Non-Alternating Innocence

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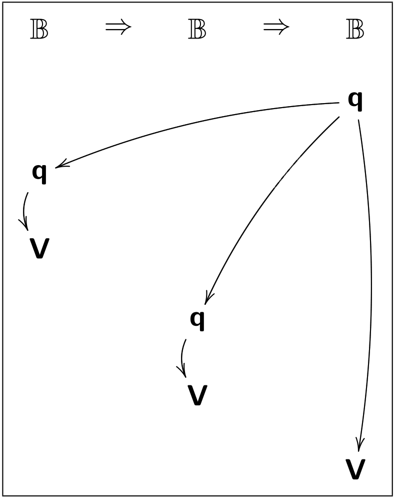
(joint work with Paul-André Melliès)

Alternating game semantics

Left and



Alternating game semantics

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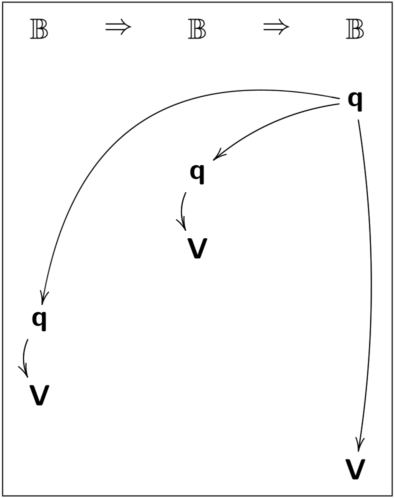
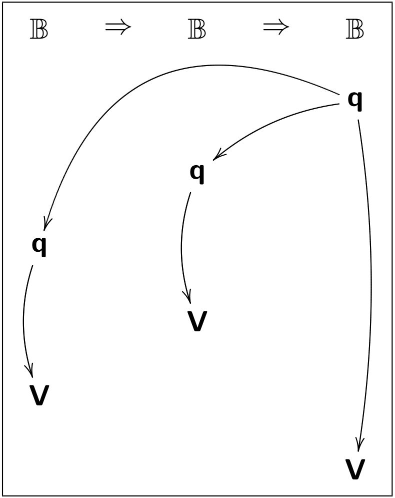


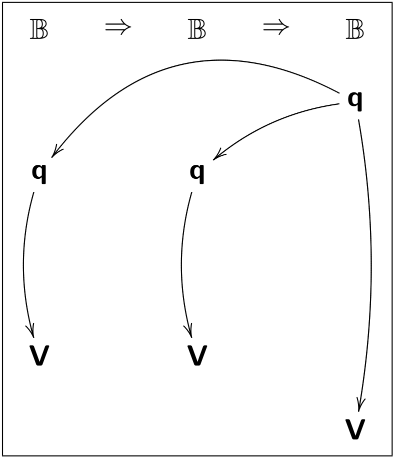
Non-alternating game semantics

Parallel and



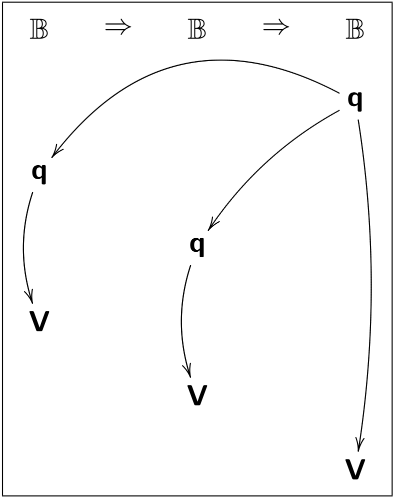
Non-alternating game semantics

Parallel and



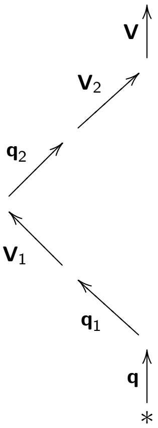
Non-alternating game semantics

Parallel and



Asynchrony: Non-alternating strategies

Left and



Asynchrony: Non-alternating strategies

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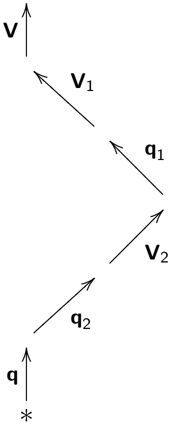

Asynchrony: Non-alternating strategies

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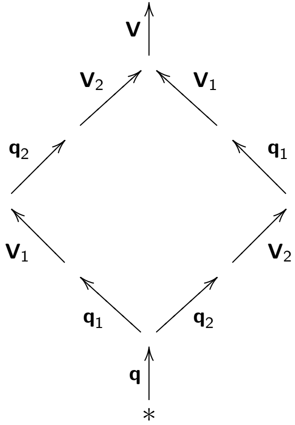

Asynchrony: Non-alternating strategies

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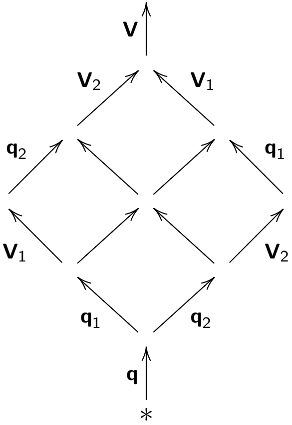

Asynchrony: Non-alternating strategies

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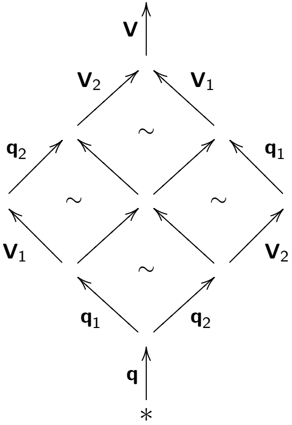

Asynchrony: Non-alternating strategies

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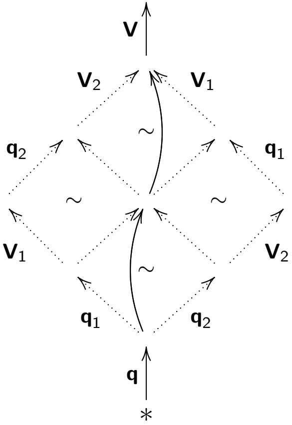
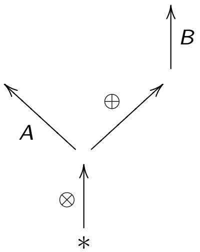


Formulas are inherently non-alternating

$$\dfrac{\dfrac{\vdots}{A} \qquad \dfrac{\dfrac{\vdots}{B}}{B \oplus C}}{A \otimes (B \oplus C)}$$



Each connective $\otimes$ and $\oplus$ is performed by a Player move

Part I

What is innocence [in alternating games]?

Innocent strategies are partial orders

In alternating games:

arena = formula = partial order

innocent strategy = Böhm tree = partial order

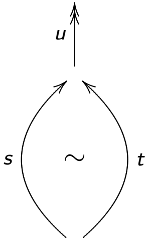
Every Böhm tree refines its formula

Innocent strategies are positional

In alternating games:

Positionality of Innocence [Melliès 2004]
Suppose that σ is innocent, and that $s \in \sigma$ and $t \in \sigma$,

$$s \sim t \quad \text{and} \quad s \cdot u \in \sigma \qquad \text{implies} \qquad t \cdot u \in \sigma$$



Innocent strategies are relational

In alternating games:

The set of halting positions of a strategy σ is defined as

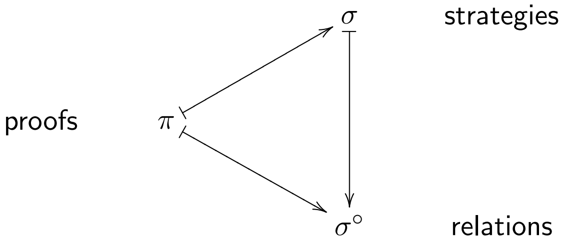
$$\sigma^{\circ} \quad = \quad \{x \quad | \quad \exists s \in \sigma, \quad s : * \twoheadrightarrow x\}$$

Relationality of Innocence [Melliès 2004]
Every innocent strategy σ is characterized by the set σ°.

Innocent strategies are relational

A strong monoidal functor $(-)^\circ$ from games to relations.

$$\begin{array}{rcl} \mathrm{Games} & \longrightarrow & \mathrm{Rel} \\ A & \mapsto & A^\circ \\ \sigma & \mapsto & \sigma^\circ \end{array}$$



$$\boxed{(\sigma \otimes \tau)^\circ \quad = \quad \sigma^\circ \otimes \tau^\circ}$$

Positions as relations

To every strategy $\sigma : A \multimap B$, we associate a relation on $A^\circ \multimap B^\circ$

$$\sigma^\circ \quad = \quad \{(x, y) \in A^\circ \times B^\circ \quad | \quad \exists s : * \twoheadrightarrow (x, y) \in \sigma\}$$

Functoriality

$$\underset{\text{dynamic composition}}{(\sigma;\tau)^\circ} \quad = \quad \underset{\text{static composition}}{\sigma^\circ;\tau^\circ}$$

The aim of this talk

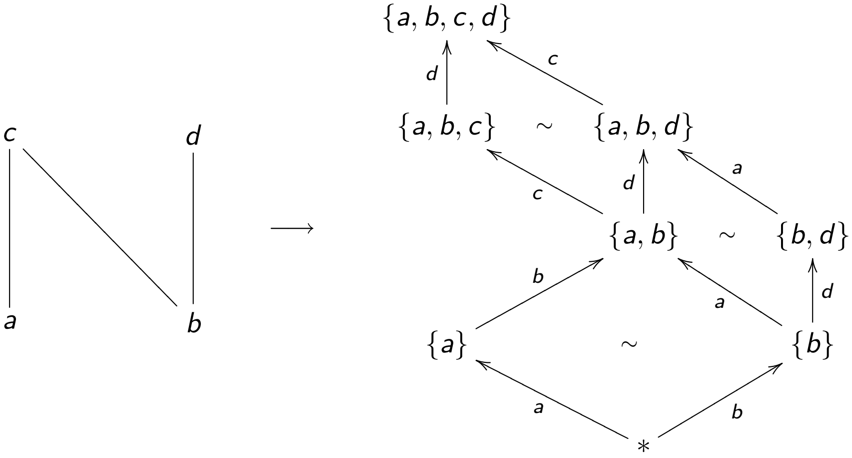
A tentative definition of innocence in non-alternating games

Methodology: extend the three properties by diagrammatic methods.

Part II

Homotopy classes are partial orders

Every partial order generates a 2-graph



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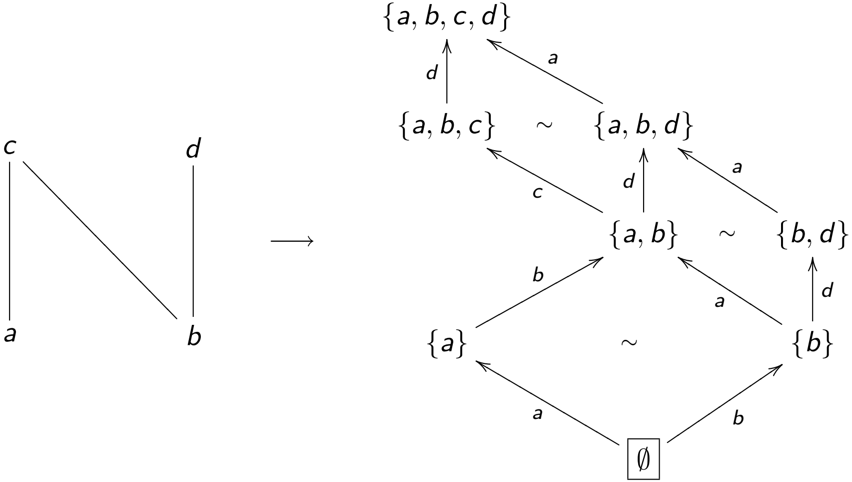

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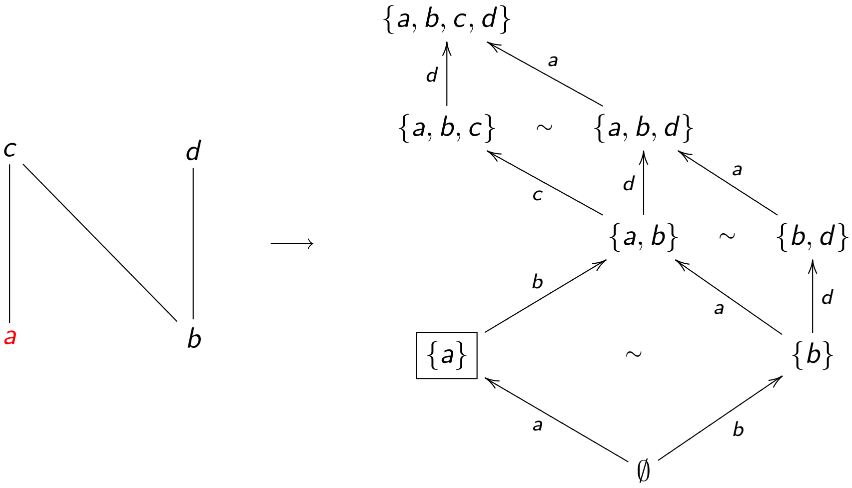

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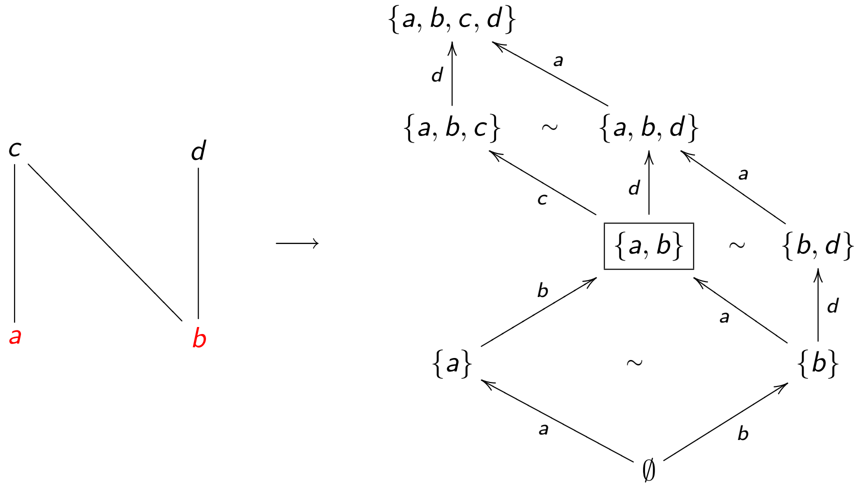

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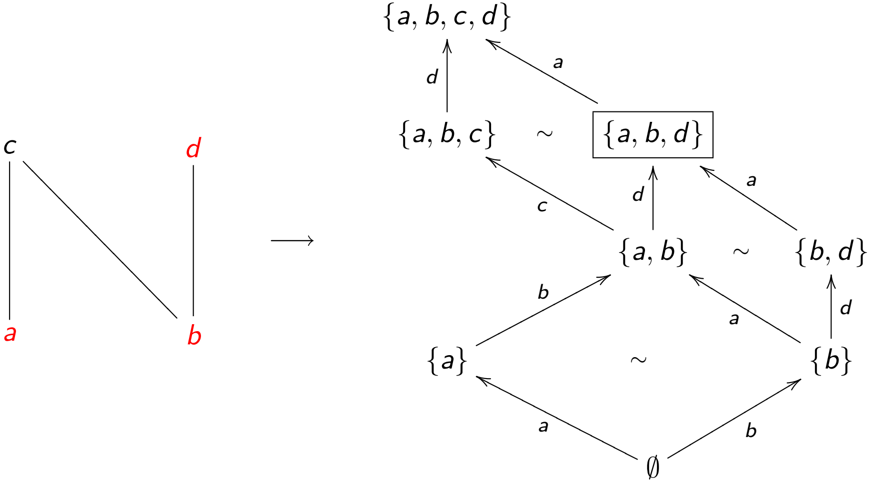

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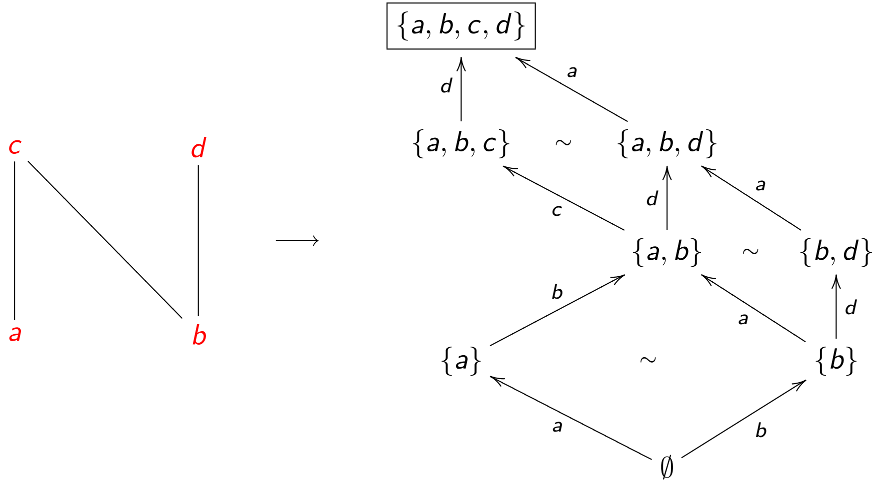
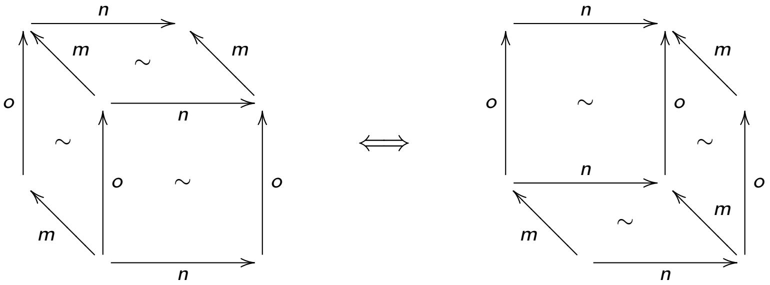
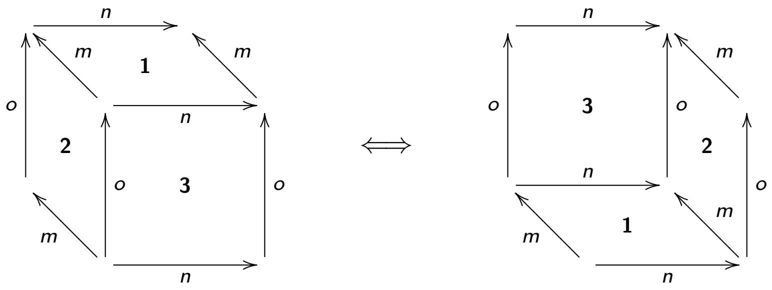


The Cube Property



The Cube Property



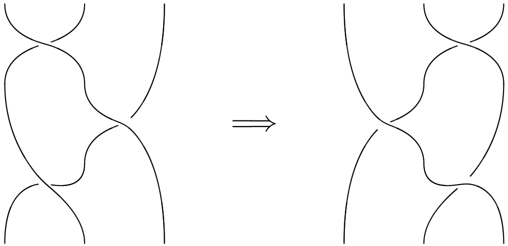
1: $m \| n$ **2**: $m \| o$ **3**: $n \| o$

Conversely...

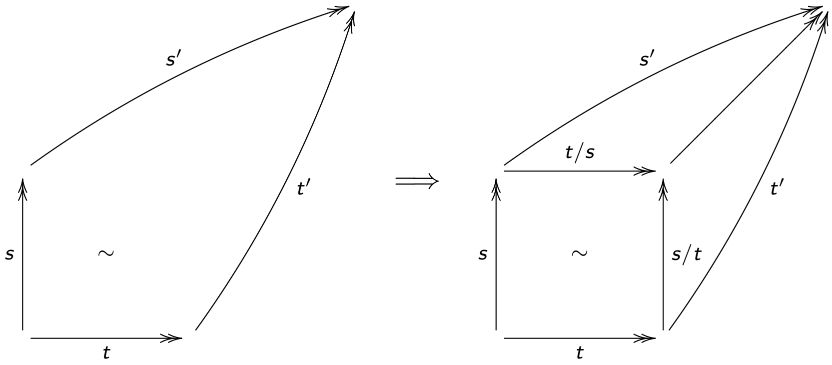
Let us consider a 2-graph satisfying the Cube Property.

Poincaré Duality: from Cubes to Braids

Yang-Baxter equations as a confluent 3-dimensional Rewriting System



Unions and intersections as normal forms



Structure of the prefixes

Consequence
The prefixes of a path f modulo homotopy form a distributive lattice.

Every homotopy class is a partial order

Every path f generates a partial order $[\![f]\!]$ on its set of moves, such that

$$g \sim f \quad \Longleftrightarrow \quad g \text{ is a linearization of } [\![f]\!].$$

An embarassingly simple notion of homotopy!

Part III

From sequentiality to positionality

Definition of asynchronous game

An asynchronous game is a 2-graph satisfying the Cube Property.

A vertex $*$ is chosen as initial position of the game.

The sequential definition of a strategy

A strategy is a set of paths

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \cdots x_{k-1} \xrightarrow{m_k} x_k$$

which is

- non-empty,
- closed under prefix.

The traditional definition of a strategy in game semantics.

Positionality

Definition
A strategy is positional when it is the set of paths

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \cdots x_{k-1} \xrightarrow{m_k} x_k$$

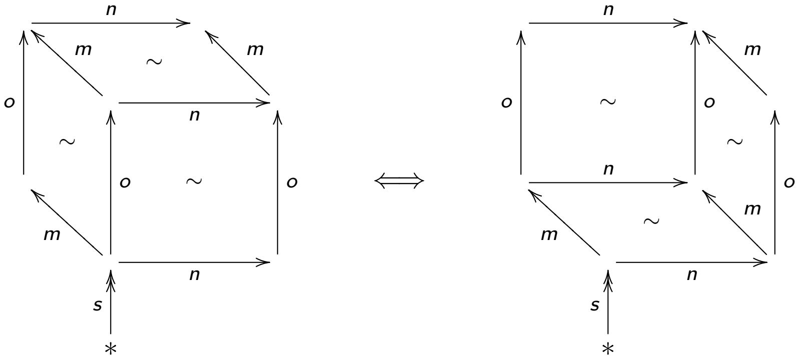
of a subgraph of the 2-graph.

Same definition as previously.

From sequentiality to positionality

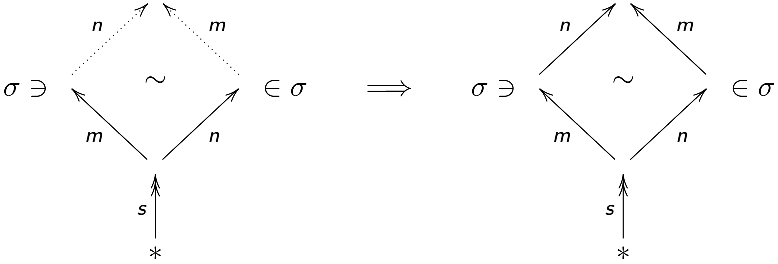
When is a sequential strategy positional?

Three properties: The Cube



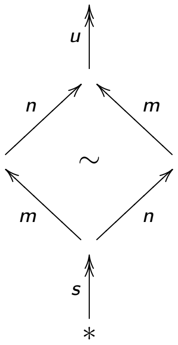
Three properties: Preservation of Compatibility

Preservation of compatibility



Three properties: Extension

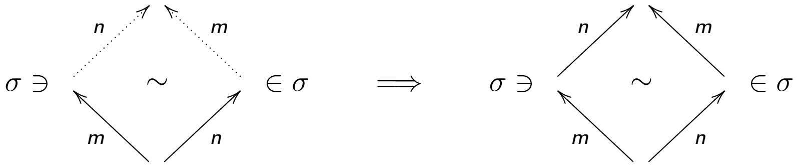
Extension property



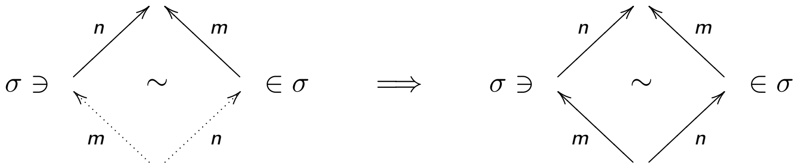
$$\frac{s \cdot m \cdot n \in \sigma \quad s \cdot n \cdot m \in \sigma \quad s \cdot m \cdot n \cdot u \in \sigma}{s \cdot n \cdot m \cdot u \in \sigma}$$

Dynamic positionality

Theorem
An innocent strategy is a subgraph of the graph of the game which satisfies



and



Part IV

From positionality to relationality

Halting positions

The set of halting positions of a strategy σ is defined as

$$\sigma^\circ = \{x \quad | \quad \forall s : * \longrightarrow x \in \sigma, \forall m \in M, \quad s \cdot m \in \sigma \Rightarrow \lambda(m) = \mathrm{P}\}$$

halting position $\quad = \quad$ the strategy has nothing left to play

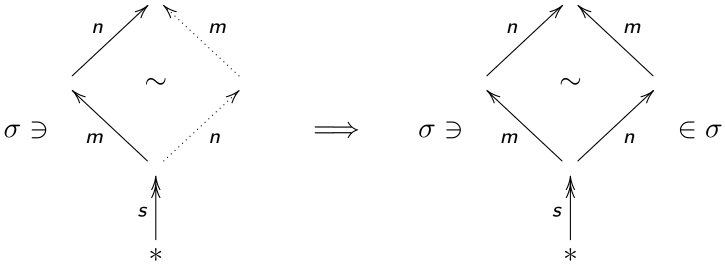
Relationality

Relationality
Strategies are characterized by their halting positions: we can recover σ from σ°.

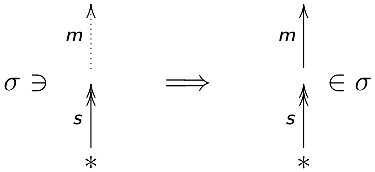
$$\boxed{\text{strategy} \quad = \quad \text{closure operator}}$$

Definition of asynchronous strategy

- *Courteous*: for every Player move m,

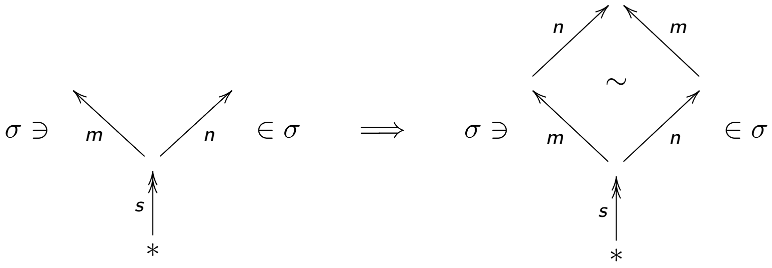


- *Receptive*: for every Opponent move m



Definition of deterministic strategy

- for every Player move m



Functoriality of relationality

Functoriality:

$$(\sigma;\tau)^{\circ} \quad = \quad \sigma^{\circ};\tau^{\circ}$$

Livelocks/deadlocks avoided by adding payoff on paths.

We get a faithful strong monoidal functor from Games to Rel.

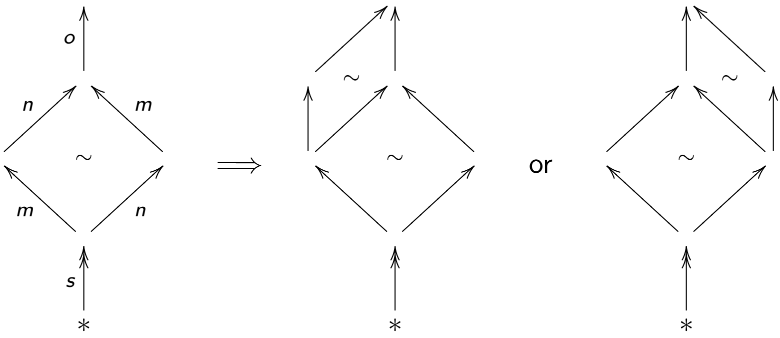
Part V

Further work

Recovering alternating innocence

The subcategory of alternating innocent strategies:

- games are alternating
- for every Opponent moves m and n, and Player move o,



Summary

Four interactive paradigms:

1. small steps (sequential)
2. big steps (sequential by clusters of moves)
3. dynamic positionality (closure operators)
4. static positionality (halting positions)

What's next?

- Construct a model of Linear Logic in which every connective is interpreted by a move, based on a lax and unbiased monoidal category with *n*-ary tensor products:

$$(A_1 \otimes \cdots \otimes A_n)$$

  and a 2-categorical notion of cartesian product.
- Reconstruct semantically focalization and correctness criteria.

$$(A_1 \otimes \cdots \otimes A_k) \otimes (A_{k+1} \otimes \cdots \otimes A_n) \quad \mapsto \quad (A_1 \otimes \cdots \otimes A_n)$$

- Exhibit truly concurrent models of concurrent languages.