Asynchronous games 1:
Uniformity by group invariance

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Ten years ago, Abramsky, Jagadeesan, Malacaria (AJM) introduced a fully abstract game model of PCF, inspired by Girard's Geometry of Interaction. The key ingredient of the model is a partial equivalence relation (per) on strategies, which captures the idea of a strategy "blind to the Opponent's thread indexing". We reveal the group-theoretic nature of this construction, and reformulate arena games accordingly. The justification pointers are replaced by thread indexing, modulo a left and right group action; a strategy is equivalent to itself (wrt. the per) when it verifies a *bi-invariance* condition, which replaces the familiar *invariance* condition of group theory.

Foreword on asynchronous games. This article opens a series of papers on *asynchronous* games semantics, which aims at a *concurrent* and *geometric* account of interference and states in programming languages. In order to develop our theory, we need to reformulate arena games in a simpler algebraic vocabulary, inspired by Girard's Geometry of Interaction and Abramsky, Jagadeesan and Malacaria (AJM) token games. This is precisely the task of this article, which prepares the field for the positional / homotopic account of innocence in (Melliès 2004).

1. Introduction: justification pointers vs. indices on moves

Linear logic. Linear logic has taught us a simple recipe for cooking up denotational models of PCF, or of richer languages like Idealized Algol or Core ML. The recipe works in two stages: (1) define the linear part of the model, generally expressed in a symmetric monoidal closed category, (2) devise an exponential modality; that is, associate to every object A of the category a commutative comonoid $(!A, d_A, e_A)$ and a morphism $!A \longrightarrow A$ (called *dereliction*) verifying the following universality property: for every morphism $!A \longrightarrow B$, there exists a unique *comonoidal* morphism $!A \longrightarrow !B$ making the diagram below commute:

$$\begin{tikzcd} !A \arrow[r, dotted] \arrow[dr] & !B \arrow[d] \\ & B \end{tikzcd} \tag{1}$$

Curiously, stage (1) is generally simpler than stage (2) which requires most of the attention, see (Benton *et al.* 1992; Bierman 1995; Hyland 1997) or (Melliès 2002) for a recent survey on the categorical models of linear logic.

In this article, we focus on the game-theoretic models of linear logic. In this class of models, the "exponential" game $!A$ is constructed by *replaying* the "linear" game A as many times as Opponent desires. Consequently, a play s of the game $!A$ is the juxtaposition of possibly several plays $s_1, ..., s_n$ of the original game A. In that case, each play s_i is called a *thread* of the play s.

Several juxtaposition policies appear in the litterature: backtracking or repetitive, uniform or non uniform, etc... whose zoology is studied extensively in (Melliès 2002). Each policy induces a model of (intuitionistic) linear logic, in which the universality property (1) amounts essentially to a property of *thread factorization*: whenever a n-tuple $(s_1, ..., s_n)$ of threads is juxtaposed in a play s of $!A$, the strategy $!A \longrightarrow !A^{\otimes n}$ obtained by co-multiplication to the n-fold tensor product of $!A$, is able to "extract" interactively the n-tuple $(s_1, ..., s_n)$ from the play s.

Thread indexing vs. justification pointers. In this article, we compare two well known juxtaposition policies of games semantics, namely: (1) by thread indexing and (2) by justification pointers. We recall them briefly.

1. Inspired by the Geometry of Interaction (Girard 1989) Abramsky, Jagadeesan and Malacaria (AJM) design a fully abstract model of PCF (Abramsky *et al.* 1994) in which threads are indexed by natural numbers. Every AJM game comes with a partial equivalence relation (per) on plays, which describes when two plays "equal modulo thread indexing." Hyland introduces in his lecture notes (Hyland 1997) another indexing policy, in which the AJM partial equivalence relation is avoided, by imposing an incremental indexing of threads: the thread of A indexed by the integer $j + 1$ in the game $!A$ starts only when the thread of A indexed by the integer j is started.

2. Inspired by Curry and Gandy's work on higher-order sequential functionals, Hyland, Ong and Nickau (HON) introduce a fully abstract model of PCF (Hyland and Ong 1994; Nickau 1994) based on *arena games*. An arena is a bipartite forest $A = (M_A, \lambda_A, \vdash_A)$ whose nodes $m \in M_A$ (=the moves) are polarized by a function $\lambda_A : M_A \longrightarrow \{+1, -1\}$ $(+1 : \text{Player}, -1 : \text{Opponent})$. A root of the forest is called an *initial* move; a move m is said to *justify* a move n when there is an oriented edge $m \vdash_A n$ in the arena. A justified play (also called justified sequence) is defined as a finite string s of moves, equipped with a pointer relation, indicating for each (occurrence of) non-initial move n of the string s, a previous (occurrence of a) move m of the string s, such that $m \vdash_A n$. In that case, one says that n points to m.

We recall the definition of a strategy in an arena game. A justified play $s = m_1 \cdots m_k$ is *legal* when

$$\forall i \in [1, ..., k], \quad \lambda_A(m_i) = (-1)^i.$$

In other words, s is legal iff $s = \epsilon$ or s is alternated and starts by an Opponent move. The set of legal plays of the arena A is denoted L_A. A strategy σ of A is defined as a set of legal plays of *even length* such that, for every play s and moves m, n, n_1, n_2:

— the empty play ϵ is element of σ,
— if $s \cdot m \cdot n \in \sigma$, then $s \in \sigma$,
— if $s \cdot m \cdot n_1 \in \sigma$ and $s \cdot m \cdot n_2 \in \sigma$, then $n_1 = n_2$.

The advantages of justification pointers. In a few years, arena games became predominant over indexed games. For two reasons at least. First, the pointer structure enables a remarkably elegant description of threads, arguably simpler than the AJM per technique. Second, the pointer structure reveals a series of fundamental constraints on strategies, hardly visible in indexed games: e.g. the classes of *innocent*, *well-bracketed*, *visible*, or *single-threaded* strategies. Strikingly, each class implements a particular programming feature, in a fully abstract fashion (Abramsky and McCusker 1999; Abramsky *et al.* 1998; Harmer 2000).

Our theory of asynchronous games starts from a concurrent reformulation of innocent strategies. This class of strategies introduced in (Hyland and Ong 1994; Nickau 1994) captures the simply-typed λ-calculus with a constant Ω for non-termination, either formulated as Böhm trees (Danos *et al.* 1996), as proofs of Polarized Linear Logic (Laurent 2001), or (after a continuation-passing style translation) as PCF programs augmented with local control (Laird 1997; Abramsky and McCusker 1999). Technically, innocence is defined using a notion of *Player view* of a legal play, deduced from the pointer structure of the play. The Player view of a legal play s is the legal play $\ulcorner s \urcorner$ defined by induction:

$$\begin{array}{rcll} \ulcorner s \cdot m \cdot n \urcorner & = & \ulcorner s \cdot m \urcorner \cdot n & \text{if } \lambda_A(n) = +1, \\ \ulcorner s \cdot m \cdot t \cdot n \urcorner & = & \ulcorner s \urcorner \cdot m \cdot n & \text{if } \lambda_A(n) = -1 \text{ and } n \text{ points to } m, \\ \ulcorner s \cdot n \urcorner & = & n & \text{if } \lambda_A(n) = -1 \text{ and } n \text{ is initial}, \\ \ulcorner \epsilon \urcorner & = & \epsilon & \text{where } \epsilon \text{ is the empty string.} \end{array}$$

A strategy is *innocent* when for every plays $s, t \in \sigma$ and moves m, n:

$$s \cdot m \cdot n \in \sigma \wedge t \cdot m \in L_A \wedge \ulcorner s \cdot m \urcorner = \ulcorner t \cdot m \urcorner \Rightarrow t \cdot m \cdot n \in \sigma. \tag{2}$$

A germ of confusion in arena games. Arena games have been extraordinarily successful in the last decade, and no doubt, we would have been happy to carry on with them, had we not bumped against a very serious difficulty when we started manipulating them with concurrency ideas in mind (e.g. permute moves in a justified sequence).

The difficulty is generally hidden by the convention to keep the justification pointers as implicit as possible in arena games. The convention is quite useful to hide the obvious details under a rhetorical carpet, and to keep the theory as concise as possible. For example, the convention applies in the section above, when we define the Player view and “forget” to mention that, for every (occurrence of) Player move n of the play s pointing to an (occurrence of) Opponent move m:

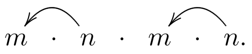
- the move m appears in the view $\ulcorner s \urcorner$, and then, the move n points to the same occurrence of m in the view $\ulcorner s \urcorner$,
- or the move m does not appear in the view $\ulcorner s \urcorner$, and then, the move n becomes initial in the view $\ulcorner s \urcorner$.

Strictly speaking, the Player view $\ulcorner s \urcorner$ defines a justified sequence of the original arena *only* when every Player move in $\ulcorner s \urcorner$ points into its Player view. A justified play in which *every* Player move points into its Player view, is called *P-visible.* And indeed, an innocent strategy is implicitly required to contain only P-visible plays.

The convention hides a very disturbing fact about arena games:

The operation of extending a justified play s with a move p is ambiguous.

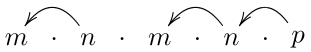
We illustrate this point. Let $\mathbb{A}$ be the arena with one initial move $m : -1$ and two moves $n : +1$ and $p : -1$ justified as: $m \vdash_{\mathbb{A}} n \vdash_{\mathbb{A}} p$. The sequence $s = m \cdot n \cdot m \cdot n$ in which the first (resp. second) occurrence of n points to the first (resp. second) occurrence of m, defines a justified play of $\mathbb{A}$ noted:



$$m \cdot n \cdot m \cdot n.$$

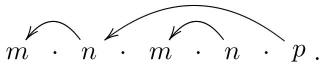
There are exactly two ways t_1 or t_2 to extend the justified sequence s with the move p, depending which occurrence of n is chosen to justify the move p:

Either



$$m \cdot n \cdot m \cdot n \cdot p$$

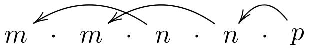
or



$$m \cdot n \cdot m \cdot n \cdot p \,. \qquad (3)$$

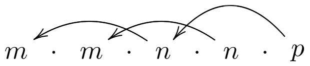
The ambiguity between the two plays t_1 and t_2 is apparently innocuous. But this germ of confusion becomes gangrenous when one starts thinking about permuting moves in a justified sequence.

Permutations as 2-dimensional cells. Suppose that one starts from the left-hand side sequence of (3) and permutes the move n of the first thread with the move m of the second thread. We draw the resulting sequence:



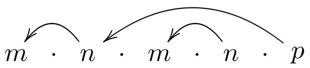
$$m \cdot m \cdot n \cdot n \cdot p \qquad (4)$$

Note that, strictly speaking, the permutation between m and n is *non-local* because it requires to alter the justification pointers of any later (occurrence of) move pointing to m or n. Now, suppose that one carries on from the sequence (4), and permutes simultaneously the two occurrences of the move m, and the two occurrences of the moves n. We draw the resulting sequence:



$$m \cdot m \cdot n \cdot n \cdot p \qquad (5)$$

Then, permuting n and m in (5) brings back to the right-hand side sequence of (3):



$$m \cdot n \cdot m \cdot n \cdot p \qquad (6)$$

So, by swapping the two threads $m \cdot n$ in the play s, the series of permutations transforms the first variant t_1 to the second variant t_2 of (3).

Reflecting on our work on Rewriting Theory (Melliès 2001a) we would like to express any series of such permutations as a *cell* in a 2-category with *positions* as objects, and *plays* as morphisms. The ongoing discussion shows that this is difficult, if not impossible, with the current formulation of arena games. Indeed, imagine that the series of permutations (4-5-6) is represented as a cell $\alpha : s \Rightarrow s$ in the 2-category we have in mind. The cell α is then drawn as a 2-dimensional arrow from the justified sequence $s = m \cdot n \cdot m \cdot n$ to itself:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "s"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, "p"] & \cdot \end{tikzcd} \tag{7}$$

The cell α may be post-composed with the move p. This induces a cell $\beta : t_1 \Rightarrow t_2$ from the play t_1 to the play t_2, left and right-hand side of (3):

$$\begin{tikzcd} \cdot \arrow[r, bend left, "t_1"] \arrow[r, bend right, "t_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \cdot \end{tikzcd} \tag{8}$$

Now, the coherence laws of a 2-category imply that the source t_1 and target t_2 of the cell β are equal to the composite $t_1 = s; p = t_2$ in the underlying category of positions and plays. This contradicts the fact that the plays t_1 and t_2 are intrinsically different.

In this article, we study a straightforward solution to that problem, which is to differentiate the source s_1 and target s_2 of the cell α, by *naming* the two threads $m \cdot n$ inside the play s. This requires a drastic reformulation of arena games, in which the source and target plays s_1 and s_2 are presented using indexed moves:

$$s_1 = (m,0) \cdot (n,0) \cdot (m,1) \cdot (n,1), \qquad s_2 = (m,1) \cdot (n,1) \cdot (m,0) \cdot (n,0).$$

Diagram (7) becomes:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "s_1"] \arrow[r, bend right, "s_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, "{(p,1)}"] & \cdot \end{tikzcd} \tag{9}$$

where the index 1 in $(p, 1)$ indicates that p points to the move $(n, 1)$. The cell $\beta : t_1 \Rightarrow t_2$ of diagram (8) is obtained by composing the cell $\alpha : s_1 \Rightarrow s_2$ with the move p. In contrast to the previous situation, post-composition with the move p is possible now, because the two equations $t_1 = s_1 \cdot (p, 1)$ and $t_2 = s_2 \cdot (p, 1)$ are verified in the indexed presentation of justified sequences.

Remark. Another peculiarity of arena games is noticed in (McCusker 1998). After constructing a category of *linear* arenas and innocent strategies, McCusker observes that the expected exponential construction *does not define* a comonad in the category. Facing this difficulty, McCusker reintroduces indices in arena games and obtains a proper model of (intuitionistic) linear logic in the spirit of AJM games. We attack the question another time in (Melliès 2001b) and construct a model of (intuitionistic) linear logic for arena games and innocent strategies, *without reintroducing indices.* Our solution is fine categorically, but requires to apply some unexpected surgery on arenas. This reveals that the pointer structure of arena games is a handy notation for another more canonical (indexed?) thread structure, and motivates the orbital reformulation exposed below.

Back to the future: replacing justification pointers by thread indexing. Motivated by our 2-categorical understanding of permutations, we decide to go against the apparent history of the subject: games semantics... and to reformulate arena games as indexed games!

We illustrate the general pattern of this reformulation on the class of *single-threaded* strategies between arena games, a model introduced by Abramsky, Honda and McCusker (Abramsky *et al.* 1998) in order to interpret a programming language with general reference à la ML. The three authors construct a cartesian closed category and show that their interpretation is fully abstract — see also (Harmer 2000).

We recall briefly the definition of single-threaded strategy. Let m, n denote two (occurrences of) moves in a justified play s and suppose that n is initial. We say that n is the *hereditary justifier* of m when following back the justification pointers from m leads to n. Note that there exists one and only one hereditary justifier of a given (occurrence of) move m in a given play s, and that this justifier is (the occurrence of) an initial move. We write $s \upharpoonright n$ for the justified subsequence of s, consisting of all (occurrences of) moves with hereditary justifier n. The *thread* of m is defined as $s \upharpoonright n$ where n is the hereditary justifier of m. Finally, the *current thread* of a justified play $s \cdot m$ is defined as the thread of m, and denoted $\lceil s \cdot m \rceil$. Now, a strategy is *single-threaded* when for every plays $s, t \in \sigma$ and moves m, n:

$$s \cdot m \cdot n \in \sigma \wedge t \cdot m \in L_A \wedge \lceil s \cdot m \rceil = \lceil t \cdot m \rceil \Rightarrow t \cdot m \cdot n \in \sigma \tag{10}$$

Note that single-threadedness is defined as innocence, except that the Player view is replaced by the thread.

We indicate in (Melliès 2002) how the single-threaded model may be formulated without any reference to arenas or justification pointers. This uses a category of Conway games (=non alternated games) equipped with an exponential modality à la Hyland — that is, based on an incremental policy. We find instructive to develop below a short account of this reformulation. The translation from justified sequences to sequences of indexed moves goes in two steps.

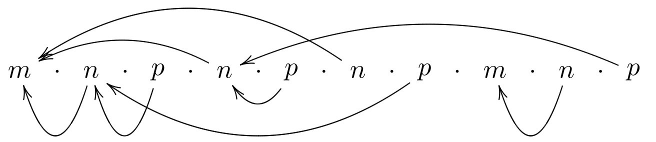
Step 1: index justification pointers. The first step is to index every justification pointer

$$\begin{tikzcd}[column sep=tiny] s_1 & \cdot & m & \cdot & s_2 & \cdot & n \arrow[ll, bend right] & \cdot & s_3 \end{tikzcd}$$

by the natural number k:

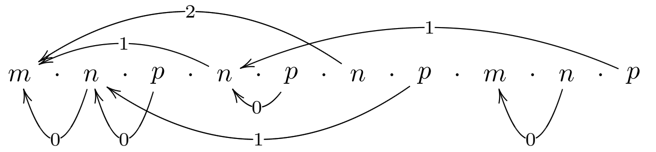
$$\begin{tikzcd}[column sep=tiny] s_1 & \cdot & m & \cdot & s_2 & \cdot & n \arrow[ll, bend right, "k"'] & \cdot & s_3 \end{tikzcd}$$

indicating the number k of (occurrences of) moves in s_2 which point to the (occurrence of) move m. We illustrate this on the arena $\mathbb{A}$, and the justified sequence below:



(11)

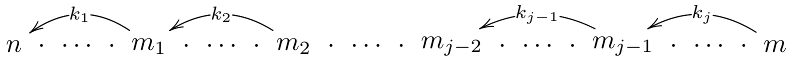
in which the justification pointers are indexed as follows:



(12)

Step 2: put the indices inside the moves. The second step of the translation is to replace every move m in the justified play s by the indexed move $(m, k_0, k_1, ..., k_j)$ obtained as follows:

— $k_1, ..., k_j$ is the sequence of indices encountered when following back the pointers from m to its hereditary justifier n:



— k_0 is the number of (occurrences of) initial moves appearing before n in the play s.

Typically, the justified play (11) is first indexed as in (12) then encoded as the string of indexed moves:

$$(m, 0) \cdot (n, 00) \cdot (p, 000) \cdot (n, 01) \cdot (p, 010) \cdot (n, 02) \cdot (p, 001) \cdot (m, 1) \cdot (n, 10) \cdot (p, 011)$$

So, we have just translated (in two steps) a justified sequence of an arena, into a string of indexed moves. The translation is one-to-one, and has the advantage of removing the ambiguity of composition in arena games. For instance, the two justified plays in (3) are translated as

$$(m, 0) \cdot (n, 00) \cdot (m, 1) \cdot (n, 10) \cdot (p, 100) \qquad (m, 0) \cdot (n, 00) \cdot (m, 1) \cdot (n, 10) \cdot (p, 000)$$

So, the arena move p is either translated as $(p, 000)$ or $(p, 100)$ depending on which occurrence of move n the move p is intended to point at.

Developing on these idea, it is not difficult to show that (a variant of) Joyal's category of Conway games equipped with a (variant of) Hyland's exponential comonad $!^{\mathrm{inc}}$ *linearizes* the single-threaded category of (Abramsky *et al.* 1998) — see (Joyal 1977; Hyland 1997; Melliès 2002). We show in this article and in the next (Melliès 2004) that, in fact, it is possible to reformulate *any* arena game model using what we call *asynchronous games* — see section 5.1 for a definition of this refinement of sequential games on decision trees. Also, as we shall see next section (section 2.4) a good account of concurrency requires to replace Hyland's incremental thread indexing by a more liberal one, inspired by AJM token games.

Synopsis. The article is composed of five sections, and a conclusion.

- in section 1, we recall the two families of games semantics studied in the article, in which threads are juxtaposed by justification pointers (HON) or by indices (AJM). We explain how justification pointers may be replaced by indices on moves — at least in the case of Abramsky, Honda and McCusker's *single-threaded* model.
- in section 2, we define a $*$-autonomous category $\mathcal{S}$ of sequential games with errors inspired by Joyal (section 2.1) ; we adapt Hyland's *incremental exponential* $!^{\mathrm{inc}}$ to its subcategory $\mathcal{G}$ of negative games (section 2.3); we justify why one needs to alter Hyland's incremental style into a more liberal style, in order to apply our homotopic ideas (section 2.4) ; finally, we recall the definition of AJM games (section 2.5.)
- in section 3, we introduce the notion of *orbital game*. An *orbital game* is defined as a sequential game equipped with a left and right group action over moves, enjoying elementary coherence properties. We define a partial equivalence relation on strategies in two different ways: by group-theoretic bi-invariance $\approx^{\mathrm{INV}}$ on one hand, and by simulation techniques $\approx^{\mathrm{SIM}}$ adapted from AJM games. We show that the two definitions are equivalent,
- in section 4, we reformulate the AJM game model of PCF as an *alternated* orbital game model of *history-free* error-free strategies,
- in section 5, we translate the lexicon of arena games (justified plays, strategies) into the lexicon of orbital games (orbits, bi-invariant strategies).

Related works. After the introduction of the AJM token game model (Abramsky *et al.* 1994) and the innocent HON arena game model (Hyland and Ong 1994; Nickau 1994), much work was devoted by the "french school" to understand the relationship between the two models of PCF (Herbelin 1995) and the abstract machines developed by Danos and Regnier (Danos *et al.* 1996). Many people (including the three authors AJM) understood at the time that the partial equivalence relation $\approx^{\mathrm{AJM}}$ on plays amounts to a group action $G \times M \longrightarrow M$ over moves. The idea appears explicitly in Baillot's PhD thesis (Baillot 1999). A model of strategies *invariant* wrt. this group action (called saturated strategies) is considered in (Baillot *et al.* 1997). The resulting model of multiplicative exponential linear logic is interesting, but slightly puzzling because the contraction strategy $!A \longrightarrow\ !A \otimes !A$ is non-deterministic — even if one understands strategies as interacting on the orbits of plays modulo the group action.

2. Incremental vs. liberal indexing of threads

2.1. *A category $\mathcal{S}$ of sequential games with errors*

In games semantics, it is customary to interpret formulas (or types) as sequential games in which plays are *alternated* (Abramsky and Jagadeesan 1994; Lamarche 1992; Curien 1993). Here, we shift to sequential games in which *non alternated plays* are also admitted, because we prepare the field for asynchronous games, in which one may permute two moves inside a play — an operation which does not preserve alternation. So, we introduce here a $*$-autonomous category of *sequential games* and *strategies* (with errors) inspired by Joyal's category of Conway games (Joyal 1977).

Definition 2.1. A sequential game is a triple $A = (M_A, \lambda_A, P_A)$ consisting of:

— a *polarized alphabet of moves* (M_A, λ_A), that is: a set M_A whose elements are called *the moves* and a function $\lambda_A : M_A \longrightarrow \{-1, +1\}$,

— a set P_A of (finite) strings of moves, whose elements are called *the plays.*

The set of plays P_A is required to verify:

— the empty string ϵ is a play,

— every prefix of a play is a play,

— every play $s = m_1 \cdots m_k$ is non repetitive:

$$\forall i, j \in [1, ..., k], \qquad i \neq j \Rightarrow m_i \neq m_j.$$

We say that a move m is Player (resp. Opponent) when $\lambda_A(m) = +1$ (resp. $\lambda_A(m) = -1$). We often use the notation $m : +1$ (resp. $m : -1$) in text and diagrams.

Definition 2.2 (alternated). A play $s = m_1 \cdots m_k$ is *alternated* when,

$$\forall i \in [1, ..., k-1], \qquad \lambda_A(m_{i+1}) = -\lambda_A(m_i)$$

A game is alternated when all its plays are alternated.

Definition 2.3 (legal play). A play s is *legal* when:

$$\forall i \in [1, ..., k], \qquad \lambda_A(m_i) = (-1)^i$$

Alternatively, a play is legal when it is alternated, and is empty or starts by an Opponent move.

Definition 2.4 (strategy). A strategy σ of A is a set of legal plays verifying that, for every play s and moves m, n_1, n_2:

1 σ is nonempty: $\epsilon \in \sigma$,

2 σ is closed under prefix: if $s \cdot m \in \sigma$, then $s \in \sigma$,

3 σ is deterministic: if $s \cdot m \cdot n_1 \in \sigma$ and $s \cdot m \cdot n_2 \in \sigma$, then $s \cdot m \cdot n_1 = s \cdot m \cdot n_2$.

So, every sequential game A admits the strategy $\{\epsilon\}$ called the *empty strategy* of A.

This definition of strategy is slightly more general than the usual one, because it enables a strategy to withdraw and play "error" at any point of the interaction. The usual definition of strategy is recovered by our definition of *error-free* strategy σ.

Definition 2.5 (deadlock,error,fixpoint). Suppose that σ is a strategy. A play s is called maximal in σ when $s \in \sigma$ and $\forall m \in M_A, s \cdot m \notin \sigma$. Then:

— a deadlock of σ is an odd-length play $s \cdot m$ such that $s \cdot m \notin \sigma$ but $s \in \sigma$,

— an error of σ is an odd-length play $s \cdot m$ maximal in σ,

— a fixpoint of σ is an error or an even-length play of σ.

Notations: We write L_A for the set of legal plays and L_A^{even} for the set of legal plays of even length of a sequential game A. We write $\sigma : A$ when σ is a strategy of A. We write $\mathrm{even}(\sigma)$, $\mathrm{odd}(\sigma)$, $\mathrm{error}(\sigma)$ and $\mathrm{fix}(\sigma) = \mathrm{even}(\sigma) \cup \mathrm{error}(\sigma)$ for the sets of even-length plays, odd-length plays, errors and fixpoints of σ respectively.

Definition 2.6 (error-free strategy). A strategy $\sigma : A$ is error-free when $\mathrm{error}(\sigma) = \emptyset$, or equivalently, when every odd-length play $s \in \sigma$ may be extended to an even-length play of σ:

$$\forall s \in P_A, \quad s \in \mathrm{odd}(\sigma) \ \Rightarrow\ \exists m \in M_A,\ s \cdot m \in \mathrm{even}(\sigma).$$

Remark. Every strategy σ is characterized by its set of fixpoints $\mathrm{fix}(\sigma)$, as a prefix-closed completion:

$$\sigma = \mathrm{fix}(\sigma) \cup \{s \in L_A, \exists m \in M_A, s \cdot m \in \mathrm{fix}(\sigma)\} \tag{13}$$

In particular, every error-free strategy is characterized by its set $\mathrm{even}(\sigma)$ of even-length plays, which coincides with $\mathrm{fix}(\sigma)$ in that case.

Negation and tensor product of sequential games, are defined as follows. The negation of a game $A = (M_A, \lambda_A, P_A)$ is the game $A^\perp = (M_A, -\lambda_A, P_A)$ obtained by reversing the role of Player and Opponent. The *tensor product* of two games A, B, is the game $A \otimes B$ obtained by "freely interleaving" the plays of A and B; formally:

— $M_{A\otimes B} = M_A + M_B$,
— $\lambda_{A\otimes B}(\mathtt{inl}(m)) = \lambda_A(m)$ and $\lambda_{A\otimes B}(\mathtt{inr}(m)) = \lambda_B(m)$,
— a play of $A \otimes B$ is a string of moves in $M_{A\otimes B}$ such that $s_{|A} \in P_A$ and $s_{|B} \in P_B$.

where $s_{|A}$ is the projection of the string s over the subalphabet M_A of the alphabet $M_A + M_B$; and similarly for $s_{|B}$. The empty game 1 is defined as the game with an empty set of moves: $M_1 = \emptyset$.

The category $\mathcal{S}$ has sequential games as objects, and strategies of $A^\perp \otimes B$ as morphisms $A \longrightarrow B$. Composition is defined by *sequential composition* + *hiding,* and identities by the usual *copycat* strategies, see e.g. (Abramsky and Jagadeesan 1994; Hyland 1997). Note that in the presence of errors, the composition and identity laws are better defined on the sets of fixpoints, rather than on the strategies directly. This presentation by fixpoints is inspired by our work on concurrent games (Abramsky and Melliès 1999). So, the identity strategy is defined as the strategy with fixpoints:

$$\mathrm{fix}(\mathtt{id}_A) = \{s \in L^{even}_{A^\perp \otimes A}, \forall t \in L^{even}_{A^\perp \otimes A}, t \text{ is prefix of } s \Rightarrow t_{|A_1} = t_{|A_2}\}$$

where the indices $1, 2$ indicate on which component of $A_1 \multimap A_2$ the play t is projected. The composite of two strategies $\sigma : A^\perp \otimes B$ and $\tau : B^\perp \otimes C$ is the strategy of $\sigma; \tau : A^\perp \otimes C$ whose set of fixpoints is given by:

$$\{s \in L_{A^\perp \otimes C}, \exists t \in P_{A\otimes B\otimes C}, t_{|A,B} \in \mathrm{fix}(\sigma), t \in_{|B,C} \in \mathrm{fix}(\tau), t_{|A,C} = s\}. \tag{14}$$

Theorem 2.7. The category $\mathcal{S}$ is $*$-autonomous category, with monoidal unit and dualizing object the game 1 with an empty set of moves.

Proof. The main difficulty is to show that composition is associative. The proof may be adapted from the sketch of proof in (Joyal 1977) or from the proof of associativity for single-threaded strategies in arena games (Abramsky *et al.* 1998; Harmer 2000). □

Remark. Several subcategories of $\mathcal{S}$ were already considered in the litterature:

— error-free strategies between sequential games define a $*$-autonomous subcategory of $\mathcal{S}$ equivalent to the category introduced in (Joyal 1977),
— the subcategory of *alternated* games and *error-free* strategies is introduced in (Abramsky and Jagadeesan 1994) and a slight variant is studied in (Baillot *et al.* 1997).

2.2. *The subcategory $\mathcal{G}$ of negative games*

The category $\mathcal{S}$ has one drawback: it is not cartesian. For that reason, we introduce its full subcategory $\mathcal{G}$ of *negative* games.

Definition 2.8 (negative, positive games). A sequential game is negative when every play is empty or starts by an Opponent move. A game A is positive when its dual $A^\perp$ is negative.

Note that the subcategory $\mathcal{G}$ is coreflective in the category $\mathcal{S}$. The counit ξ is given by the family of strategies $\xi_A : \mathrm{neg}(A) \longrightarrow A$ below, indexed by sequential games A:

- the negative game $\mathrm{neg}(A)$ has the polarized alphabet of A,

$$(M_{\mathrm{neg}(A)}, \lambda_{\mathrm{neg}(A)}) = (M_A, \lambda_A)$$

and the plays of A which do not start by a Player move:

$$P_{\mathrm{neg}(A)} = \{\epsilon\} \;\cup\; \{s \in P_A,\;\; s = m_1 \cdots m_k \text{ and } \lambda_A(m_1) = -1\},$$

- the strategy $\xi_A : (\mathrm{neg}(A))^\perp \otimes A$ has the same set of fixpoints as the identity on $\mathrm{neg}A$:

$$\{s \in L^{even}_{(\mathrm{neg}(A))^\perp \otimes A}, \forall t \in L^{even}_{\mathrm{neg}(A)^\perp \otimes A}, t \text{ is prefix of } s \Rightarrow t_{|\mathrm{neg}(A)} = t_{|A}\}.$$

The functor $\mathrm{neg} : \mathcal{S} \longrightarrow \mathcal{G}$ is useful to define the monoidal closed structure of $\mathcal{G}$. Given two negative games A, B, the negative game $A \multimap B$ is defined as:

$$A \multimap B = \mathrm{neg}(A^\perp \otimes B).$$

Observe that the strategies of $A \multimap B$ coincide with the strategies of $A^\perp \otimes B$, thus with the morphisms $A \longrightarrow B$ of $\mathcal{G}$. It follows easily from that and from the isomorphism

$$(A \otimes B) \multimap C \cong A \multimap (B \multimap C)$$

between negative games, that:

Theorem 2.9. The category $(\mathcal{G}, \otimes, 1)$ is symmetric monoidal closed, with the functor $(- \multimap - : \mathcal{G}^{op} \times \mathcal{G} \longrightarrow \mathcal{G})$ as monoidal closure.

The category $\mathcal{G}$ of negative games is not only symmetric monoidal closed: it is also cartesian. The *cartesian product* $A\&B$ of two negative non-alternated games A, B is given by the negative game:

— $M_{A\&B} = M_A + M_B$,
— $\lambda_{A\&B}(\mathtt{inl}(m)) = \lambda_A(m)$ and $\lambda_{A\&B}(\mathtt{inr}(m)) = \lambda_B(m)$,
— a play of $A\&B$ is a string of moves in $M_{A\&B}$ such that:
  – $s_{|A} \in P_A$ and $s_{|B} = \epsilon$, or

– $s_{|B} \in P_B$ and $s_{|A} = \epsilon$.

The terminal object $\top$ of the category is the same as the monoidal unit 1: the game with an empty set of moves.

Remark. There exists a *faithful* functor from the category of negative *alternated* games and error-free strategies, to the category Rel of sets and relations, see for instance (Hyland and Schalk 1999). The functor transports every game A to the set $\mathrm{Rel}(A) = P_A$ of plays of A, and every strategy $\sigma : A \longrightarrow B$ to the relation:

$$\mathrm{Rel}(\sigma) = \{(s_1, s_2) \in P_A \times P_B, \exists s \in \sigma, s_1 = s_{|A} \text{ and } s_2 = s_{|B}\} \tag{15}$$

Functoriality breaks when one extends this definition (15) to non-alternated games. Given two strategies $\sigma : A^{\perp} \otimes B$ and $\tau : B^{\perp} \otimes C$, the property $\mathrm{Rel}(\sigma; \tau) \subset \mathrm{Rel}(\sigma); \mathrm{Rel}(\tau)$ is still verified, but the counter-example below shows that the converse inclusion does not hold. Take three sequential games A, B, C and the smallest strategies σ and τ containing the plays of $A^{\perp} \otimes B$ and $B^{\perp} \otimes C$ below:

$$\begin{array}{cccccc}
A & \xrightarrow{\sigma} & B & B & \xrightarrow{\tau} & C \\
 & & & & & m : -1 \\
 & & m' : -1 & m' : +1 & & \\
 & & n' : +1 & n' : -1 & & \\
p : -1 & & & & & \\
q : +1 & & & & & \\
 & & m'' : -1 & m'' : +1 & & \\
 & & n'' : +1 & n'' : -1 & & \\
 & & & & & n : +1
\end{array} \tag{16}$$

By definition (14) the play $s = m \cdot p \cdot q \cdot n$ is not element of the composite strategy $\sigma; \tau$: formally, because the play s is not alternated in $A^{\perp} \otimes C$; intuitively, because $s \in \sigma; \tau$ would mean that the *two* threads $m \cdot m' \cdot n' \cdot m'' \cdot n'' \cdot n$ and $p \cdot q$ are running simultaneously during the interaction of σ and τ. On the other hand, each strategy σ, τ and $\sigma; \tau$ induces by (15) a relation between the plays of A, B and C:

$$\mathrm{Rel}(\sigma) : P_A \longrightarrow P_B, \qquad \mathrm{Rel}(\tau) : P_B \longrightarrow P_C, \qquad \mathrm{Rel}(\sigma; \tau) : P_A \longrightarrow P_C.$$

The relations $\mathrm{Rel}(\sigma)$ and $\mathrm{Rel}(\tau)$ contain respectively the pairs $(p \cdot q, m' \cdot n' \cdot m'' \cdot n'')$ and $(m' \cdot n' \cdot m'' \cdot n'', p \cdot n)$. It follows that the pair $(p \cdot q, m \cdot n)$ is element of the composite $\mathrm{Rel}(\sigma); \mathrm{Rel}(\tau)$ but not element of the relation $\mathrm{Rel}(\sigma; \tau)$. Thus, definition (15) does not define a functor $\mathcal{S} \longrightarrow \mathrm{Rel}$. The example is easily adapted to show that the definition is not functorial from the subcategory $\mathcal{G}$ either.

Functoriality of definition (15) may be regained by shifting to the category of asynchronous games and innocent strategies defined in (Melliès 2004), see also section 5. It will be shown that innocent strategies are *positional*, just like the concurrent strategies formulated in (Abramsky and Melliès 1999). It follows that equation (15) defines a *faithful* functor from the category of innocent strategies to the category Rel.

	$!^{\mathrm{inc}}\mathsf{bool}$	$\longrightarrow$	$!^{\mathrm{inc}}\mathsf{bool}$	$\otimes$	$!^{\mathrm{inc}}\mathsf{bool}$
(1)			$(*, 0)$		
(2)	$(*, 0)$				
(3)	$(\mathsf{true}, 0)$				
(4)			$(\mathsf{true}, 0)$		
(5)					$(*, 0)$
(6)	$(*, 1)$				
(7)	$(\mathsf{false}, 1)$				
(8)					$(\mathsf{false}, 0)$

Fig. 1. A typical play of the comultiplication of $!^{\mathrm{inc}}\mathsf{bool}$

2.3. *Hyland's incremental indexing in $\mathcal{G}$*

A model of intuitionistic linear logic is introduced in (Hyland 1997) based on negative alternated games and the exponential $!^{\mathrm{inc}}$. In the game $!^{\mathrm{inc}}A$, each thread is indexed by an index $i \in \mathbb{N}$ which is incremented each time a new thread of A is opened by Opponent. We adapt this construction to our category $\mathcal{G}$ of negative Conway games.

Definition 2.10 (Hyland). Suppose that A is a negative non-alternated game. Then, the negative non-alternated game $!^{\mathrm{inc}}A$ is defined as:

- $M_{!^{\mathrm{inc}}A} = M_A \times \mathbb{N}$,
- $\lambda_{!^{\mathrm{inc}}A}(m, i) = \lambda_A(m)$,
- a string s on the alphabet $M_{!^{\mathrm{inc}}A}$ is element of $P_{!^{\mathrm{inc}}A}$ iff for every index $i \in \mathbb{N}$:
  - its projection $s_{|i}$ over the i-th copy of A, is element of P_A,
  - $s_{|i} = \epsilon \Rightarrow s_{|i+1} = \epsilon$.

The game $!^{\mathrm{inc}}A$ defines a commutative comonoid in $\mathcal{G}$ when equipped with the two strategies

$$d_A^{\mathrm{inc}} :!^{\mathrm{inc}}A \multimap !^{\mathrm{inc}}A \otimes !^{\mathrm{inc}}A \qquad\qquad e_A^{\mathrm{inc}} :!^{\mathrm{inc}}A \multimap 1$$

defined below:

- the strategy d_A^{inc} contains a legal play s of $!^{\mathrm{inc}}A_1 \multimap !^{\mathrm{inc}}A_2 \otimes !^{\mathrm{inc}}A_3$ precisely when all the even-length prefixes of s verify property (*),
- the morphism e_A^{inc} is defined as the empty strategy $\{\epsilon\}$ of $!^{\mathrm{inc}}A \multimap 1$.

A play t of $!^{\mathrm{inc}}A_1 \multimap !^{\mathrm{inc}}A_2 \otimes !^{\mathrm{inc}}A_3$ verifies property (*) when its projections t_1 over $!^{\mathrm{inc}}A_1$ and t_{23} over $!^{\mathrm{inc}}A_2 \otimes !^{\mathrm{inc}}A_3$ are equal, modulo renaming of every move $(\mathtt{inl}(a), i)$ and $(\mathtt{inr}(a), i)$ in t_{23} by a move $(a, \varphi(inl(i)))$ and $(a, \varphi(\mathtt{inr}(i)))$ respectively, for some injective map $\varphi : \mathbb{N} + \mathbb{N} \longrightarrow \mathbb{N}$.

We illustrate in figure 1 the definition of the comultiplication strategy d_A^{inc} on the boolean game bool. The game admits three moves $* : -1$, $\mathsf{true} : +1$ and $\mathsf{false} : +1$, and three nonempty plays: $*$ and $* \cdot \mathsf{false}$ and $* \cdot \mathsf{true}$.

Proposition 2.11. The category $\mathcal{G}$ equipped with Hyland's exponential defines a model of intuitionistic linear logic over the category $\mathcal{G}$, see (Hyland 1997; Melliès 2002):

- the category $\mathcal{G}$ is symmetric monoidal closed, and cartesian,
- the triple $(!^{\mathrm{inc}}A, d_A^{\mathrm{inc}}, e_A^{\mathrm{inc}})$ defines a commutative comonoid in $(\mathcal{G}, \otimes, 1)$,

- there exists a family of *dereliction* strategies $!^{\mathrm{inc}}A \multimap A$ enjoying the universality property (1) mentioned in the introduction,
- there exists comonoidal isomorphisms:

$$!^{\mathrm{inc}}(A\&B) \cong !^{\mathrm{inc}}A \otimes !^{\mathrm{inc}}B \qquad\qquad !^{\mathrm{inc}}\top \cong \mathbf{1}.$$

We explain in the introduction and (Melliès 2002) that the category of arenas and single-threaded strategies (Abramsky *et al.* 1998) embeds fully and faithfully (as a ccc) inside the co-kleisli category associated to the (error-free) model. It is worth noting that this reformulation of arena games works with *non-alternated* games, not with *alternated* games. Why? Well, alternation of justified plays in arena games is not preserved by projection in general. Take for instance the boolean arena $\mathbb{B}$, with two moves $\mathsf{true}, \mathsf{false} : +1$ justified by an initial move $* : -1$. The justified play $*_2 \cdot *_1 \cdot \mathsf{true}_1 \cdot \mathsf{true}_2 \cdot \mathsf{false}_1 \cdot \mathsf{false}_2$ is alternated in $\mathbb{B}_1 \multimap \mathbb{B}_2$, but its projection on each component $\mathbb{B}$ is the non-alternated justified play $* \cdot \mathsf{true} \cdot \mathsf{false}$.

2.4. *Permuting moves requires a liberal indexing of threads*

We indicate briefly why we need to shift away from Hyland's exponential in order to develop an asynchronous theory of games. Consider the play

$$s = (m, 0) \cdot (n, 0) \cdot (m, 1) \cdot (n, 1)$$

in the sequential game $!^{\mathrm{inc}}A$ introduced above. Suppose that one wants to permute the first move $(n, 1)$ of the second thread, before the last move $(n, 0)$ of the first thread. Then, the resulting sequence

$$s' = (m, 0) \cdot (m, 1) \cdot (n, 0) \cdot (n, 1)$$

is a play of $!A$ which starts the first and second threads of A in a row. Now, permuting $(m, 1)$ before $(m, 0)$ induces the sequence

$$s'' = (m, 1) \cdot (m, 0) \cdot (n, 0) \cdot (n, 1)$$

which is *not* a play of $!A$, because by definition, a play of $!A$ cannot start the *second* thread of A before the *first* one.

This means that we need to relax our definition, and enable a play of $!A$ to interleave the threads of A in any order. The comultiplication and counit maps

$$d :!A \longrightarrow !A \otimes !A \qquad\qquad e :!A \longrightarrow 1$$

are defined as follows:

- by a copy-cat strategy between each thread i of the left component of $!A \otimes !A$, and the thread $2i$ of the codomain $!A$;
- by a copy-cat strategy between each thread j of the right component of $!A \otimes !A$, and the thread $2j + 1$ of the codomain $!A$;
- by the empty strategy $\{\epsilon\}$ between $!A$ and the unit game 1.

There is a price to pay for relaxing Hyland's definition: the game $!A$ is not a commutative comonoid anymore in the category $\mathcal{G}$. In fact, the comultiplication operation is not associative, that is, the diagram below does not commute:

$$\begin{tikzcd}
!A \arrow[r, "d"] \arrow[d, "d"'] & !A\otimes!A \arrow[r, "d\otimes!A"] & (!A\otimes!A)\otimes!A \arrow[d, "\alpha"] \\
!A\otimes!A \arrow[rr, "!A\otimes d"] & & !A \otimes (!A\otimes!A)
\end{tikzcd} \tag{17}$$

2.5. *Towards orbital games: AJM games*

With diagram (17), we are back at the position of Abramsky, Jagadeesan and Malacaria (AJM) in the early nineties, when the three authors decided to extract a game semantics from Girard's Geometry of Interaction (GoI). The role of the GoI is played in our case by the pseudo-comonoid $!A$: keeping the definition of $!A$ as it is (pseudo-comonoidal) induces a semantics of proofs which is preserved only by particular cut-elimination steps, called *special* in (Girard 1989), see also (Laurent 2001).

So, extracting a game semantics from the Geometry of Interaction (or equivalently from a *liberal, non incremental* indexing of threads) amounts grossly to making diagram (17) commutative. Observe that the two sides σ and τ of diagram (17) are different, but only modulo an automorphism on the thread indices. It is therefore tempting to introduce an equivalence relation $\approx$ on plays, and to deduce from that a partial equivalence relation $\approx$ on the strategies, in such a way that $\sigma \approx \tau$ in our example. This is precisely how Abramsky, Jagadeesan and Malacaria proceed in (Abramsky *et al.* 1994). We recall below their notion of games.

Definition 2.12 (AJM game). An AJM game $(A, \approx_A^{\mathrm{AJM}})$ is a negative alternated game A equipped with an equivalence relation $\approx_A^{\mathrm{AJM}}$ on plays, which is required to verify, for every plays $s, t \in P_A$, and moves $m, n \in M_A$:

— $s \approx_A^{\mathrm{AJM}} t$ implies that the two plays s and t are of same length,

— $s \cdot m \approx_A^{\mathrm{AJM}} t \cdot n$ implies that $s \approx_A^{\mathrm{AJM}} t$,

— $s \approx_A^{\mathrm{AJM}} t$ and $s \cdot m$ is a play implies that there exists a move n such that $t \cdot n$ is a play and $s \cdot m \approx_A^{\mathrm{AJM}} t \cdot n$.

Remark. We forget the Question/Answer discipline of the original definition of AJM games (Abramsky *et al.* 1994) because this discipline has little to do with the discussion here.

Abramsky, Jagadeesan and Malacaria define for every AJM game A, a preorder $\precapprox_A^{\mathrm{AJM}}$ between the *error-free* strategies of A. Intuitively, $\sigma \precapprox^{\mathrm{AJM}} \tau$ means that every interaction of σ may be simulated by an interaction of τ, modulo the equivalence relation $\approx_A^{\mathrm{AJM}}$ between plays. The definition goes as follows.

Definition 2.13 ($\precapprox^{\mathrm{AJM}}$). Two error-free strategies σ and τ verify $\sigma \precapprox_A^{\mathrm{AJM}} \tau$ when for every plays $s, s' \in L_A^{even}$ of even-length, and every moves $m, n, m' \in M_A$:

$$s \cdot m \cdot n \in \sigma, s' \in \tau, s \cdot m \approx_A^{\mathrm{AJM}} s' \cdot m' \Rightarrow \exists n', s \cdot m \cdot n \approx_A^{\mathrm{AJM}} s' \cdot m' \cdot n', s' \cdot m' \cdot n' \in \tau.$$

The partial equivalence relation $\approx^{\mathrm{AJM}}$ between the strategies of A follows from the preorder $\precapprox^{\mathrm{AJM}}$ in the usual way:

Definition 2.14 ($\approx^{\mathrm{AJM}}$). Two error-free strategies σ and τ verify $\sigma \approx_A^{\mathrm{AJM}} \tau$ when $\sigma \precapprox_A^{\mathrm{AJM}} \tau$ and $\tau \precapprox_A^{\mathrm{AJM}} \sigma$.

Abramsky, Jagadeesan and Malacaria deduce from these definitions a model of intuitionistic linear logic, which we discuss further in section 4, after introducing in section 3 our orbital model of games.

3. Orbital games

In this section, we introduce the notion of orbital games, and define two categories $(\mathcal{S}/\approx)$ and $(\mathcal{G}/\approx)$ of orbital games and strategies with errors.

3.1. *The definition of an orbital game*

An orbital game is a sequential game (M_A, λ_A, P_A) as formulated in definition 2.1 equipped with

— two groups G_A and H_A,
— a left group action on moves: $G_A \times M_A \longrightarrow M_A$,
— a right group action on moves: $M_A \times H_A \longrightarrow M_A$,

verifying that the left and right actions commute:

$$\forall m \in M_A, \forall g \in G_A, \forall h \in H_A, \qquad (g \bullet m) \bullet h = g \bullet (m \bullet h).$$

The action on moves induces an action on strings of moves. Given two elements $g \in G_A$ and $h \in H_A$ and a string of moves $s = m_1 \cdots m_k$, the string of moves $g \bullet s \bullet h$ is defined by:

$$g \bullet s \bullet h = (g \bullet m_1 \bullet h) \cdots (g \bullet m_k \bullet h).$$

We require that every orbital game verifies *four* coherence axioms; namely, that for every $g \in G_A$ and $h \in H_A$:

(i) $\lambda_A(g \bullet m \bullet h) = \lambda_A(m)$ for every move $m \in M_A$,
(ii) $g \bullet s \bullet h \in P_A$ when $s \in P_A$,
(iii) $m = g \bullet m$ when $s \cdot m \in P_A$ and $s = g \bullet s$ and $\lambda_A(m) = -1$,
(iv) $m = m \bullet h$ when $s \cdot m \in P_A$ and $s = s \bullet h$ and $\lambda_A(m) = +1$.

The two first axioms are elementary: axiom (i) states that the group action preserves the polarities of moves, and axiom (ii) ensures that the group action on moves lifts to a group action on plays. The axioms (iii) and (iv) are dual, and more interesting. Take axiom (iii) for instance. It indicates that the left action of an element $g \in G_A$ which keeps invariant a play s keeps also invariant any play $s \cdot m$ in which the move m is Opponent. Let us explain this informally. The left action g is meant to be an action on the thread indices chosen by Player; and every such thread index which appears in the Opponent move m appears already in one move of the play s, typically in the Player move n which "justifies" the fact that the move m induces a play $s \cdot m$. See also section 5.5 for a related axiomatics, this time on asynchronous games.

3.2. *An equivalence relation $\approx$ on plays*

Definition 3.1 ($\approx$). The equivalence relation $\approx_A \subset P_A \times P_A$ between plays $s, t \in P_A$ is defined as

$$s \approx_A t \iff \exists (g,h) \in G_A \times H_A, \qquad t = g \bullet s \bullet h.$$

Remark. Every negative alternated orbital game A induces an AJM game $U(A)$ with same underlying sequential game, and $\approx^{\mathrm{AJM}}_{U(A)}$ defined as $\approx_A$. Let us check definition 2.12. (1) $t = g \bullet s \bullet h$ implies that the two plays s and t are of same length, (2) $t \cdot n = g \bullet (s \cdot m) \bullet h$ implies that $t = g \bullet s \bullet h$, and (3) $t = g \bullet s \bullet h$ and $s \cdot m$ is a play implies that $t \cdot n = g \bullet (s \cdot m) \bullet h$ is a play for $n = (g \bullet m \bullet h)$.

3.3. *The strategies*

The strategies of an orbital game A are defined as the strategies of its underlying sequential game, formulated in definition 2.4.

3.4. *Two pers $\approx^{\mathrm{INV}}$ and $\approx^{\mathrm{SIM}}$ on strategies of an orbital game*

We introduce two partial equivalence relations (pers) on the strategies of an orbital game A:

- the per $\approx^{\mathrm{INV}}_A$ is defined by a group-theoretic notion of *bi-invariance* inspired by geometry, and the notion of invariant variety wrt. a group action. It is interesting that the usual geometric notion of invariance mutates here to a bi-invariance property. This may have to do with the interactive nature of the logical universe.
- the per $\approx^{\mathrm{SIM}}_A$ is defined by simulation in the spirit of AJM games, which generalizes the usual definition to non alternated games, and error-aware strategies.

The two definitions are different in nature, but we will show in section 3.5 that they coincide.

3.4.1. *The definition by bi-invariance.* Suppose that σ, τ are two strategies of an orbital game A.

Definition 3.2 ($\precapprox^{\mathrm{INV}}, \approx^{\mathrm{INV}}$). We write $\sigma \precapprox^{\mathrm{INV}}_A \tau$ when

$$\forall s \in \sigma, \quad \forall h \in H_T, \quad \exists g \in G_T, \quad g \bullet s \bullet h \in \tau.$$

We write $\sigma \approx^{\mathrm{INV}}_A \tau$ when $\sigma \precapprox^{\mathrm{INV}}_A \tau \precapprox^{\mathrm{INV}}_A \sigma$.

3.4.2. *The definition by simulation.* We adapt to orbital games the original definitions 2.13 and 2.14 of self-equivalence in AJM games.

Definition 3.3 ($\precapprox^{\mathrm{SIM}}, \approx^{\mathrm{SIM}}$). We write $\sigma \precapprox^{\mathrm{SIM}}_A \tau$ when for every plays $s, s' \in P_A$ and move $m \in M_A$ verifying:

$$s \approx_A s' \quad \text{and} \quad s \cdot m \in \sigma \quad \text{and} \quad s' \in \tau,$$

the property (18) holds when $\lambda_A(m) = -1$,

$$\forall m' \in M_A, \qquad s \cdot m \approx_A s' \cdot m' \Rightarrow s' \cdot m' \in \tau, \tag{18}$$

and the property (19) holds when $\lambda_A(m) = +1$,

$$\exists m' \in M_A, \qquad s \cdot m \approx_A s' \cdot m' \text{ and } s' \cdot m' \in \tau. \tag{19}$$

We write $\sigma \approx_A^{\mathrm{SIM}} \tau$ when $\sigma \precapprox_A^{\mathrm{SIM}} \tau \precapprox_A^{\mathrm{SIM}} \sigma$.

Remark. In section 3.2 we observed that every negative alternated orbital game A induces an AJM game $(A, \approx_A)$. Note that the preorder $\precapprox_A^{\mathrm{AJM}}$ and the per $\approx_A^{\mathrm{AJM}}$ defined for the AJM game $(A, \approx_A)$ in section 2.5 coincides with the preorder $\precapprox_A^{\mathrm{INV}}$ and the per $\approx_A^{\mathrm{INV}}$ restricted to *error-free* strategies of A.

3.5. *Equivalence by bi-invariance = equivalence by simulation*

We prove here that the two formulations of self-equivalence in section 3.4 coincide. That is, for every two strategies σ, τ of an orbital game:

$$\sigma \precapprox^{\mathrm{INV}} \tau \iff \sigma \precapprox^{\mathrm{SIM}} \tau.$$

We prove the claim in two preliminary lemmas, and a proposition. Throughout the section, A denotes an orbital game.

Lemma 3.4. Suppose that s is a play of A, and that $g \in G_A$ and $h \in H_A$. Then,

$$s = g \bullet s \bullet h \Rightarrow s = g \bullet s = s \bullet h. \tag{20}$$

Proof. Suppose that property (20) is proved for every play $s \in P_A$ of length k, and every elements $g \in G_A$ and $h \in H_A$. We establish property (20) for any play $s \cdot m$ of length $k+1$. Suppose that $s \cdot m = g \bullet (s \cdot m) \bullet h$. Then, $s = g \bullet s \bullet h$ and $m = g \bullet m \bullet h$. The equalities $s = g \bullet s = s \bullet h$ hold by induction hypothesis. We proceed by case analysis on the polarity of m. Suppose that $\lambda_A(m) = +1$. Axiom (iv) of orbital games and $s = g \bullet s$ and $s \cdot m \in P_A$ imply that $m = g \bullet m$. From this, it follows that:

$$s \cdot m = (g \bullet s) \cdot (g \bullet m) = g \bullet (s \cdot m). \tag{21}$$

This proves one part of property (20). The other part follows from (21) and $s \cdot m = g \bullet (s \cdot m) \bullet h$:

$$s \cdot m = (g \bullet (s \cdot m)) \bullet h = (s \cdot m) \bullet h.$$

We proceed similarly when $\lambda_A(m) = -1$. We conclude lemma 3.4 □

Lemma 3.5. Suppose that s is a play of A, that σ is a strategy of A, and that $g_1, g_2 \in G_A$. Then,

$$g_1 \bullet s \in \sigma \text{ and } g_2 \bullet s \in \sigma \;\Rightarrow\; g_1 \bullet s = g_2 \bullet s.$$

Proof. By induction on the length of s. Suppose that the property is proved for every play s of length less than k. Consider a play $s \cdot m$ of length $k+1$, such that $g_1 \bullet (s \cdot m) \in \sigma$

and $g_2 \bullet (s \cdot m) \in \sigma$. The equality $g_1 \bullet s = g_2 \bullet s$ holds by induction hypothesis. We proceed by case analysis on the polarity of m. Case 1: $\lambda_A m = -1$. Then, axiom (iii) of orbital games and $s = (g_1^{-1} g_2) \bullet s$ and $s \cdot m \in P_A$ implies that $m = (g_1^{-1} g_2) \bullet m$, or equivalently, that $g_1 \bullet m = g_2 \bullet m$. It follows that $g_1 \bullet (s \cdot m) = g_2 \bullet (s \cdot m)$. Case 2: $\lambda_A(m) = +1$. In that case, $g_1 \bullet (s \cdot m) = g_2 \bullet (s \cdot m)$ follows from $g_1 \bullet s = g_2 \bullet s$ and determinism of σ. We conclude. □

Proposition 3.6. Suppose that σ and τ are two strategies of A. Then,

$$\sigma \precapprox_A^{\mathrm{INV}} \tau \iff \sigma \precapprox_A^{\mathrm{SIM}} \tau$$

Proof. ($\Rightarrow$) Suppose that $\sigma \precapprox_A^{\mathrm{INV}} \tau$; and that $s \cdot m \in \sigma$, $s' \in \tau$ and $s \approx_A s'$ for some plays s, s' and move m of A. We proceed by case analysis on the polarity of m.

(A) Suppose that $\lambda_A(m) = +1$. The equivalence $s \approx_A s'$ means that there exists $g \in G_A$ and $h \in H_A$ such that

$$s' = g \bullet s \bullet h$$

By $\sigma \precapprox_A^{\mathrm{INV}} \tau$ and $s \cdot m \in \sigma$ and $h \in H_A$, there exists $g' \in G_A$ such that

$$g' \bullet (s \cdot m) \bullet h \in \tau \tag{22}$$

As prefix of the play $g' \bullet (s \cdot m) \bullet h \in \tau$, the play $g' \bullet s \bullet h$ is element of the strategy τ. On the other hand, the play $s' = g \bullet s \bullet h$ is also element of τ. It follows from lemma 3.5 that

$$g \bullet s \bullet h = g' \bullet s \bullet h \tag{23}$$

Let m' denote the move $(g' \bullet m \bullet h)$. It follows from (23) that

$$s' \cdot m' = (g \bullet s \bullet h) \cdot (g' \bullet m \bullet h) = (g' \bullet s \bullet h) \cdot (g' \bullet m \bullet h) = g' \bullet (s \cdot m) \bullet h$$

We obtain that $s \cdot m \approx_A s' \cdot m'$ and that $s' \cdot m' \in \tau$ from (22). We conclude that the move m' verifies the property required by (18).

(B) Now, suppose that $\lambda_A(m) = -1$ and let m' be any move such that $s \cdot m \approx_A s' \cdot m'$. The equivalence $s \cdot m \approx_A s' \cdot m'$ means that there exists $g \in G_A$ and $h \in H_A$ such that

$$s' \cdot m' = g \bullet (s \cdot m) \bullet h$$

By $\sigma \precapprox_A^{\mathrm{INV}} \tau$ and $s \cdot m \in \sigma$ and $h \in H_A$, there exists $g' \in G_A$ such that

$$g' \bullet (s \cdot m) \bullet h \in \tau \tag{24}$$

We claim that

$$g \bullet (s \cdot m) \bullet h = g' \bullet (s \cdot m) \bullet h$$

The two plays $s' = g \bullet s \bullet h$ and $g' \bullet s \bullet h$ are element of τ. It follows from lemma 3.5 that

$$g \bullet s \bullet h = g' \bullet s \bullet h \tag{25}$$

Axiom (iii) of orbital games and $\lambda_A(m) = -1$ imply that $g \bullet m \bullet h = g' \bullet m \bullet h$. This proves the claim, and shows that m' verifies the property required by (19):

$$s' \cdot m' = g' \bullet (s \cdot m) \bullet h \in \tau$$

This concludes the proof by induction that $\sigma \precapprox_A^{\mathrm{SIM}} \tau$.

($\Leftarrow$) Suppose that $\sigma \precsim_A^{\mathrm{SIM}} \tau$, that $s \in \sigma$ and $h \in H$. We prove by induction on the length of s that there exists $g \in G$, such that $g \centerdot s \centerdot h \in \tau$. The property is obvious when s is the empty play. Suppose now that the property is established for $s \in \sigma$; and let us prove it for the play $s \cdot m \in \sigma$. By induction hypothesis, there exists $g \in G_A$ such that $g \centerdot s \centerdot h \in \tau$. Let s' denote $g \centerdot s \centerdot h$. We proceed by case analysis on the polarity of m.

Suppose that $\lambda_A(m) = -1$. The proof is particularly easy in that case. By axiom (ii) on orbital games, the string $g \centerdot (s \cdot m) \centerdot h$ is a play of A. Besides, $s \cdot m \approx_A g \centerdot (s \cdot m) \centerdot h$. It follows from $\sigma \precsim_A^{\mathrm{SIM}} \tau$ and $\lambda_A(m) = -1$ that $s' \cdot m' \in \tau$. This is the content of (18). We conclude that $g \centerdot (s \cdot m) \centerdot h \in \tau$.

Now, suppose that $\lambda_A(m) = +1$. Then, it follows from $\sigma \precsim_A^{\mathrm{SIM}} \tau$ and $s \cdot m \in \sigma$, that there exists a move $m' \in M_A$ such that $s' \cdot m' \in \tau$ and $s \cdot m \approx_A s' \cdot m'$. This is the content of (19). The equivalence $s \cdot m \approx_A s' \cdot m'$ means that there exists $g' \in G_A$ and $h' \in H_A$ such that

$$s' \cdot m' = g' \centerdot (s \cdot m) \centerdot h'$$

We claim that

$$g' \centerdot (s \cdot m) \centerdot h' = g' \centerdot (s \cdot m) \centerdot h \tag{26}$$

The equality

$$g' \centerdot s \centerdot h' = g \centerdot s \centerdot h$$

follows from $s' = g \centerdot s \centerdot h$ and $s' = g' \centerdot s \centerdot h'$. By lemma 3.4, the equality implies that $s \centerdot h = s \centerdot h'$ ($\bigstar$) which implies in turn, by axiom (iv) of orbital games, that $m \centerdot h = m \centerdot h'$ ($\bigstar\bigstar$). We conclude that:

$$\begin{array}{rcll} g' \centerdot (s \cdot m) \centerdot h' & = & (g' \centerdot s \centerdot h') \cdot (g' \centerdot m \centerdot h') & \text{def. of the action on plays} \\ & = & (g' \centerdot s \centerdot h) \cdot (g' \centerdot m \centerdot h) & \text{equations } (\bigstar) \text{ and } (\bigstar\bigstar) \\ & = & g' \centerdot (s \cdot m) \centerdot h & \text{def. of the action on plays} \end{array}$$

This proves our claim (26), and exhibits an element $g' \in G_A$ such that $g' \centerdot (s \cdot m) \centerdot h \in \tau$. This concludes our proof by induction that $\sigma \precsim_A^{\mathrm{INV}} \tau$. ☐

3.6. *Constructions on orbital games*

In section 2, we introduced several constructions on sequential games:

— the dual $A^{\perp}$ and the tensor product $A \otimes B$ of two sequential games A, B,
— the negative game $\mathrm{neg}(A)$ associated to a sequential game,
— the cartesian product $A \& B$ of two negative games,
— Hyland's incremental exponential $!^{\mathrm{inc}} A$ of a negative game.

These constructions induce the $*$-autonomous category $\mathcal{S}$ of sequential games, and the cartesian symmetric monoidal closed category $\mathcal{G}$ of negative games. Besides, the exponential modality $!^{\mathrm{inc}}$ induces a model of intuitionistic linear logic over $\mathcal{G}$.

Here, we adapt all these constructions to orbital games. Our constructions are *conservative* in the sense that they define the same underlying sequential games as before. The only exception is the exponential $!^{\mathrm{inc}}$ for which we shift to the more liberal exponential $!$ indicated in section 2.4.

Opposite group action. We recall that the opposite of a group $(G, \times, e)$ is the group denoted $(G^{op}, \times^{op}, e)$ with same elements as G, and with product law defined as:

$$\forall g_1, g_2 \in G^{op}, \qquad g_1 \times^{op} g_2 = g_2 \times g_1$$

Observe that every right group action $\centerdot : X \times G \longrightarrow X$ of G over a set X induces a left group action $\centerdot^{op} : G^{op} \times X \longrightarrow X$ of G^{op} over the set X, defined as:

$$\forall g \in G^{op}, \forall x \in X, \qquad g \centerdot^{op} x = x \centerdot g$$

and conversely, every left group action $\centerdot$ of G over X induces a right group action $\centerdot^{op}$ of G^{op} over X.

Dual. The *dual* of an orbital game (A, G_A, H_A) is defined as the orbital game below:
— its underlying sequential game $A^{\perp}$ is the dual of the sequential game A,
— its left group action is the opposite $\centerdot^{op}$ of the right group action $\centerdot$ of A,
— its right group action is the opposite $\centerdot^{op}$ of the left group action $\centerdot$ of A.

So, $G_{A^{\perp}} = (H_A)^{op}$ and $H_{A^{\perp}} = (G_A)^{op}$, and:

$$\forall g \in G_{A^{\perp}}, \forall h \in H_{A^{\perp}}, \forall m \in M_{A^{\perp}}, \qquad g \centerdot^{op} m \centerdot^{op} h = h \centerdot m \centerdot g$$

Tensor product. The tensor product of two orbital games (A, G_A, H_A) and (B, G_B, H_B) is defined as the orbital game below:
— its underlying sequential game is the tensor product $A \otimes B$ of the two sequential games A and B,
— the left action of an element (g_A, g_B) of the group $G_{A\otimes B} = G_A \times G_B$ over $M_{A\otimes B} = M_A + M_B$ is defined as:

$$(g_A, g_B) \centerdot_{A\otimes B} \texttt{inl}(m) = \texttt{inl}(g_A \centerdot m) \;\; (g_A, g_B) \centerdot_{A\otimes B} \texttt{inr}(m) = \texttt{inr}(g_B \centerdot m)$$

— the right action of an element (h_A, h_B) of the group $H_{A\otimes B} = H_A \times H_B$ over $M_{A\otimes B} = M_A + M_B$ is defined as:

$$\texttt{inl}(m) \centerdot_{A\otimes B} (h_A, h_B) = \texttt{inl}(m \centerdot h_A) \;\; \texttt{inr}(m) \centerdot_{A\otimes B} (h_A, h_B) = \texttt{inr}(m \centerdot h_B)$$

Negative orbital game associated to an orbital game. An orbital game is negative when its underlying sequential game is negative, that is, when every non-empty play starts by an Opponent move.

To every orbital game (A, G_A, H_A) we associate the *negative* orbital game $(\mathrm{neg}(A), G_A, H_A)$ with underlying sequential game $\mathrm{neg}(A)$ as defined in section 2.2, and same left and right actions on moves as in the orbital game A.

Cartesian product. The cartesian product of two negative orbital games (A, G_A, H_A) and (B, G_B, H_B) is defined as the orbital game below:
— its underlying sequential game is the cartesian product $A\&B$ of the two sequential games A and B,

- the left action of an element (g_A, g_B) of the group $G_{A\&B} = G_A \times G_B$ over $M_{A\&B} = M_A + M_B$ is defined as:

$$(g_A, g_B) \bullet_{A\&B} \mathtt{inl}(m) = \mathtt{inl}(g_A \bullet m) \quad (g_A, g_B) \bullet_{A\&B} \mathtt{inr}(m) = \mathtt{inr}(g_B \bullet m)$$

- the right action of an element (h_A, h_B) of the group $H_{A\&B} = H_A \times H_B$ over $M_{A\&B} = M_A + M_B$ is defined as:

$$\mathtt{inl}(m) \bullet_{A\&B} (h_A, h_B) = \mathtt{inl}(m \bullet h_A) \quad \mathtt{inr}(m) \bullet_{A\&B} (h_A, h_B) = \mathtt{inr}(m \bullet h_B)$$

Exponentials. The exponential $!A$ of a negative orbital game is defined as the negative orbital game below:

- its underlying sequential game is defined as follows:
  - $M_A = M_A \times \mathbb{N}$,
  - $\lambda_{!A}(m, i) = \lambda_A(m)$,
  - a string s over the alphaber $M_{!A}$ is a play of $!A$ precisely when $s \upharpoonright i$ is a play of A for every index $i \in \mathbb{N}$,
- the group $G_{!A} = \Pi_{i \in \mathbb{N}} G_A$ has elements $(g_i)_{i \in \mathbb{N}}$ the families of elements of G_A indexed by natural numbers. Product of two families $g = (g_i)_{i \in \mathbb{N}}$ and $g' = (g'_i)_{i \in \mathbb{N}}$ is defined pointwise:

$$gg' = (g_i g'_i)_{i \in \mathbb{N}}$$

- the group $H_{!A}$ is defined by wreath product. It has elements the pairs $((g_i)_{i \in \mathbb{N}}, \pi)$ where:
  - $(g_i)_{i \in \mathbb{N}}$ is a family of elements of $g_i \in H_A$, indexed by natural numbers $i \in \mathbb{N}$,
  - π is a permutation of $\mathbb{N}$.

  Product is defined as

$$((g_i)_{i \in \mathbb{N}}, \pi)((h_i)_{i \in \mathbb{N}}, \pi') = ((h_{\pi(i)} g_i)_{i \in \mathbb{N}}, \pi' \circ \pi).$$

- the left action of an element $g = (g_i)_{i \in \mathbb{N}}$ of $G_{!A}$ on a move $(m, i) \in M_{!A}$ is defined pointwise:

$$g \bullet (m, i) = (g_i \bullet m, i)$$

- the right action of an element $g = ((g_i)_{i \in \mathbb{N}}, \pi)$ of $H_{!A}$ on a move $(m, i) \in M_{!A}$ is defined as:

$$(m, i) \bullet g = (m \bullet g_i, \pi(i))$$

Constants. The units 1 and $\top$ are defined as the orbital group with empty set of moves, and trivial groups.

3.7. *A $*$-autonomous category $(\mathcal{S}/\approx)$ of orbital games and strategies with errors*

The coincidence of $\precapprox^{\mathrm{INV}}$ and $\precapprox^{\mathrm{SIM}}$ established in proposition 3.6 enables to prove that the relation $\precapprox^{\mathrm{INV}}$ is preserved by composition in the category $\mathcal{S}$.

Lemma 3.7. Suppose that A, B, C, D are sequential games, that $\rho, \sigma, \tau, \upsilon$ are morphisms in the category $\mathcal{S}$ with source and target as indicated below:

$$\begin{tikzcd} A \arrow[r, "\rho"] & B \arrow[r, bend left, "\sigma"] \arrow[r, bend right, "\tau"'] & C \arrow[r, "\upsilon"] & D \end{tikzcd}$$

Then,

$$\sigma \precapprox^{\mathrm{INV}}_{B^{\perp} \otimes C} \tau \Rightarrow \rho; \sigma; \upsilon \precapprox^{\mathrm{INV}}_{A^{\perp} \otimes D} \rho; \tau; \upsilon$$

Proof. Once recognized that $\precapprox^{\mathrm{INV}}$ and $\precapprox^{\mathrm{SIM}}$ coincide, the proof proceeds as in Abramsky et al (Abramsky *et al.* 1994). ☐

The category $(\mathcal{S}/\!\approx)$ of orbital games and strategies with errors, is defined as the category $\mathcal{S}$ quotiented by the partial equivalence relation $\approx^{\mathrm{INV}}$ on strategies. Its objects are the objects of $\mathcal{S}$ and its morphisms $A \longrightarrow B$ are the equivalence classes of $\approx^{\mathrm{INV}}_{A^{\perp} \otimes B}$. It is not difficult to prove that:

Theorem 3.8. The category $(\mathcal{S}/\!\approx)$ is $*$-autonomous, with monoidal unit and dualizing object the game 1 with an empty set of moves.

Remark. Note that $\mathcal{S}$ embeds fully and faithfully in thecategory $(\mathcal{S}/\!\approx)$ by identifying every sequential game (M_A, λ_A, P_A) to the orbital game $(M_A, \lambda_A, P_A, G_A, H_A)$ with trivial groups $G_A = H_A = \{e\}$.

3.8. *An orbital model* $(\mathcal{G}/\!\approx)$ *of intuitionistic linear logic*

The category $(\mathcal{S}/\!\approx)$ has the same drawback as the category $\mathcal{S}$: it is not cartesian. We thus proceed as in section 2.2, and consider the full subcategory $(\mathcal{G}/\!\approx)$ of *negative* games. Just like $\mathcal{G}$ is coreflective in $\mathcal{S}$, the category $(\mathcal{G}/\!\approx)$ is coreflective in $(\mathcal{S}/\!\approx)$, with counit $\mathrm{neg}(A) \longrightarrow A$ defined as in section 2.2, with left and right action on $\mathrm{neg}(A)$ inherited from A. Like the category $\mathcal{G}$, the category $(\mathcal{G}/\!\approx)$ is symmetric monoidal closed, with same tensor product and unit as in $(\mathcal{S}/\!\approx)$, and monoidal closure given by:

$$A \multimap B = \mathrm{neg}(A^{\perp} \otimes B).$$

The category $(\mathcal{G}/\!\approx)$ is also cartesian. The cartesian product of two negative games A and B is given by $A \& B$. The terminal object $\top$ is the game with an empty set of moves, and a trivial group action.

We show that the exponential ! defined in section 3.6. induces a model of intuitionistic linear logic, in the sense of (Hyland 1997; Melliès 2002). We proceed as in (Abramsky *et al.* 1994).

Contraction and weakening. In order to obtain a commutative comonoid in $(\mathcal{G}/\!\approx)$, one equips every negative orbital game A with two (error-free) strategies, playing the role of the comultiplication and the counit of $!A$:

$$d^{\varphi}_{A} :!A \multimap !A \otimes !A \qquad\qquad e_A :!A \multimap 1$$

The two strategies are defined below. Note that the strategy d_A^{φ} depends on a given injective function $\varphi : \mathbb{N} + \mathbb{N} \longrightarrow \mathbb{N}$.

— the strategy d_A^{φ} contains a legal play s of $!A_1 \multimap !A_2 \otimes !A_3$ precisely when all the even-length prefixes of s verify property (*)

— the strategy e_A is the empty strategy $\{\epsilon\}$ of $!A \multimap 1$.

A play t of $!A_1 \multimap !A_2 \otimes !A_3$ verifies property (*) when its projections t_1 over $!A_1$ and t_{23} over $!A_2 \otimes !A_3$ are equal, modulo renaming of every move $(\mathtt{inl}(a), i)$ and $(\mathtt{inr}(a), i)$ in t_{23} by a move $(a, \varphi(\mathtt{inl}(i)))$ and $(a, \varphi(\mathtt{inr}(i)))$ respectively. One proves easily the two next lemmas.

Lemma 3.9. Suppose that $\varphi_1, \varphi_2 : \mathbb{N} + \mathbb{N} \longrightarrow \mathbb{N}$ are two injective functions. Then, $d_A^{\varphi_1} \approx^{\mathrm{INV}} d_A^{\varphi_2}$.

Lemma 3.10. The triple $(!A, d_A^{\varphi}, e_A)$ defines a commutative comonoid in the category $(\mathcal{G}/\approx)$.

Dereliction. Given an integer $i \in \mathbb{N}$, the (error-free) strategy $\epsilon_A^i : !A \multimap A$ is defined as follows: it contains a legal play s of $!A \multimap A$ precisely when, for every even-length prefix t of the legal play s:

— the projection of $t_{|!A}$ over the thread j is the empty play when $j \neq i$,

— the projection of $t_{|!A}$ over the thread i is equal to the projection $t_{|A}$.

Lemma 3.11. Suppose that i and j are two non negative integers. Then, $\epsilon_A^i \approx^{\mathrm{INV}} \epsilon_A^j$.

Lemma 3.12. The family $\epsilon_A^i : !A \multimap A$ verifies the universality property indicated in diagram (1), that is: for every morphism $\sigma : !A \longrightarrow B$ in the category $(\mathcal{G}/\approx)$, there exists a unique comonoidal morphism $\sigma^{\dagger} : !A \longrightarrow !B$ such that $\sigma = \sigma^{\dagger} ; \epsilon_A^i$.

Canonical isomorphism. There exists for every orbital games A, B, comonoidal isomorphisms $!A \otimes !B \cong !(A \& B)$ and $1 \cong !\top$ in the category $(\mathcal{G}/\approx)$.

Theorem 3.13. The category $(\mathcal{G}/\approx)$ equipped with the structures introduced above, defines a model of intuitionistic linear logic

Proof. We refer the reader to the definition of Lafont-Seely category in (Melliès 2002). □

3.9. *Side remark: the liberal and the incremental exponentials are equivalent*

Another model of intuitionistic linear logic is obtained by adapting to the category $(\mathcal{G}/\approx)$ the exponential $!^{\mathrm{inc}}$ of definition 2.10. This is done in the expected way: the functor $!^{\mathrm{inc}}$ transports every orbital game A to the sequential game of definition 2.10 equipped with two groups $G_{!^{\mathrm{inc}}A} = \Pi_{i \in \mathbb{N}} G_A$ and $H_{!^{\mathrm{inc}}A} = \Pi_{i \in \mathbb{N}} H_A$ and their pointwise action on the set of moves $M_{!^{\mathrm{inc}}A} = M_A \times \mathbb{N}$. It is not difficult to show that the resulting comonoid $!^{\mathrm{inc}}A$ is *isomorphic* to the comonoid $!A$ in the category $(\mathcal{G}/\approx)$. It follows that the liberal exponential ! and the incremental exponential $!^{\mathrm{inc}}$ induce the same model of intuitionistic linear logic, and alternative linearizations of the single-threaded model.

4. An orbital formulation of AJM games

An orbital game is *alternated* when its underlying sequential game is *alternated* in the sense of definition 2.2. The full subcategory of (negative) alternated games in $(\mathcal{G}/\approx)$ is denoted $(\mathcal{G}^{\mathsf{alt}}/\approx)$. There is a functor $\mathrm{alt} : (\mathcal{G}/\approx) \longrightarrow (\mathcal{G}^{\mathsf{alt}}/\approx)$ which:

- transports every orbital game A to the game $\mathrm{alt}(A)$ with same moves and same group actions, and with plays the alternated plays of A,
- transports every morphism $\sigma : A \longrightarrow B$ of $(\mathcal{S}/\approx)$, or equivalently every strategy $\sigma : A \multimap B$, to the morphism $\mathrm{alt}(\sigma) : \mathrm{alt}(A) \multimap \mathrm{alt}(\sigma)$ defined as expected:

$$\mathrm{alt}(\sigma) = \sigma \cap L_{\mathrm{alt}(A) \multimap \mathrm{alt}(B)}.$$

The category $(\mathcal{G}^{\mathsf{alt}}/\approx)$ defines a model of intuitionistic linear logic with same cartesian product as in the category $(\mathcal{G}/\approx)$, and tensor product, linear implication and exponentials defined as:

$$A \otimes_{\mathrm{alt}} B = \mathrm{alt}(A \otimes B), \qquad A \multimap_{\mathrm{alt}} B = \mathrm{alt}(A \multimap B), \qquad !^{\mathrm{alt}} A = \mathrm{alt}(!A).$$

This and the interpretation of recursion induces a model of the language PCF, which we would like to compare to the fully abstract model of PCF delivered in (Abramsky *et al.* 1994). Recall that a PCF type T is constructed by the grammar:

$$T \quad = \quad o \quad | \quad \iota \quad | \quad T \Rightarrow T$$

where o and ι denote the boolean and integer base types respectively. Every PCF type T is interpreted in our orbital model $(\mathcal{G}^{\mathsf{alt}}/\approx)$ as an alternated orbital game denoted $[T]_{\mathrm{alt}}$ and defined by structural induction on T. The boolean and integer types o and ι are interpreted as the sequential games bool and nat below:

$$(M_{\mathsf{bool}}, \lambda_{\mathsf{bool}}) = \{* : -1, \mathsf{true} : +1, \mathsf{false} : +1\}, \qquad P_{\mathsf{bool}} = \{\epsilon, *, * \cdot \mathsf{true}, * \cdot \mathsf{false}\}.$$

$$(M_{\mathsf{nat}}, \lambda_{\mathsf{nat}}) = \{* : -1\} \cup \{\mathsf{n} : +1 \mid n \in \mathbb{N}\}, \qquad P_{\mathsf{nat}} = \{\epsilon, *\} \cup \{* \cdot \mathsf{n} \mid n \in \mathbb{N}\}.$$

equipped with the trivial groups $G_{\mathsf{bool}} = H_{\mathsf{bool}} = G_{\mathsf{nat}} = H_{\mathsf{nat}} = \{e\}$ and thus, with the trivial group actions. The PCF type $T_1 \Rightarrow T_2$ is interpreted by Girard's formula:

$$[T_1 \Rightarrow T_2]_{\mathrm{alt}} \;=\; (!^{\mathrm{alt}}[T_1]_{\mathrm{alt}}) \;\multimap_{\mathrm{alt}}\; [T_2]_{\mathrm{alt}}.$$

We mentioned in section 3.2 that every negative alternated orbital game A defines an AJM game $U(A)$ with same underlying sequential game (M_A, λ_A, P_A), and partial equivalence relation $\approx^{\mathrm{AJM}}_{U(A)}$ defined as the partial equivalence relation $\approx_A$ in definition 3.1:

$$s \approx_A t \iff \exists (g,h) \in G_A \times H_A, \qquad t = g \centerdot s \centerdot h.$$

It follows from proposition 3.6 that the translation defines a functor U from the category $(\mathcal{G}^{\mathsf{alt}}/\approx)$ to the category $(\mathcal{G}^{\mathsf{alt}}/\approx^{\mathrm{AJM}})$ of AJM games and self-equivalent *error-aware* strategies. Besides, the functor is full and faithful, and transports the logical structure $(\otimes_{\mathrm{alt}}, \multimap_{\mathrm{alt}}, !^{\mathrm{alt}})$ from the first category to the second. It follows that the orbital model of PCF coincides with an error-aware and history-sensitive variant of the AJM game model. The argument may be adapted to *history-free* and *well-bracketed* strategies in both categories $(\mathcal{G}^{\mathsf{alt}}/\approx)$ and $(\mathcal{G}^{\mathsf{alt}}/\approx^{\mathrm{AJM}})$. We conclude:

Lemma 4.1. The fully abstract AJM token game model of PCF (Abramsky *et al.* 1994). coincides with an alternated, history-free, error-free and well-bracketed variant of the orbital game model.

5. An orbital formulation of arena games

First, we introduce the notion of *asynchronous game* in section 5.1 and explain in sections 5.2, 5.3 and 5.4 that every asynchronous game induces a sequential game enriched with a notion of *justification* in the spirit of arena games. Then, we equip every asynchronous game with an orbital structure, and deduce a model of intuitionistic linear logic, in sections 5.5 and 5.6.

5.1. *Asynchronous games*

Definition 5.1 (asynchronous game). An asynchronous game is a triple

$$A = (M_A, \lambda_A, \leq_A)$$

consisting of:

— a polarized alphabet (M_A, λ_A) whose elements are called *moves*,

— an ordered set $(M_, \leq_A)$ verifying that every move $m \in M_A$ defines a *finite* downward closed subset $m{\downarrow}= \{m' \in M_A, m' \leq_A m\}$.

5.2. *The position graph.*

Every asynchronous game induces an position graph, defined as follows.

Definition 5.2 (position). We call *position* of A any *finite* downward closed subset of $(M_, \leq_A)$.

Definition 5.3 (position graph $\mathcal{G}(A)$). The graph $\mathcal{G}(A)$ has:

— positions of A as vertices,

— an edge $x \xrightarrow{m} y$ labelled by a move $m \in M_A$ for every pair of positions x, y such that $y = x \cup \{m\}$ and $m \notin x$.

Note that the graph $\mathcal{G}(A)$ is *pointed* by the empty position, noted $*$.

5.3. *The plays.*

Definition 5.4 (play). A play is a path

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} \cdots \xrightarrow{m_{k-1}} x_{k-1} \xrightarrow{m_k} x_k$$

in the position graph $\mathcal{G}(A)$. The set of plays is denoted P_A.

Remark. Alternatively, a play of A is a finite string $s = m_1 \cdots m_k$ without repetition such that the set $\{m_1, ..., m_j\}$ is downward closed in $(M_A, \leq_A)$ for every $1 \leq j \leq k$.

5.4. *The justification pointers.*

Every asynchronous game A defines in this way a sequential game (M_A, λ_A, P_A). This enables to import from section 2.1 the definitions of legal play, of strategy, of error-free strategy, etc. The main novelty is a notion of *justification* in the spirit of arena games.

Definition 5.5 (justification). We write $m \vdash_A n$ when $m \leq_A n$ and:

$$\forall p \in M_A, \qquad m \leq_A p \leq_A n \Rightarrow m = p \text{ or } p = n.$$

We say in that case that the move m justifies the move n.

Note that the situation here is slightly simpler than in arena games, because the play s is non repetitive. In particular, we do not need to distinguish a move m from its occurrences. The situation is also more general, because several moves $n_1, ..., n_k$ may justify a move m — whereas there is at most one such justifying move in the original definition of arena games.

5.5. *Asynchronous orbital games*

An asynchronous orbital game is an asynchronous game $(M_A, \lambda_A, \leq_A)$ equipped with
— two groups G_A and H_A,
— a left group action on moves: $G_A \times M_A \longrightarrow M_A$,
— a right group action on moves: $M_A \times H_A \longrightarrow M_A$,
verifying that the left and right actions commute:

$$\forall m \in M_A, \forall g \in G_A, \forall h \in H_A, \qquad (g \centerdot m) \centerdot h = g \centerdot (m \centerdot h).$$

We require that every asynchronous orbital game verifies four *coherence axioms*; namely, that for every $g \in G_A$ and $h \in H_A$ and $m, n \in M_A$:

(i) $m \leq_A n \Rightarrow g \centerdot m \centerdot h \leq_A g \centerdot n \centerdot h$,
(ii) $\lambda_A(g \centerdot m \centerdot h) = \lambda_A(m)$,
(iii) $m = g \centerdot m$ when $\lambda_A(m) = -1$ and $n = g \centerdot n$ for every $n \leq_A m$,
(iv) $m = m \centerdot h$ when $\lambda_A(m) = +1$ and $n = n \centerdot h$ for every $n \leq_A m$.

Every asynchronous orbital game A defines an orbital game $(M_A, \lambda_A, P_A, G_A, H_A)$ in the sense of section 3.1. This enables to define a *strategy* of A as a strategy of the underlying sequential game (M_A, λ_A, P_A) and to import from section 3 the two definitions of $\approx^{\text{INV}}$ and $\approx^{\text{SIM}}$, which coincide by proposition 3.6.

5.6. *A model of intuitionistic linear logic*

From now on, we call "negative ao-game" any asynchronous orbital game A verifying three additional properties:
— for every move $n \in M_A$, there exists at most one move $m \in M_A$ such that $m \vdash_A n$,
— the miminal moves of $(M_A, \leq_A)$ are Opponent (that is, of polarity -1),
— the move $m \in M_A$ and the move $n \in M_A$ are of opposite polarity when $m \vdash_A n$.

We construct a model of intuitionistic linear logic (with additives).

Tensor product. The tensor product $A \otimes B$ of two negative games A and B is the negative game with asynchronous game the disjoint sum $(M_A + M_B, \lambda_A + \lambda_B, \leq_A + \leq_B)$ and group action defined in the same way as for orbital games in section 3.6.

Linear implication. The linear implication $A \multimap B$ of two negative ao-games A and B is the negative ao-game defined as:

— $M_{A \multimap B} = M_A + M_B$,
— $\lambda_{A \multimap B}(\mathtt{inl}(m)) = -\lambda_A(m)$ and $\lambda_{A \multimap B}(\mathtt{inr}(m)) = \lambda_B(m)$,
— $m \leq_{A \multimap B} n$ iff
  - $m = \mathtt{inl}(m_A)$ and $n = \mathtt{inl}(n_A)$ and $m_A \leq_A n_A$, or
  - $m = \mathtt{inr}(m_B)$ and $n = \mathtt{inr}(n_B)$ and $m_B \leq_B n_B$, or
  - $m = \mathtt{inl}(m_B)$ and $n = \mathtt{inr}(n_A)$ and m_B is minimal in $(M_B, \leq_B)$.

— the left action of an element (h_A, g_B) of the group $G_{A \multimap B} = H_A^{op} \times G_B$ and the right action of an element (g_A, h_B) of the group $H_{A \multimap B} = G_A^{op} \times H_B$ are defined as expected on the set of moves $M_{A \multimap B}$:

$$(h_A, g_B) \bullet_{A \multimap B} \mathtt{inl}(m) = \mathtt{inl}(h_A \bullet^{op} m), \qquad (h_A, g_B) \bullet_{A \multimap B} \mathtt{inr}(m) = \mathtt{inr}(g_B \bullet m),$$

$$\mathtt{inl}(m) \bullet_{A \multimap B} (g_A, h_B) = \mathtt{inl}(m \bullet^{op} g_A), \qquad \mathtt{inr}(m) \bullet_{A \multimap B} (h_A, h_B) = \mathtt{inr}(m \bullet h_B).$$

Exponentials. The exponential $!A$ of a negative ao-game A is the negative ao-game defined as:

— $M_{!A} = M_A \times \mathbb{N}$,
— $\lambda_{!A}(m, i) = \lambda_A(m)$ for every $m \in M_A$ and $i \in \mathbb{N}$,
— $(m, i) \leq_{!A} (n, j)$ iff $i = j$ and $m \leq_A n$, for every $m, n \in M_A$ and $i, j \in \mathbb{N}$,
— the group actions on $M_{!A}$ are defined exactly in the same way as for orbital games in section 3.6.

The category. The category $\mathcal{H}$ has objects the negative ao-games, morphisms $A \longrightarrow B$ the strategies $A \multimap B$, and composition defined as in the category $\mathcal{S}$ of section 2. The category $(\mathcal{H}/\approx)$ is obtained as quotient of the category $\mathcal{H}$ modulo bi-invariance, in the same way as the category $(\mathcal{S}/\approx)$ is obtained from the category $\mathcal{S}$ in section 3. We prove that:

Lemma 5.6. The category $(\mathcal{H}/\approx)$ equipped with the structures introduced above, defines a model of intuitionistic linear logic (without additives).

Proof. We refer the reader to the definition of new-Lafont category in (Melliès 2002). □

Remark. It is not really difficult to define an asynchronous and orbital model of intuitionistic linear logic *with additives*. It is sufficient to enrich the definition of asynchronous games with a binary relation # of *incompatibility* between moves, as is quite common in asynchronous transition systems.

Remark. Like the category $\mathcal{G}$ in section 2 and the category $(\mathcal{G}/\approx)$ in section 3, the category $(\mathcal{H}/\approx)$ is another presentation of the single-threaded model of intuitionistic linear logic given in (Abramsky *et al.* 1998). In other words, shifting from an incremental to a locative indexing does not alter the model, and is thus pointless, when one works with single-threaded strategies. It is only justified when one moves to other classes of strategies, like the innocent ones. This brings us to the category of arena games and innocent strategies introduced in (Hyland and Ong 1994; Nickau 1994), already discussed in section 1. In that case, using a locative indexing enables to express innocence using simple concurrency ideas inspired by Mazurkiewicz. This leads us to formulate a new definition of innocence in asynchronous games in (Melliès 2004). We establish there that the usual notion of innocence in arena games, is captured as the combination of asynchronous innocence, and bi-invariance. Thus, every innocent strategy in an arena game corresponds to a strategy σ in an asynchronous orbital game, verifying that:

1. the strategy σ is innocent in the asynchronous sense,
2. the strategy σ is bi-invariant in the orbital sense.

Reformulated in this way, innocence becomes a very elementary notion, which lives naturally in a much wider class of arena games than the usual one. Typically, the asynchronous description of arena games enables to extend them to arena games with a dag (directed acyclic graph) structure, enabling several moves to justify the same move; or to arena games with independence, that is, admitting interfering moves.

6. Conclusion

The main technical contribution of the article is a group-theoretic formulation of the partial equivalence relation (per) of AJM games, which enables to capture when a strategy is “blind to the Opponent’s thread indexing”. We believe that the idea of a left and right group action may be adapted to other models of linear logic, like the Geometry of Interaction, or coherence space models. We are currently working in this direction.

By avoiding “justification sequences” in our formulation of arena games, we get closer to mainstream automata theory, and closer to an algebraic description of copy and locality mechanisms in denotational semantics. This prepares the field for later concurrent analysis of games semantics.

References

S. Abramsky and R. Jagadeesan. Games and full completeness for multiplicative linear logic. *Journal of Symbolic Logic*, 59(2):543-574, June 1994.

S. Abramsky, R. Jagadeesan and P. Malacaria. Full abstraction for PCF (extended abstract). In M. Hagiya and J.C. Mitchell, editors, *Theoretical Aspects of Computer Software*, page 1-15. Springer Verlag, 1994.

S. Abramsky, K. Honda, G. McCusker. Fully abstract game semantics for general reference. In *Proceedings of IEEE Symposium on Logic in Computer Science*, Computer Society Press, 1998.

S. Abramsky and G. McCusker. Game Semantics. In *Computational Logic.* U. Berger and H. Schwichtenberg eds, Springer Verlag, 1999.

S. Abramsky, P.-A. Melliès. Concurrent games and full completeness. In *Proceedings of the Fourteenth Annual IEEE Symposium on Logic in Computer Science,* LICS '99, IEEE Computer Society Press, 1999.

P. Baillot, V. Danos, T. Ehrhard, L. Regnier. Timeless Games. *Proceedings CSL'97.*

P. Baillot. *Approches dynamiques en sémantique de la logique linéaire: jeux et géometrie de l'interaction.* PhD thesis Universite Aix-Marseille 2, January 1999 (in french)

P.N. Benton, G.M. Bierman, V.C.V. de Paiva and J.M.E. Hyland. *Term assignment for intuitionistic linear logic.* Technical Report 262, Computer Laboratory, University of Cambridge, 1992.

G. Bierman. What is a categorical model of intuitionistic linear logic? *Typed Lambda-Calculus and Applications*, 1995.

P.-L. Curien. On the symmetry of sequentiality. In *Proceedings of Mathematical Foundations of Programming Semantics 1993*, LNCS 802, Springer Verlag, 1993.

V. Danos, H. Herbelin, L. Regnier. Games Semantics and Abstract Machines. *Proceedings of LICS'96*, 1996.

J.-Y. Girard. Geometry of interaction I. 1989.

R. Harmer. *Games and Full Abstraction for Nondeterministic Languages.* PhD thesis. University of London, 2000.

H. Herbelin. *Stratégies innocentes et stratégies sans mémoire.* Personal communication. 1995.

M. Hyland and L. Ong. On full abstraction for PCF. Submitted to Information and Computation, 1994.

M. Hyland. Game Semantics. In *Semantics and logics of computation.* A. Pitts and P. Dybjer editors. Publications of the Newton Institute, Cambridge University Press, 1997.

M. Hyland and A. Schalk. Abstract Games for Linear Logic. Extended Abstract. *Proceedings for CTCS '99,* Volume 29 of Electronic Notes in Theoretical Computer Science, 1999.

A. Joyal. Remarques sur la théorie des jeux à deux personnes. Gazette des Sciences Mathématiques du Québec, Vol I, no 4, pp.46-52, 1977.

J. Laird. Full abstraction for functional languages with control. Extended abstract, in the *Proceedings of the 12th Annual Symposium on Logic in Computer Science,* LICS '97, 1997.

F. Lamarche. Sequentiality, games and linear logic. *Manuscript*, 1992.

O. Laurent. A Token Machine for Full Geometry of Interaction (extended abstract).In Samson Abramsky, editor, *Typed Lambda Calculi and Applications* (TLCA '01), volume 2044 of LNCS, pages 283-297. Springer-Verlag. 2001

S. McCusker. *Games and full abstraction for a functional metalanguage with recursive types.* Distinguished Dissertation Series, Springer Verlag, 1998.

P-A. Melliès. Axiomatic Rewriting Theory 1: A diagrammatic standardization theorem. Prépublication de l'équipe PPS (December 2002, number 13). Submitted to Information and Computation. 2001.

P-A. Melliès. Reconciling Hyland-Ong games and intuitionistic linear logic. Manuscript. 2001.

P-A. Melliès. Categorical models of linear logic revisited. Prépublication de l'équipe PPS (March 2003, number 22). To appear in Theoretical Computer Science. 2002.

P-A. Melliès. Comparing hierarchies of types in models of linear logic. Prépublication de l'équipe PPS (December 2002, number 14). Information and Computation, Volume 189, Issue 2, Pages 202-234, March 2004.

P-A. Melliès. Asynchronous games 2: the true concurrency of innocence. Submitted. 2004.

P-A. Melliès. Asynchronous games 3: an innocent model of linear logic. Submitted. 2004.

H. Nickau. Hereditarily Sequential Funtionals. In *Proc. Symp. Logical Foundations of Computer Science: Logic at St. Petersburg*, Eds. A. Nerode and Yu. V. Matiyasevich, Lecture Notes in Computer Science, volume 813, pages 253-264, Springer-Verlag, 1994.