Homology of Higher Categories

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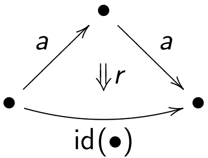
A presentation of $\mathbb{Z}_2$

generator	a
relation	$aa = 1$
rewriting rule	$r : aa \to 1$



Rewriting paths build a 2-category

$\Sigma = \{a\}$ — graph $\quad 1 \leftleftarrows \Sigma$

free category $\quad 1 \leftleftarrows \Sigma^*$

$\mathcal{R} = \{r\}$ — 2-graph $\quad 1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}$

free 2-category $\quad 1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}^*$

Remark

The monoid $\mathbb{Z}_2$ is a category with one object, two arrows.

Globular sets

O category with:

- objects = integers $0, 1, 2, \ldots$
- arrows generated by $\mathrm{s}_n, \mathrm{t}_n : n \to n{+}1$ satisfying

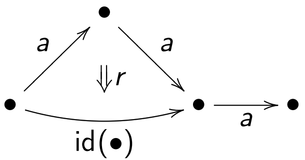
$$\begin{aligned} \mathrm{s}_{n+1}\mathrm{s}_n &= \mathrm{t}_{n+1}\mathrm{s}_n \\ \mathrm{t}_{n+1}\mathrm{t}_n &= \mathrm{s}_{n+1}\mathrm{t}_n \end{aligned}$$

$$0 \underset{\mathrm{t}_0}{\overset{\mathrm{s}_0}{\rightrightarrows}} 1 \underset{\mathrm{t}_1}{\overset{\mathrm{s}_1}{\rightrightarrows}} 2 \underset{\mathrm{t}_2}{\overset{\mathrm{s}_2}{\rightrightarrows}} \cdots$$

Definition

A globular set is a presheaf on **O**, that is a functor $\mathbf{O}^{op} \to \mathbf{Sets}$.

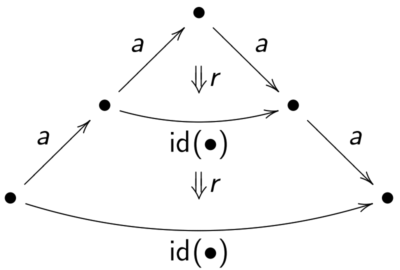
2-cells compose horizontally



$$r *_0 a \quad : \quad a *_0 a *_0 a \longrightarrow a$$

$r *_0 a$ is short for $r *_0 \mathrm{id}^2(a)$.

… and vertically



$$(a *_0 r *_0 a) {\color{red} *_1} r \quad : \quad a *_0 a *_0 a *_0 a \to \mathrm{id}(\bullet)$$

Higher dimensional categories

- globular set $C_0 \leftleftarrows C_1 \leftleftarrows C_2 \leftleftarrows \cdots$
- compositions and units satisfying axioms
- $\omega\mathbf{Cat}$ = (strict) ω-categories + ω-functors

Examples

1. monoid

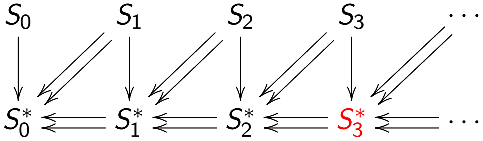
$$1 \leftleftarrows \mathbb{Z}_2 \leftleftarrows () \leftleftarrows \cdots$$

2. presentation

$$1 \leftleftarrows \Sigma^* \leftleftarrows \mathcal{R}^*_{/\|} \leftleftarrows () \leftleftarrows \cdots$$

What is a polygraph?

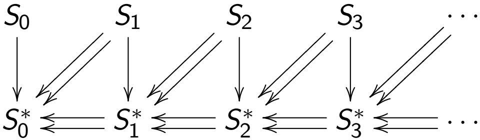
- 0-polygraph = set
- 1-polygraph = graph
- 2-polygraph = computad (Street, 1976)
- *n*-computad (Power, 1991) or polygraph (Burroni, 1991)



- $(.)^*$: **Pol** $\to$ ω**Cat** left-adjoint

Linearization

- polygraph



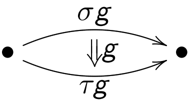
- linearization map

$$[.] : S_i^* \to \mathbb{Z}S_i$$

Example

$$[(a *_0 r *_0 a) *_1 r] = [\mathrm{id}^2(a)] + [r] + [\mathrm{id}^2(a)] + [r] = 2[r]$$

Boundary operator



$g \in S_{i+1}$

$\partial_i[g] = [\tau g] - [\sigma g]$

extends by linearity: $\mathbb{Z}S_i \leftarrow \mathbb{Z}S_{i+1}$

$$\mathbb{Z}S_0 \xleftarrow{\partial_0} \mathbb{Z}S_1 \xleftarrow{\partial_1} \mathbb{Z}S_2 \xleftarrow{\partial_2} \mathbb{Z}S_3 \xleftarrow{\partial_3} \cdots$$

- chain complex $\partial_{i-1}\partial_i = 0$
- homology $\mathrm{H}_i(\mathbb{Z}S) = \ker \partial_{i-1} / \operatorname{im} \partial_i$

Functoriality

- $\mathbf{F}$ = full subcategory of $\omega\mathbf{Cat}$ on objects S^*
- $\mathbf{F}$ = Kleisli category of the adjunction $\mathbf{Pol} \rightleftharpoons \omega\mathbf{Cat}$
- $[.] : \mathbf{F} \to \mathbf{Ch}$ functor

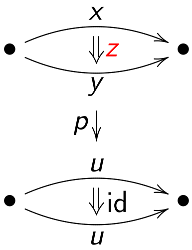
$$\begin{array}{rcl} S^* & \mapsto & [S^*] = \mathbb{Z}S \\ u : S^* \to T^* & \mapsto & [u] : \mathbb{Z}S \to \mathbb{Z}T \end{array}$$

- Homology functors $\mathrm{H}_i(.) : \mathbf{F} \to \mathbf{Ab}$

What is a resolution?

Stretching property

- $p : B \to C$
- If x, y are parallel cells and $p(x) = p(y) = u$, then there is a $z : x \to y$ such that $p(z) = \mathrm{id}(u)$.

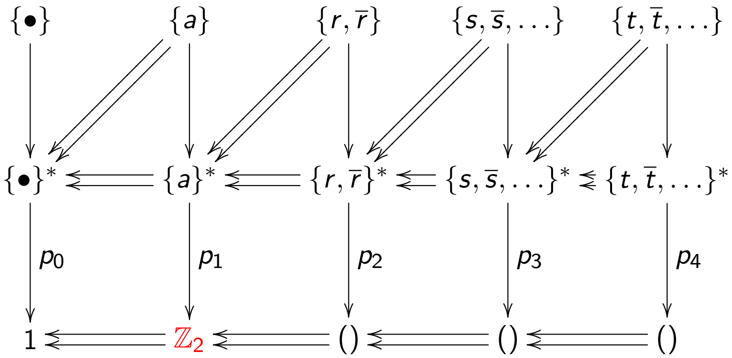


Definition

A polygraphic resolution of C is a morphism $p : S^* \to C$ satisfying:

1. p is surjective;
2. p has the stretching property.

A partial resolution of $\mathbb{Z}_2$



Critical pair

- Two parallel rewriting paths:

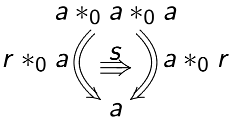
$$\begin{aligned} r *_0 a &: \quad a *_0 a *_0 a \longrightarrow a \\ a *_0 r &: \quad a *_0 a *_0 a \longrightarrow a \end{aligned}$$

- Same image:

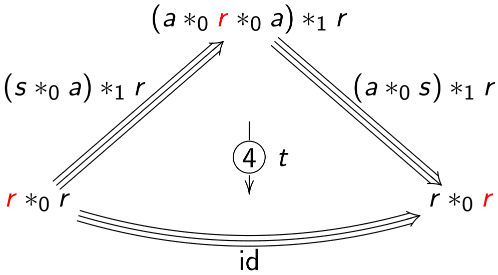
$$p_2(r *_0 a) = p_2(a *_0 r) = \mathrm{id}^2(a)$$

- Filling a critical pair:

$$s \;:\; r *_0 a \longrightarrow a *_0 r$$



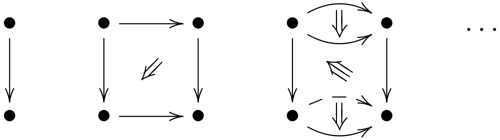
Filling a three dimensional loop



- $l \;=\; (((s *_0 a) *_1 r) *_2 (a *_0 s) *_1 r)$ is a loop;
- $\partial[t] = -[l] = -2[s]$, $\mathrm{H}_3(\mathbb{Z}_2) = \mathbb{Z}_2$.

Path object

n-cylinders

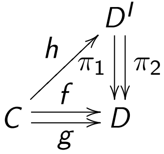


- $(C^I)_n = \mathrm{Hom}(\mathrm{cyl}[n], C)$
- C^I strict ω-category
- natural transformations $\pi_1, \pi_2 : C^I \to C$

Homotopy

Definition

A (directed) homotopy from f to g is an ω-functor $h : C \to D^I$ such that $\pi_1 h = f$ and $\pi_2 h = g$.



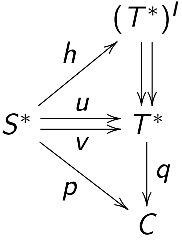
- notation $f \rightsquigarrow g$

Proposition

Let $u, v : S^* \to T^*$. If $u \rightsquigarrow v$, then $[u], [v] : \mathbb{Z}S \to \mathbb{Z}T$ are chain-homotopic: $[u] \sim [v]$.

Homotopical invariance

$p : S^* \to C$, $q : S^* \to C$ resolutions



Theorem

Each ω-category has a polygraphic resolution, which is unique up to homotopy.

Homology

Theorem

Let $p : S^* \to C$ be a polygraphic resolution. $\mathsf{H}_*(\mathbb{Z}S)$ *only depends on C.*

$$\mathsf{H}^{\mathsf{pol}}_*(C) =_{\mathsf{def}} \mathsf{H}_*(\mathbb{Z}S)$$

Proof.

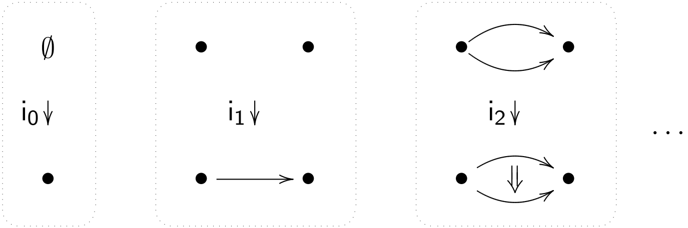
Let $p : S^* \to C$ and $q : T^* \to C$ be resolutions of C.

1. There are $u : S^* \to T^*$, $v : T^* \to S^*$ such that $qu = p$ and $pv = q$, hence $uv \rightsquigarrow \mathrm{id}$ and $vu \rightsquigarrow \mathrm{id}$.
2. By applying linearization, $[u][v] \sim \mathrm{id}$ and $[v][u] \sim \mathrm{id}$, whence an isomorphism $[u]_* : \mathsf{H}_*(\mathbb{Z}S) \to \mathsf{H}_*(\mathbb{Z}T)$.

□

Canonical inclusions

- globes $O[n]$ and boundaries $\partial O[n]$
- canonical inclusions $\mathsf{i}_n : \partial O[n] \to O[n]$



Right lifting property

Proposition

For any ω-functor p, the following properties are equivalent:

1. p is surjective and has the stretching property;
2. p has the right lifting property with respect to the canonical inclusions.

$$\begin{array}{ccc} \partial O[n] & \longrightarrow & C \\ {\scriptstyle \mathrm{i}_n}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ O[n] & \longrightarrow & D \end{array}$$

In particular polygraphic resolutions belong to $\mathrm{rlp}(\{\mathrm{i}_n, n \geq 0\})$

Model categories

Model structures

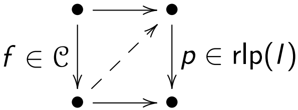
- $\mathbf{C}$ complete and cocomplete
- three classes of morphisms: $\mathcal{W}$, $\mathcal{F}$, $\mathcal{C}$
  - weak equivalences
  - fibrations
  - cofibrations
- axioms: stability, lifting properties, factorizations

Homotopy category

- $\mathrm{Ho}(\mathbf{C}) = \mathcal{W}^{-1}\mathbf{C}$ localization by weak equivalences
- $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ universal among functors taking w.e. to isos

Generating cofibrations

In *cofibrantly generated* model structures, there is a set $I \subset \mathcal{C}$ such that $\mathcal{C} = \mathrm{llp}(\mathrm{rlp}(I))$.

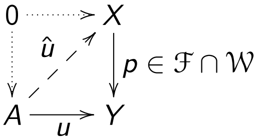


Remark

In any model structure, $\mathcal{F} = \mathrm{rlp}(\mathcal{C} \cap \mathcal{W})$.
A cofibrantly generated structure is entirely determined by $\mathcal{W}$ and the set I of generating cofibrations.

Cofibrant objects

- A is cofibrant if $0 \to A$ is a cofibration:



- duality: X fibrant if $X \to 1 \in \mathcal{F}$
- $\mathbf{C}_{cf}$ full subcategory of $\mathbf{C}$ on objects both fibrant and cofibrant:

$$\mathrm{Ho}(\mathbf{C}) \simeq \pi\mathbf{C}_{cf}$$

- morphisms in $\pi\mathbf{C}_{cf}$ = homotopy classes of morphisms in $\mathbf{C}_{cf}$

Folk model structure

Theorem (Lafont, Worytkiewicz and FM, 2007)

There is a model structure on $\omega\mathbf{Cat}$ such that:

1. *the canonical inclusions are generating cofibrations;*
2. *the class $\mathcal{W}$ of weak equivalences is minimal.*

Properties

- all objects are fibrant;
- cofibrant objects are exactly polygraphs.

$$(\omega\mathbf{Cat})_{cf} = \mathbf{F}$$

Abelian group objects

Denormalization theorem (Bourn)

There is an equivalence of categories between:

$$\begin{array}{llcl} & \omega\mathbf{Cat}^{ab} & = & \text{abelian group objects in } \omega\mathbf{Cat} \\ \text{and} & \mathbf{Ch} & = & \text{chain complexes} \end{array}$$

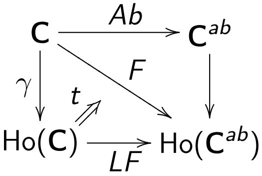
Abelianization functor

$$Ab : \omega\mathbf{Cat} \to \mathbf{Ch}, \qquad C \mapsto (A, \partial)$$

$A_i = \mathbb{Z}C_i/_{\approx}$, where $\mathrm{id}(x) \approx 0$ and $x *_j y \approx x + y$

Homology as a derived functor

Derived functor



$t : LF \circ \gamma \to F$

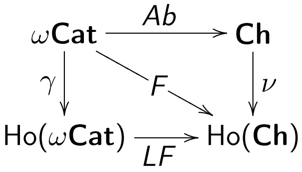
Model structure on **Ch**

- Weak equivalences induce isomorphisms in homology
- $\nu : \mathbf{Ch} \to \mathrm{Ho}(\mathbf{Ch})$

Deriving the abelianization functor

Theorem

Let $F = \nu \circ Ab$. There is a left derived functor LF and for any polygraph S, $(LF \circ \gamma)(S^*) \simeq F(S^*)$.



Proof.

- on cofibrant objects $Ab(S^*) = [S^*] = \mathbb{Z}S$;
- If $f : S^* \to T^*$ is a weak equivalence, then $Ab(f)$ is a quasi-isomorphism.

□