Keywords

- Analytic functors
- Trace
- Lagrange-Good inversion formula
- Twiners

Matrices over sets

$\boldsymbol{m \times n}$ matrix $\begin{pmatrix} \mathbf{0} & \mathbf{3} & \mathbf{2} & \cdots \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \cdots \\ & \vdots & & \end{pmatrix} \quad \to \quad \mathbb{C}^{\boldsymbol{m \times n}}$

Matrices over sets

$$A \times B \text{ matrix} \begin{pmatrix} \mathbf{0} & \mathbf{3} & \mathbf{2} & \cdots \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \cdots \\ \vdots & & & \end{pmatrix} \qquad \to \qquad \mathbb{C}^{A \times B}$$

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$$\Downarrow$$

$$A \times B \text{ matrix} \begin{pmatrix} \boldsymbol{\emptyset} & \boldsymbol{\aleph_0} & \mathbf{1} & \cdots \\ \cdots & \cdots & & \\ & \vdots & & \end{pmatrix} \text{ over sets}$$

Matrices over sets

$$A \times B \text{ matrix} \begin{pmatrix} 0 & 3 & 2 & \cdots \\ 1 & 2 & 0 & \cdots \\ & \vdots & & \end{pmatrix} \quad \to \quad \mathbb{C}^{A \times B}$$

$$\Downarrow$$

$$A \times B \text{ matrix} \begin{pmatrix} \emptyset & \aleph_0 & 1 & \cdots \\ \cdots & \cdots & & \\ & \vdots & & \end{pmatrix} \text{ over sets}$$

$$\to \quad \mathbf{Set}^{A \times B}$$

Interpretation of lollipop

Interpretation of linear logic

$$t : A \quad \rightsquigarrow \quad [\![t]\!] \in \mathbf{Set}^{[\![A]\!]} \quad \text{(vector)}$$

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Interpretation of $\multimap$

$$t : A \multimap B \quad \rightsquigarrow \quad t \in \mathbf{Set}^{B \times A} \qquad \text{(matrix)}$$

Power series

Matrix (a_{ij}) as a linear map

$$
\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots \\
y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \cdots \\
\cdots
\end{aligned}
$$

$$\Downarrow$$

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$$\Downarrow$$

extend to formal power series

$$\begin{aligned} y_1 &= a + b x_1 x_3 + c x_1^3 x_2 x_3^2 \cdots \\ y_2 &= d x_1^3 + e x_2 x_4 + f x_1^4 x_2^2 \cdots \\ &\cdots \end{aligned}$$

Multisets

Regard $x_1^3 x_2 x_3^2$ as multiset $\{x_1, x_1, x_1, x_2, x_3, x_3\}$

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$\exp A$ $\cdots$ the set of multisets of A

$\Downarrow$

formal power series $\leftrightsquigarrow$ matrix in $\mathbf{Set}^{B \times \exp A}$

Interpretation of bang

Interpretation of $!A$ $\quad\cdots\quad$ $\boldsymbol{exp\, A}$

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Linear map in $!A \multimap B$

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$\leftrightsquigarrow$ formal power series $\mathbf{Set}^{A} \rightarrow \mathbf{Set}^{B}$

$=_{def}$ analytic functor

(special case of)

Interpreting linear logic

$$A^{\perp} \cdots A$$
$$A \otimes B \cdots A \times B$$
$$A \,\&\, B \cdots A + B$$
$$!A \cdots exp\, A$$

comonad $!A \xrightarrow{\delta} !!A$, $!A \xrightarrow{\varepsilon} A$

comonoid $!A \xrightarrow{d} !A \otimes !A$, $!A \xrightarrow{e} 1$

$!A \multimap B$ corresponds to analytic functors (i.e., formal power series)

Trace

Trace in monoidal categories

$$A \otimes X \xrightarrow{f} B \otimes X$$

$$\rightsquigarrow \quad A \xrightarrow{tr^X(f)} B$$

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$$A \otimes X \xrightarrow{f} B \otimes X \quad \rightsquigarrow \quad A \xrightarrow{tr^X(f)} B$$

Diagrammatically

Diagonal sum

Diagonal sum of matrices is trace

$$M : (A \otimes X) \multimap (B \otimes X)$$

Diagonal sum

Diagonal sum of matrices is trace

$$M : \mathbf{Set}^{(A\times X)\times(B\times X)}$$

Diagonal sum

Diagonal sum of matrices is trace

$$M \ : \ \mathbf{Set}^{(A\times X)\times(B\times X)}$$

$$\rightsquigarrow \quad (tr^X M)_{a,b} = \sum_{x \in X} M_{(a,x),(b,x)}$$

Fixed points

Provided tensor $\otimes$ is cartesian

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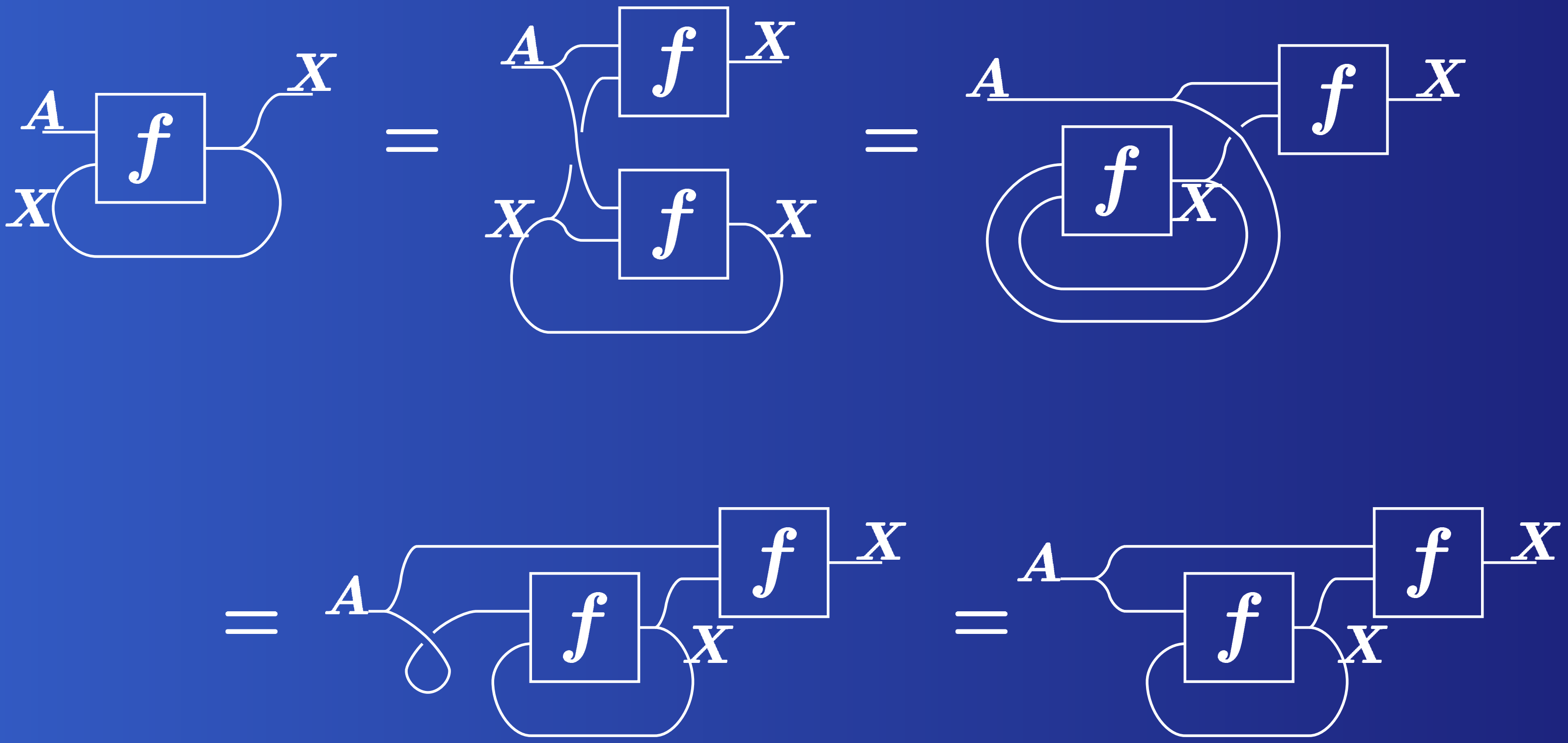
trace $\leftrightsquigarrow$ fixed point combinator

(with Bekič's condition)

Fixed point from trace



Trace of analytic functors

Diagonal sum $\sum_{x \in X} M_{(a,x),(b,x)}$

is trace for tensor $A \otimes X$

not for cartesian $A \& X$

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We need trace of analytic functors

$$M : !(A \,\&\, X) \multimap B \,\&\, X$$

$$\rightsquigarrow \quad \tau^X M : !A \multimap B$$

Coalgebras

Category of analytic functors

$$!A \xrightarrow{f} B$$

$$\cong$$

(Sub)category of coalgebras

$$\begin{array}{ccc} !A & \xrightarrow{g} & !B \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ !!A & \xrightarrow[!g]{} & !!B \end{array}$$

Coalgebraic trace

Coalgebra map $!A \otimes !X \xrightarrow{g} !B \otimes !X$

$$\rightsquigarrow \quad \begin{array}{ccc} !A & \xrightarrow{tr^X g} & !B \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ !!A & \xrightarrow[!tr^X g]{} & !!B \end{array}$$

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Diagonal sum does not satisfy this

We must fi nd new construction

Comonoid map

In the categorical model of linear logic

coalgebraic map $\rightsquigarrow$ comonoid map

$$
\begin{array}{ccc}
!A & \xrightarrow{g} & !B \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\
!A \otimes !A & \xrightarrow[g \otimes g]{} & !B \otimes !B
\end{array}
\qquad\qquad
\begin{array}{ccc}
!A & \xrightarrow{g} & !B \\
{\scriptstyle e}\searrow & & \swarrow{\scriptstyle e} \\
& 1 &
\end{array}
$$

Vanishing case

Required condition:

$$!A \otimes !X \xrightarrow{g} !X$$

$$\rightsquigarrow \quad !A \xrightarrow{tr^X g} \mathbf{1} \text{ (comonoid map)}$$

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i.e., $tr^X g = e$.

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i.e., $tr^X g = e$.

Reading as a formal power series, $tr^X g(x) = 1$

Normalized trace

Given $!A \otimes !X \xrightarrow{g} !B \otimes !X$,

i.e., $M \in \mathbf{Set}^{(exp\,A \times exp\,X) \times (exp\,B \times exp\,X)}$

Define $!A \xrightarrow{tr^X g} !B$

i.e., analytic functor $\mathbf{Set}^A \to \mathbf{Set}^{exp\,B}$

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as a formal power series

$$\beta \quad \mapsto \quad \underline{\textstyle\sum_\alpha (\sum_\gamma M_{(\alpha,\gamma),(\beta,\gamma)}) x^\alpha}$$

$$(\alpha \in exp\,A, \beta \in exp\,B, \gamma \in exp\,X)$$

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$(\alpha \in exp\,A, \beta \in exp\,B, \gamma \in exp\,X)$

Well-defined?

Division of formal power series

$$\frac{\sum_{\alpha}(\sum_{\gamma} M_{(\alpha,\gamma),(\beta,\gamma)})x^{\alpha}}{\sum_{\alpha}(\sum_{\gamma} M_{(\alpha,\gamma),(\emptyset,\gamma)})x^{\alpha}}$$

Well-defined as a formal power series?

(coeffi cients are sets : possibly infi nite, no negative)

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Well-defined as a formal power series?

(coeffi cients are sets : possibly infi nite, no negative)

Trick: Add dummy variables to ensure fi niteness

Imbed coeffi cients in $\mathbb{Q}$

Fixed points of analytic functors

$$tr^X g : \beta \mapsto \frac{\sum_\alpha (\sum_\gamma M_{(\alpha,\gamma),(\beta,\gamma)}) x^\alpha}{\sum_\alpha (\sum_\gamma M_{(\alpha,\gamma),(\emptyset,\gamma)}) x^\alpha}$$

is a coalgebraic trace

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$\rightsquigarrow$ fixed points of analytic functors

Given $!A \otimes !X \xrightarrow{f} X$, set

$$M_{(\alpha,\gamma),(\beta,\gamma')} = [x^\alpha y^\gamma] y^\beta f(x,y)^{\gamma'}$$

(coeffi cient of $x^\alpha y^\gamma$)

Fixed point formula

Fixed point of $f(x, y)$

$$\textbf{fix}\, f(x) = \frac{\sum_\gamma [y^\gamma] y f(x,y)^\gamma}{\sum_\gamma [y^\gamma] f(x,y)^\gamma}$$

Application to mathematical analysis

Solve $y = x \cdot h(y)$ (i.e., $y = \mathbf{fix}(x \cdot h(\text{-}))$)

Its solution $y = h^{\dagger}(x)$

$$h^{\dagger}(x) = \frac{\sum_{\gamma}[y^{\gamma}]y \cdot (x \cdot h(y))^{\gamma}}{\sum_{\gamma}[y^{\gamma}](x \cdot h(y))^{\gamma}}$$

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denominator $= det(I - N(x, h^{\dagger}(x)))^{-1}$

where $N(x, y)$ is matrix $(x_i(\partial h_i(y)/\partial y_j))_{ij}$

Lagrange-Good inversion formula

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$$\rightsquigarrow \quad \frac{y}{1 - xe^{y}} = \sum_{n} [y^{n}] y (xe^{y})^{n} = \sum_{n} \frac{n^{n}}{n!} x^{n}$$

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$\rightsquigarrow$ $y' = \sum \frac{n^{n}}{n!} x^{n-1}$, i.e., $y = \sum \frac{n^{n-1}}{n!} x^{n}$

Towards 2nd-order linear logic

Types affect terms in 2nd-order

$\leadsto$ Analysis of types are needed

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1st-order types are (extended) analytic functors

$$\mathit{exp}\, X = \sum_n X^n / S_n$$

(orbits of the S_n-action on X^n)

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$$\mathit{exp}\, X \;=\; \sum_n X^n / S_n$$

(orbits of the S_n-action on X^n)

But, quotient is ill-behaved

Groupoids

Switch to groupoids (in place of $\sum_n X^n/S_n$)

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$$X \; wr \; S = \int_{n \in S} \mathbf{Hom}(n, X)$$

X: groupoid

$S = \coprod S_n$

Hom: in 2-category of groupoids

$\int$: Grothendieck construction

Wreath product

Given $G \xrightarrow{\varphi} \mathbf{C}$

G: groupoid

φ: pseudo-functor

C groupoid-enriched category

$$X \; wr \; G \; =_{def} \int_{z \in G} \mathbf{Hom}(\varphi(z), X)$$

extension of wreath product in group theory

Twiner

Twiner is a 2-functor

$$(\text{-})\ wr\ G\ :\ \mathbf{C} \rightarrow \mathbf{Gpoid}$$

satisfying a finite presentability condition

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satisfying a finite presentability condition

discrete twiner $\cdots$ if it preserves surjections

quasi-discrete twiner $\cdots$ a weaker condition

Interpreting 2nd-order

Variable type $F(X)$

$\cdots$ discrete twiner **Gpoid** $\rightarrow$ **Gpoid**

Variable term $t_X : F(X)$

$\cdots$ quasi-discrete twiner $Gr(F) \rightarrow$ **Gpoid**

$Gr(F)$: Grothendieck construction

Further topics

Linear parametricity on the twiner model