into $\mathcal{P}(\tilde{\mathbb{N}})$ ” is realized in the model \mathcal{N} .

Corollary 30. The formula: “for every integer n there exists an injection of $\mathbb{J}n$ into $\mathcal{P}(\tilde{\mathbb{N}})$ ” is realized in the model \mathcal{N} .

Using theorem 27 we see, by recurrence on m , that the model \mathcal{N} realizes the formula:

“ $\forall m \in \tilde{\mathbb{N}} ((\mathbb{J}2)^m \text{ is equipotent to } \mathbb{J}(2^m))$ ”; and therefore also the formula:

“ $\forall m \in \tilde{\mathbb{N}}$ (there exists an injection of $\mathbb{J}(2^m)$ into $\mathcal{P}(\tilde{\mathbb{N}})$)”.

Finally, by theorem 26(ii), we see that the following formula is realized:

“ $\forall n \in \tilde{\mathbb{N}}$ (there exists an injection of $\mathbb{J}n$ into $\mathcal{P}(\tilde{\mathbb{N}})$)”.

Q.E.D.

Realizability models in which \mathbb{R} is not well ordered

$\mathbb{J}2$ atomless

Theorem 31. We suppose there exist two proof-like terms ω_0, ω_1 such that, for every $\pi \in \Pi$, we have $\omega_0 k_\pi \Vdash \perp$ or $\omega_1 k_\pi \Vdash \perp$. Then, the Boolean algebra $\mathbb{J}2$ is non trivial. Indeed: $\theta \Vdash \forall x (x \neq 1, x \neq 0 \rightarrow x \notin \mathbb{J}2) \rightarrow \perp$ with $\theta = \lambda f (\text{cc}) \lambda k ((f)(\omega_1)k)(\omega_0)k$.

Let $\xi \Vdash \forall x (x \neq 1, x \neq 0 \rightarrow x \notin \mathbb{J}2)$ and $\pi \in \Pi$. We must show:

$\theta \star \xi \cdot \pi \in \perp$, that is $\xi \star \omega_1 k_\pi \cdot \omega_0 k_\pi \cdot \pi \in \perp$.

But, by hypothesis on ξ , we have $\xi \Vdash \top, \perp \rightarrow \perp$ and $\xi \Vdash \perp, \top \rightarrow \perp$. Hence the result, by hypothesis on ω_1, ω_0 .

Q.E.D.

Remark. When the Boolean algebra $\mathfrak{J}2$ is non trivial, there are necessarily non standard integers in the realizability model \mathcal{N} , i.e. integers which are not in \mathcal{M} . Indeed, let $a \in \mathfrak{J}2$, $a \neq 0, 1$; by theorem 28, there is an integer n such that $\Delta(n) \neq 0, \Delta(n) \leq a$; thus $\Delta(n) \neq 1$. The integer n cannot be standard, since $\Delta(m) = 0$ or 1 if m is in \mathcal{M} .

Theorem 32. *We suppose that there exists three proof-like terms $\alpha_0, \alpha_1, \alpha_2$ such that, for every $\xi \in \Lambda$ and $\pi \in \Pi$, we have $k_\pi \xi \alpha_0 \Vdash \perp$ or $k_\pi \xi \alpha_1 \Vdash \perp$ or $k_\pi \xi \alpha_2 \Vdash \perp$.*

Then, the Boolean algebra $\mathfrak{J}2$ is atomless. Indeed :

$\theta \Vdash \forall x[\forall y(x \wedge y \neq 0, x \wedge y \neq x \rightarrow y \notin \mathfrak{J}2), x \neq 0 \rightarrow x \notin \mathfrak{J}2]$

with $\theta = \lambda x \lambda y(cc) \lambda k((x)(k)y \alpha_0)((x)(k)y \alpha_1)(k)y \alpha_2$.

By a simple computation, we see that we must show :

i) $\theta \Vdash (\perp, \perp \rightarrow \perp), \perp \rightarrow \perp$.

ii) $\theta \Vdash |\top, \perp \rightarrow \perp| \cap |\perp, \top \rightarrow \perp|, \top \rightarrow \perp, \top \rightarrow \perp$.

Proof of (i) : let $\eta \in |\perp, \perp \rightarrow \perp|$ and $\xi \in |\perp|$. We must show $\theta \star \eta \cdot \xi \cdot \pi \in \perp$, that is :

$\eta \star k_\pi \xi \alpha_0 \cdot ((\eta)(k_\pi) \xi \alpha_1)(k_\pi) \xi \alpha_2 \cdot \pi \in \perp$.

But, from $\xi \Vdash \perp$, we deduce $k_\pi \xi \zeta \Vdash \perp$ for every $\zeta \in \Lambda_c$.

Since $\eta \Vdash \perp, \perp \rightarrow \perp$, we have $((\eta)(k_\pi) \xi \alpha_1)(k_\pi) \xi \alpha_2 \Vdash \perp$ and therefore :

$\eta \star k_\pi \xi \alpha_0 \cdot ((\eta)(k_\pi) \xi \alpha_1)(k_\pi) \xi \alpha_2 \cdot \pi \in \perp$.

Proof of (ii) : let $\eta \in |\top, \perp \rightarrow \perp| \cap |\perp, \top \rightarrow \perp|$ and $\xi \in \Lambda_c$. Again, we must show that :

$\eta \star k_\pi \xi \alpha_0 \cdot ((\eta)(k_\pi) \xi \alpha_1)(k_\pi) \xi \alpha_2 \cdot \pi \in \perp$. If this is false, then :

$k_\pi \xi \alpha_0 \not\Vdash \perp$ (because $\eta \Vdash \perp, \top \rightarrow \perp$) and $((\eta)(k_\pi) \xi \alpha_1)(k_\pi) \xi \alpha_2 \not\Vdash \perp$ (because $\eta \Vdash \top, \perp \rightarrow \perp$).

But, since $\eta \Vdash \perp, \top \rightarrow \perp$ (resp. $\top, \perp \rightarrow \perp$), we have $k_\pi \xi \alpha_1 \not\Vdash \perp$ (resp. $k_\pi \xi \alpha_2 \not\Vdash \perp$).

This contradicts the hypothesis of the theorem.

Q.E.D.

\mathbb{R} not well orderable

Theorem 33.

We suppose that there exists a proof-like term ω such that, for every $\xi, \xi' \in \Lambda$, $\xi \neq \xi'$ and $\pi \in \Pi$, we have $\omega k_\pi \xi \Vdash \perp$ or $\omega k_\pi \xi' \Vdash \perp$.

Then we have, for every formula F with three free variables :

$\theta \Vdash \forall m^{\mathbb{N}} \forall n^{\mathbb{N}} \forall z[(m < n) = 1 \leftrightarrow$

$(\forall x \forall y \forall y'(F(x, y, z), F(x, y', z), y \neq y' \rightarrow \perp), \forall y^{\mathbb{N}} \neg \forall x^{\mathbb{N}} \neg F(x, y, z) \rightarrow \perp)]$

with $\theta = \lambda x \lambda x'(cc) \lambda k(x') \lambda z(xzz)(\omega) kz$.

Remark. This shows that, if $(m < n) = 1$, then $(\mathfrak{J}m \subset \mathfrak{J}n)$ and there is no surjection of $\mathfrak{J}m$ onto $\mathfrak{J}n$: indeed, it suffices to take, for $F(x, y, z)$, the formula $\langle x, y \rangle \varepsilon z$.

Assume this is false ; then, there exist $m, n \in \mathbb{N}$ with $m < n$, an individual c , two terms $\xi, \xi' \in \Lambda$ and a stack $\pi \in \Pi$ such that :

$\theta \star \xi \cdot \xi' \cdot \pi \notin \perp$;

$\xi \Vdash \forall x \forall y \forall y'[F(x, y, c), F(x, y', c), y \neq y' \rightarrow \perp]$;

$\xi' \Vdash \forall y^{\mathbb{N}} \neg \forall x^{\mathbb{N}} \neg F(x, y, c)$.

Therefore, we have $\xi' \star \eta \cdot \pi \notin \perp$ with $\eta = \lambda z(\xi z z)(\omega) k_\pi z$. By hypothesis on ξ' we have, for every integer $i < n$: $\eta \not\Vdash \forall x^{\mathbb{N}} \neg F(x, i, c)$. Thus, there exists an integer $m_i < m$ such that $\eta \not\Vdash \neg F(m_i, i, c)$. It follows that there exist $\xi_i \in \Lambda$ and $\pi_i \in \Pi$ such that $\xi_i \Vdash F(m_i, i, c)$ and

$\eta \star \xi_i \bullet \pi_i \notin \perp$. By definition of η , we get $\xi \star \xi_i \bullet \xi_i \bullet \omega k_\pi \xi_i \bullet \pi_i \notin \perp$. By hypothesis on ξ , it follows that $\omega k_\pi \xi_i \not\vdash i \neq i$; in other words, we have $\omega k_\pi \xi_i \not\vdash \perp$ for every integer $i < n$.

By the hypothesis of the theorem, it follows that we have $\xi_i = \xi_j$ for every $i, j < n$.

But, since $m_i < m < n$ and $i < n$, there exist $i, j < n, i \neq j$ such that $m_i = m_j = k$.

Then, $\xi_i = \xi_j \Vdash F(k, i, c), F(k, j, c)$ and $\omega k_\pi \xi_i \Vdash i \neq j$ since $\|i \neq j\| = \emptyset$.

Therefore, by hypothesis on ξ , we have $\xi \star \xi_i \bullet \xi_i \bullet \omega k_\pi \xi_i \bullet \pi_i \in \perp$, which is a contradiction.

Q.E.D.

Now, we see that, with the hypothesis of theorem 33, there is no surjection from $\mathbb{J}2$ onto $\mathbb{J}2 \times \mathbb{J}2$. Indeed, by theorem 27, there exists a bijection from $\mathbb{J}2 \times \mathbb{J}2$ onto $\mathbb{J}4$ and, by theorem 33, there is no surjection from $\mathbb{J}2$ onto $\mathbb{J}4$. But, by theorem 32, $\mathbb{J}2$ is infinite; it follows that $\mathbb{J}2$ cannot be well ordered.

Now, by theorem 29, $\mathbb{J}2$ is equipotent with a subset of $\mathcal{P}(\tilde{\mathbb{N}})$. Therefore, the hypothesis of theorems 32 and 33 are sufficient in order that the following formula be realized in the model \mathcal{N} :

There is no well ordering on the set of reals.

In fact, the hypothesis of theorem 33 is sufficient: this follows from theorem 34.

Theorem 34.

Same hypothesis as theorem 33: there exists a proof-like term ω such that, for every $\pi \in \Pi$ and $\xi, \xi' \in \Lambda, \xi \neq \xi'$, we have $\omega k_\pi \xi \Vdash \perp$ or $\omega k_\pi \xi' \Vdash \perp$.

Then we have, for every formula F with three free variables:

$\theta \Vdash \forall z \{ \forall x [\forall n^{\text{ent}} F(n, x, z) \rightarrow x \notin \mathbb{J}2], \forall n \forall x \forall y [\neg F(n, x, z) \neg F(n, y, z), x \neq y \rightarrow \perp] \rightarrow \perp \}$

with $\theta = \lambda x \lambda x' (\text{cc}) \lambda k(x) \lambda n (\text{cc}) \lambda h(x' h h) (\omega k) \lambda f(f) h n$.

Remark. This formula means that, in the realizability model \mathcal{N} , there is no surjection from the set of integers $\tilde{\mathbb{N}}$ onto $\mathbb{J}2$: it suffices to take for $F(x, y, z)$ the formula $\langle x, y \rangle \notin z$ (the graph of an hypothetical surjection being $\langle x, y \rangle \in z$).

Reasoning by contradiction, we suppose that there is an individual c , a stack $\pi \in \Pi$, and two terms ξ, ξ' such that:

$\xi \Vdash \forall x [\forall n^{\text{ent}} F(n, x, c) \rightarrow x \notin \mathbb{J}2]$; $\xi' \Vdash \forall n \forall x \forall y [\neg F(n, x, c) \neg F(n, y, c), x \neq y \rightarrow \perp]$ and

$\theta \star \xi \bullet \xi' \bullet \pi \notin \perp$.

Therefore, we have $\xi \star \eta \bullet \pi \notin \perp$, with $\eta = \lambda n (\text{cc}) \lambda h(\xi' h h) (\omega k_\pi) \lambda f(f) h n$.

By hypothesis on ξ , we have $\eta \not\vdash \forall n^{\text{ent}} F(n, 0, c)$ and $\eta \not\vdash \forall n^{\text{ent}} F(n, 1, c)$. Thus, we see that there exist $n_0, n_1 \in \mathbb{N}, \pi_0 \in \|F(n_0, 0, c)\|$ and $\pi_1 \in \|F(n_1, 1, c)\|$ such that $\eta \star \underline{n}_0 \bullet \pi_0 \notin \perp$ and $\eta \star \underline{n}_1 \bullet \pi_1 \notin \perp$. By performing these two processes, we obtain:

$\xi' \star k_{\pi_0} \bullet k_{\pi_0} \bullet \zeta_0 \bullet \pi_0 \notin \perp$ et $\xi' \star k_{\pi_1} \bullet k_{\pi_1} \bullet \zeta_1 \bullet \pi_1 \notin \perp$,

with $\zeta_0 = (\omega k_\pi) \lambda f(f) k_{\pi_0} \underline{n}_0$ and $\zeta_1 = (\omega k_\pi) \lambda f(f) k_{\pi_1} \underline{n}_1$.

By hypothesis on ξ' , we have $\xi' \Vdash \neg F(n_0, 0, c), \neg F(n_0, 0, c), 0 \neq 0 \rightarrow \perp$. Since $k_{\pi_0} \Vdash \neg F(n_0, 0, c)$, we see that $\zeta_0 \not\vdash \perp$ and, in the same way, $\zeta_1 \not\vdash \perp$.

Thus, by the hypothesis of the theorem, we have:

$\lambda f(f) k_{\pi_0} \underline{n}_0 = \lambda f(f) k_{\pi_1} \underline{n}_1$, and therefore $n_0 = n_1$ and $\pi_0 = \pi_1$.

But, we have $\xi' \Vdash \neg F(n_0, 0, c), \neg F(n_0, 1, c), 0 \neq 1 \rightarrow \perp$. Moreover, we have:

$\pi_0 \in \|F(n_0, 0, c)\|$ and $\pi_1 \in \|F(n_1, 1, c)\|$, thus $\pi_0 \in \|F(n_0, 1, c)\|$ since $n_0 = n_1, \pi_0 = \pi_1$.

Therefore $k_{\pi_0} \Vdash \neg F(n_0, 0, c)$ and $\neg F(n_0, 1, c)$. Moreover, we have obviously $\zeta_0 \Vdash 0 \neq 1$, since $\|0 \neq 1\| = \emptyset$. Therefore, we have $\xi' \star k_{\pi_0} \bullet k_{\pi_0} \bullet \zeta_0 \bullet \pi_0 \in \perp$, which is a contradiction.

Q.E.D.

Theorems 33 and 34 show that \mathbb{N} is infinite and not equipotent with $\mathbb{N} \times \mathbb{N}$, thus not well orderable. Since \mathbb{N} is equipotent with a subset of $\mathcal{P}(\tilde{\mathbb{N}})$ (theorem 29), we have shown that $\mathcal{P}(\tilde{\mathbb{N}})$ is not well orderable, with the hypothesis of theorem 33.

More precisely, by corollary 30, we know that \mathbb{N}_n is equipotent with a subset of $\mathcal{P}(\tilde{\mathbb{N}})$ for each integer n . Therefore, we have :

Theorem 35. *With the hypothesis of theorem 33, the following formula is realized :*

“ There exists a sequence \mathcal{X}_n of infinite subsets of $\mathcal{P}(\tilde{\mathbb{N}})$ such that, for every integers $m, n \geq 2$:

- there is an injection from \mathcal{X}_n into \mathcal{X}_{n+1} ;
- there is no surjection from \mathcal{X}_n onto \mathcal{X}_{n+1} ;
- $\mathcal{X}_m \times \mathcal{X}_n$ and \mathcal{X}_{mn} are equipotent ”.

For each integer $n \geq 2$, the set $\mathbf{n} = \{0, 1, \dots, n-1\}$ is a ring : the ring of integers modulo n ; the Boolean algebra $\{0, 1\}$ is a set of idempotents in this ring. These ring operations extend to the realizability model, giving a ring structure on \mathbb{N}_n , and \mathbb{N} is a set of idempotents in \mathbb{N}_n .

For each $a \in \mathbb{N}$, the equation $ax = x$ defines an ideal in \mathbb{N}_n , which we denote as $a\mathbb{N}_n$.

The application $x \mapsto ax$ is a retraction from \mathbb{N}_n onto $a\mathbb{N}_n$.

Proposition 36. *The following formulas are realized in \mathcal{N} :*

- i) $\forall n \exists a \in \mathbb{N} (the\ application\ x \mapsto (ax, (1-a)x)\ is\ a\ bijection$
from \mathbb{N}_n onto $a\mathbb{N}_n \times (1-a)\mathbb{N}_n$).
- ii) $\forall m \exists n \exists a \in \mathbb{N} (the\ application\ (x, y) \mapsto my + x\ is\ a\ bijection$
from $a\mathbb{N}_m \times a\mathbb{N}_n$ onto $a\mathbb{N}_{(mn)}$).

i) Trivial : the inverse is $(y, y') \mapsto y + y'$.

ii) By theorem 27, this application is injective ; clearly, it sends $a\mathbb{N}_m \times a\mathbb{N}_n$ into $a\mathbb{N}_{(mn)}$. Conversely, if $z \in a\mathbb{N}_{(mn)}$, then there exists $x \in a\mathbb{N}_m$ and $y \in \mathbb{N}_n$ such that $z = my + x$; thus, we have $z = az = may + ax$ with $ax \in a\mathbb{N}_m$ and $ay \in a\mathbb{N}_n$.

Q.E.D.

Theorem 37. *We suppose that, for each $\alpha \in \Lambda$, $\pi \in \Pi$, and every distinct $\zeta_0, \zeta_1, \zeta_2 \in \Lambda$, we have $k_\pi \alpha \zeta_0 \Vdash \perp$ or $k_\pi \alpha \zeta_1 \Vdash \perp$ or $k_\pi \alpha \zeta_2 \Vdash \perp$.*

Then, for each formula $F(x, y, z)$ with three free variables, we have :

$\theta \Vdash \forall z \forall m \exists n \exists a \in \mathbb{N} [(2m < n) = 1 \leftrightarrow$

$$(a \neq 0, \forall x \forall y \forall y' (F(x, y, z), F(x, y', z), y \neq y' \rightarrow \perp), \forall y \exists x \exists x' F(x, ay, z) \rightarrow \perp)]$$

with $\theta = \lambda a \lambda x \lambda y (cc) \lambda k (y) \lambda z (xzz) (k) az$.

Remark. This formula means that, if $n > 2m$, $a \in \mathbb{N}$, $a \neq 0$, then there is no surjection from \mathbb{N}_m onto $a\mathbb{N}_n$: it suffices to take $F(x, y, z) \equiv \langle x, y \rangle \varepsilon z$.

Reasoning by contradiction, let us consider $m, n \in \mathbb{N}$ with $n > 2m$, $a \in \{0, 1\}$, an individual c , three terms $\alpha, \xi, \eta \in \Lambda$ and $\pi \in \Pi$ such that :

$\theta \star \alpha \bullet \xi \bullet \eta \bullet \pi \notin \perp$, $\alpha \Vdash a \neq 0$, $\xi \Vdash \forall x \forall y \forall y' (F(x, y, c), F(x, y', c), y \neq y' \rightarrow \perp)$,

$\eta \Vdash \forall y \exists x \exists x' F(x, ay, c)$.

We have $\theta \star \alpha \bullet \xi \bullet \eta \bullet \pi > \eta \star \theta' \bullet \pi$ and therefore $\eta \star \theta' \bullet \pi \notin \perp$ with $\theta' = \lambda z (\xi z z) (k_\pi) \alpha z$.

It follows that, for every $y \in \{0, \dots, n-1\}$, we have $\theta' \Vdash \forall x \exists x' F(x, ay, c)$.

Thus, there exist two functions $y \mapsto x_y$ (resp. $y \mapsto \zeta_y$) from $\{0, \dots, n-1\}$ into $\{0, \dots, m-1\}$ (resp. into Λ), such that $\zeta_y \Vdash F(x_y, ay, c)$ and $\theta' \star \zeta_y \bullet \omega_y \notin \perp$ (for some suitable stacks ω_y).

Now, we have $\theta' \star \zeta_y \cdot \omega_y > \xi \star \zeta_y \cdot \zeta_y \cdot \kappa_y \cdot \omega_y$ with $\kappa_y = k_\pi \alpha \zeta_y$; therefore, we have :
 $\xi \star \zeta_y \cdot \zeta_y \cdot \kappa_y \cdot \omega_y \notin \perp$ for each $y \in \{0, \dots, n-1\}$.

By hypothesis on ξ (with $y = y'$), it follows that $\kappa_y \not\vdash \perp$ for every $y < n$.

It follows first that $\alpha \not\vdash \perp$ and therefore, we have $a = 1$; thus $\zeta_y \Vdash F(x_y, y, c)$.

Moreover, since $n > 2m$, there exist $y_0, y_1, y_2 < n$ distinct, such that $x_{y_0} = x_{y_1} = x_{y_2}$.

But, following the hypothesis of the theorem, the terms $\zeta_{y_0}, \zeta_{y_1}, \zeta_{y_2}$ cannot be distinct, because $\kappa_{y_0}, \kappa_{y_1}, \kappa_{y_2} \not\vdash \perp$. Therefore we have, for instance, $\zeta_{y_0} = \zeta_{y_1}$; then, we apply the hypothesis on ξ with $y = y_0, y' = y_1$, which gives $\xi \star \zeta_{y_0} \cdot \zeta_{y_1} \cdot \kappa \cdot \omega \in \perp$ for every $\kappa \in \Lambda$ and $\omega \in \Pi$. But it follows that $\xi \star \zeta_{y_0} \cdot \zeta_{y_0} \cdot \kappa_{y_0} \cdot \omega_{y_0} \in \perp$ which is a contradiction.

Q.E.D.

Corollary 38. *With the hypothesis of theorem 37, the following formulas are realized :*

- i) $\forall n^{\tilde{\mathbb{N}}} \forall a^{\mathbb{J}2} (a \neq 0 \rightarrow \text{there is no surjection from } \mathbb{J}n \text{ onto } a\mathbb{J}(n+1))$.
- ii) $\forall n^{\tilde{\mathbb{N}}} \forall a^{\mathbb{J}2} \forall b^{\mathbb{J}2} (a \wedge b = 0, b \neq 0 \rightarrow \text{there is no surjection from } a\mathbb{J}n \text{ onto } b\mathbb{J}2)$.
- iii) $\forall n^{\tilde{\mathbb{N}}} \forall a^{\mathbb{J}2} \forall b^{\mathbb{J}2} (a \wedge b = a, a \neq b \rightarrow \text{there is no surjection from } a\mathbb{J}n \text{ onto } b\mathbb{J}2)$.

i) Suppose that there is a surjection from $\mathbb{J}n$ onto $a\mathbb{J}(n+1)$. Then, by the recurrence scheme (theorem 18(ii)), we see that, for each integer $k \in \tilde{\mathbb{N}}$, there exists a surjection from $(\mathbb{J}n)^k$ onto $(a\mathbb{J}(n+1))^k$; and, by proposition 36(ii) and the recurrence scheme, it follows that there is a surjection from $\mathbb{J}(n^k)$ onto $a\mathbb{J}((n+1)^k)$.

But, for $k > n$, we have $(n+1)^k > 2n^k$ and this contradicts theorem 37.

ii) Since $a \wedge b = 0$, the rings $(a+b)\mathbb{J}n$ and $a\mathbb{J}n \times b\mathbb{J}n$ are isomorphic. Reasoning by contradiction, there would exist a surjection from $(a+b)\mathbb{J}n$ onto $b\mathbb{J}2 \times b\mathbb{J}n$, thus also onto $b\mathbb{J}(2n)$ (proposition 36(ii)), thus a surjection from $\mathbb{J}n$ onto $b\mathbb{J}(2n)$, which contradicts (i).

iii) Otherwise, there would exist a surjection from $a\mathbb{J}n$ onto $(b-a)\mathbb{J}2$, which contradicts (ii).

Q.E.D.

Applications.

i) By DC, since $\mathbb{J}2$ is atomless, there exists in $\mathbb{J}2$ a strictly decreasing sequence. Hence, by corollary 38(iii) and theorem 29, there exists a sequence of infinite subsets of $\mathcal{P}(\tilde{\mathbb{N}})$, the “cardinals” of which are strictly decreasing.

ii) Applying corollary 38(ii) with $n = 2$, we see that there exist two subsets of $\mathcal{P}(\tilde{\mathbb{N}})$ the “cardinals” of which are incomparable; which means that there is no surjection of one of them onto the other.

More precisely, let \mathcal{B} be the image of $\mathbb{J}2$ by the injection in $\mathcal{P}(\tilde{\mathbb{N}})$ given by theorem 29; then we have :

Theorem 39. *With the hypothesis of theorem 37, the following formula is realized in \mathcal{N} :*

There exists a subset \mathcal{B} of $\mathcal{P}(\tilde{\mathbb{N}})$ (the real line of the model \mathcal{N}), such that

\mathcal{B} is an atomless Boolean algebra for the usual order \subseteq on $\mathcal{P}(\tilde{\mathbb{N}})$,

with $\emptyset, \tilde{\mathbb{N}} \in \mathcal{B}$; $a, b \in \mathcal{B} \Rightarrow a \cap b \in \mathcal{B}$.

If $a \in \mathcal{B}, a \neq \emptyset$ then there is no surjection from \mathcal{B} onto $a\mathcal{B} \times a\mathcal{B}$ (where $a\mathcal{B}$ means $\{x \in \mathcal{B}; x \subseteq a\}$).

If $a, b \in \mathcal{B}, a, b \neq \emptyset$ and $a \cap b = \emptyset$, then there is no surjection from $a\mathcal{B}$ onto $b\mathcal{B}$ (the “cardinals” of $a\mathcal{B}, b\mathcal{B}$ are incomparable).

If $a, b \in \mathcal{B}, a \subseteq b$ and $a \neq b$, then there is no surjection from $a\mathcal{B}$ onto $b\mathcal{B}$ (the “cardinal” of $a\mathcal{B}$ is strictly less than the “cardinal” of $b\mathcal{B}$).

In other words, for $a, b \in \mathcal{B}$, we have : $a \subseteq b \Leftrightarrow$ there exists a surjection from $b\mathcal{B}$ onto $a\mathcal{B}$. The order, in the atomless Boolean algebra \mathcal{B} , is the order on the “cardinals” of its initial segments.

The model of threads

This model is the canonical instance of a non trivial coherent realizability model. It is defined as follows :

Let $n \mapsto \pi_n$ be an enumeration of the *stack constants* and let $n \mapsto \theta_n$ be a recursive enumeration of the *proof-like terms*. For each $n \in \mathbb{N}$, the *thread with number n* is the set of processes which appear during the execution of the process $\theta_n \star \pi_n$. In other words, it is the set of all processes $\xi \star \pi$ such that $\theta_n \star \pi_n > \xi \star \pi$.

Note that every term which appears in the n -th thread contains the only stack constant π_n .

We define \perp^c (the complement of \perp) as the union of all threads. Thus, a process $\xi \star \pi$ is in \perp^c iff $(\exists n \in \mathbb{N}) \theta_n \star \pi_n > \xi \star \pi$.

Therefore, we have $\xi \star \pi \in \perp$ iff the process $\xi \star \pi$ never appears in any thread.

For every term ξ , we have $\xi \Vdash \perp$ iff ξ never appears in head position in any thread.

If ξ is a proof-like term, we have $\xi = \theta_n$ for some integer n , and therefore $\xi \star \pi_n \notin \perp$, by definition of \perp . It follows that *the model of threads is coherent*.

If $\xi \in \Lambda$, $\xi \not\Vdash \perp$ then ξ appears in head position in at least one thread. This thread is unique, unless ξ is a proof-like term, because it is determined by the number of any stack constant which appears in ξ .

Theorem 40. *The hypothesis of theorems 31, 32, 33 and 37 are satisfied in the model of threads.*

The hypothesis of theorems 33 and 31 are trivially satisfied if we take :

$\omega = (\lambda x x x) \lambda x x x$, $\omega_0 = (\omega) \underline{0}$, and $\omega_1 = (\omega) \underline{1}$.

Moreover, the hypothesis of theorem 37 is obviously stronger than the hypothesis of theorem 32.

We check the hypothesis of theorem 37 by contradiction : we suppose $k_\pi \alpha \zeta_0 \not\Vdash \perp$, $k_\pi \alpha \zeta_1 \not\Vdash \perp$ and $k_\pi \alpha \zeta_2 \not\Vdash \perp$. Therefore, these three terms appear in head position, and moreover in the same thread : indeed, since they contain the stack π , this thread has the same number as the stack constant of π .

Let us consider their first appearance in head position, for instance with the order 0, 1, 2.

Therefore we have, in this thread : $k_\pi \alpha \zeta_0 \star \rho_0 > \alpha \star \pi > \dots > k_\pi \alpha \zeta_1 \star \rho_1 > \alpha \star \pi > \dots$

But, at the second appearance of $\alpha \star \pi$, the thread enters into a loop, and the term $k_\pi \alpha \zeta_2$ can never arrive in head position, since $\zeta_1 \neq \zeta_2$.

Q.E.D.

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