Playing for truth on mutable graphs

Hirschowitz[3]

Gerard Huet– 2007/06/23

Gödel's black hole

Mathematics work (till now) but

- No justification for mathematics
- No hope for justification
- feel comfortable in this (Gödel's) black hole

Try to escape!

What is problematic?

- Consistency
- Excluded-middle
- Choice

. . . without axioms!

Try with two-player games!

Find a game where

- P wants to prove S
- O wants to prove $S^{\perp}$

- Obviously, at most one of them can win
- Hope: at least one of them can win

Logical games already exist!

- Blass
- Abramsky Jagadeesan Malacaria
- Hyland Ong
- Abramsky Melliès
- Girard Curien Faggian

Proof nets = geometry of proofs (for formulas)
Ludics = geometry of proofs (for sequents)
So what? Geometry of proofs for hypersequents ?

Plan

- Position = mutable graph
- Move = mutation
- Embedding LL
- Sequential plays
- The stack of plays on a position
- Strategies as substack
- with a glance toward cut-elimination

Graphs as topological spaces

A graph is a topological space where:

- some points are closed: vertices and free edges
- if x is not closed (bound edge), then its boundary consists of one or two closed points (ends)
- closed subset: with each edge, contains its ends
- the graph knows which closed points are vertices

A morphism of graphs is a continuous application.

Example: sequents

Definition

- A sequent is a graph with a closed point which is an end of each edge
- An open sequent is a graph with only one closed point
- A closed sequent is a sequent where each edge has two ends
- The center of the sequent is the common end

Weights

Definition

A weight space $W := W^+ \cup W^-$ is a set (W) equipped with an involution (-) exchanging W^+ and W^-

Weighted graph

- given: a weight space W,
- a W-graph is a graph H equipped with
- for each edge, a positive weight and
- for each bound edge, a direction

Arrows in a weighted graph

- Edges are directed by their weight
- arrow = directed edge (with positive weight)

Why and which weights?

Our graphs will evolve.
Our weights will prescribe locally how our graph may evolve.
Our weights will be called themes.
The sets T of themes and T^+ of positive themes are defined recursively as follows:

- a theme is a pair of a sign and a positive theme
- a positive theme is a family of themes (label-theme
- or is a pointed T-graph (edge-theme)

Exercise: figure out initial cases!

Definition

A mutable graph is a T-graph, where T is the set of themes.

Mutations

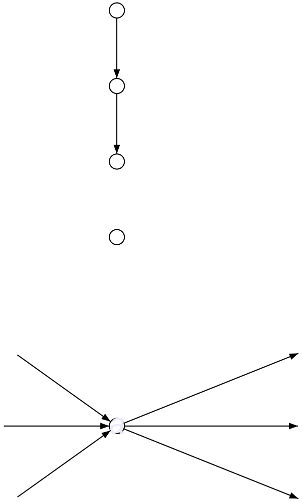
- Given an edge e of G,
- a mutation of G at e is an embedding of $G - e^+$ into another W-graph H
- which is open on $G - e$ and where orphan edges are shared
- where $G - e^+$ denotes the W-graph obtained from G by deleting e and its source (if any).

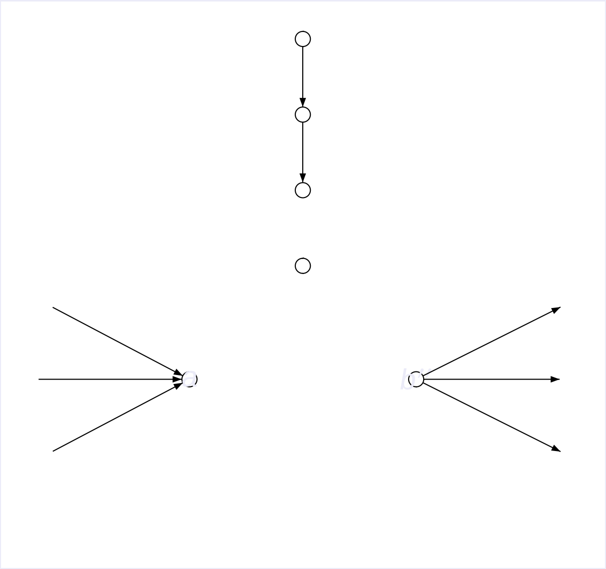
Later on, we will glue G and H along a common part (orphan edges are duplicated and we will add a 2-cell).

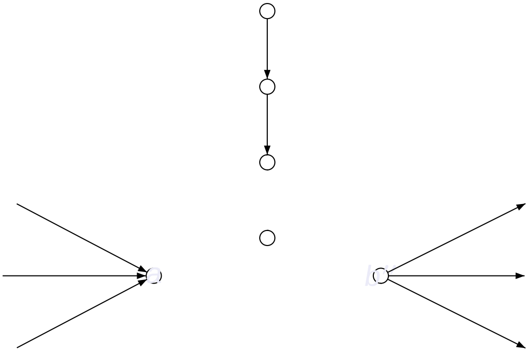
We will have
vertex-mutations, label-mutations and edge-mutations

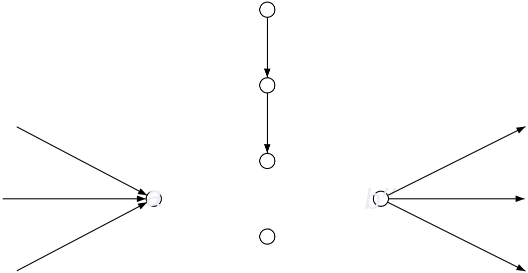
Vertex-mutations

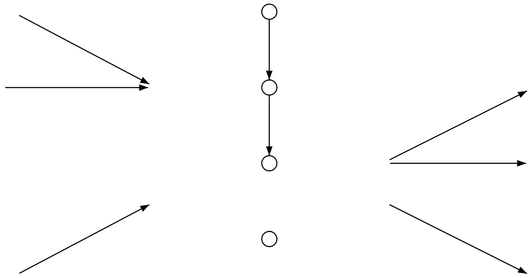
- perform a W-vertex-mutation = replace a vertex by a W-graph F and share orphan edges
- F is called the central fiber of the vertex-mutation
- a vertex-mutation is cut-free if the central fiber has no edge

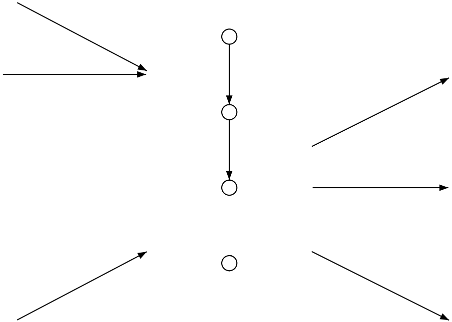


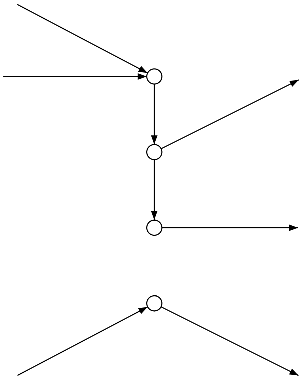






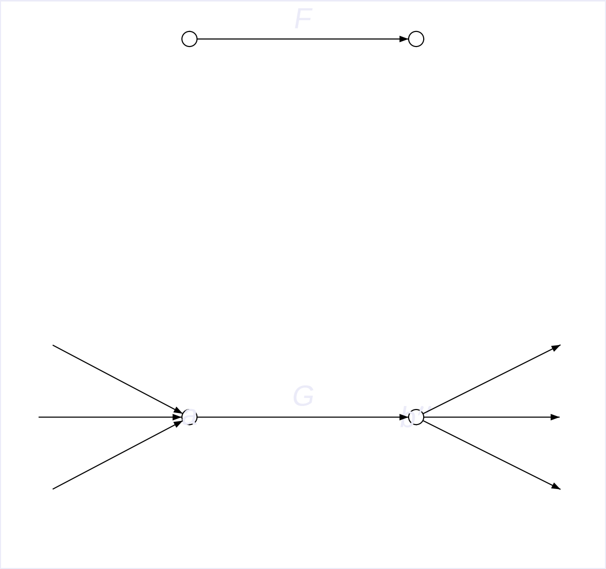






Label-mutations

- perform a W-label-mutation: change the weight of an edge
- the new weight may be of opposite sign
- all label-mutations are cut-free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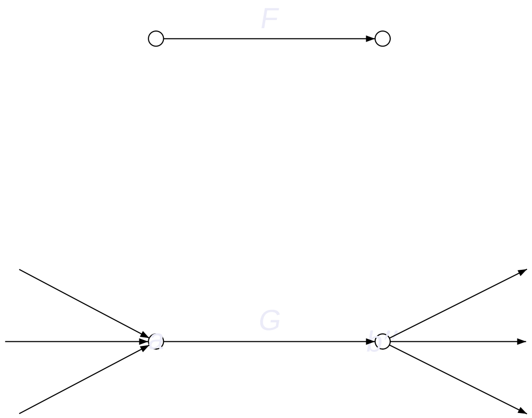
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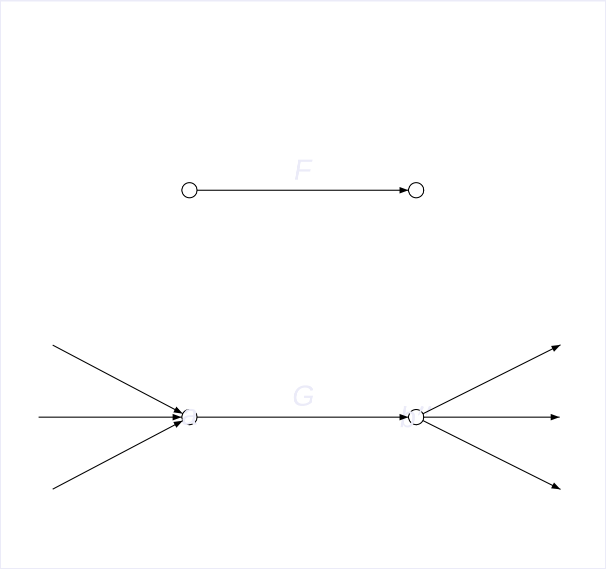
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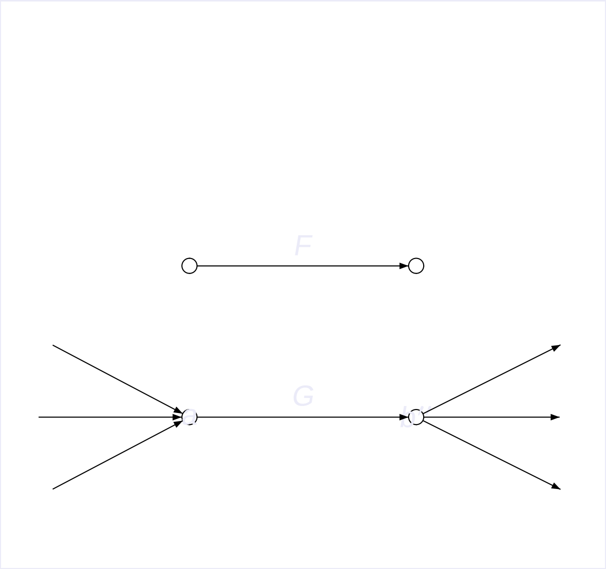

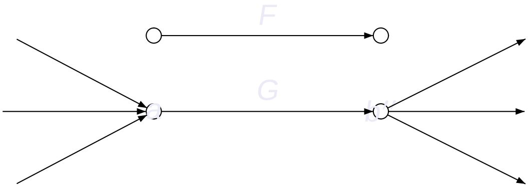
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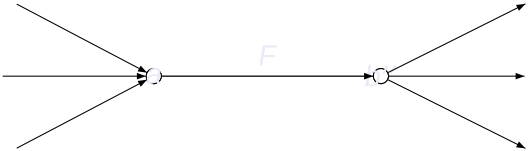
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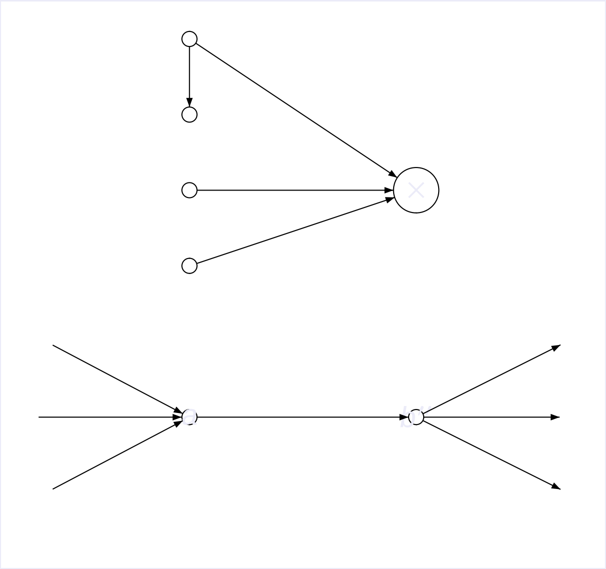
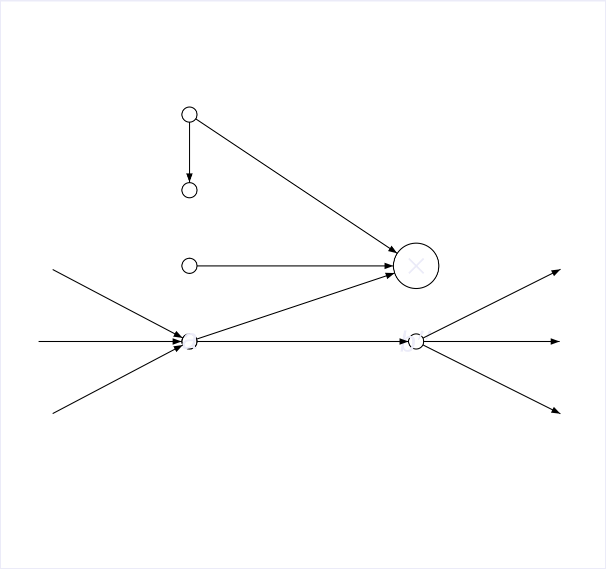
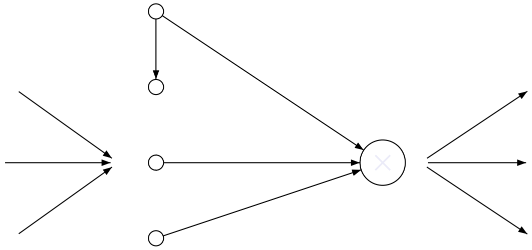
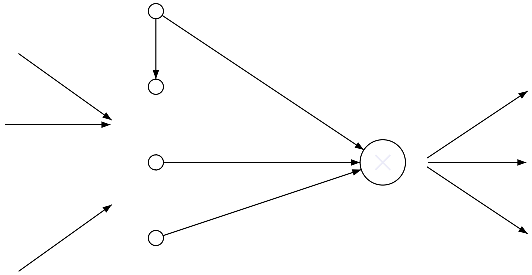
Edge-mutations

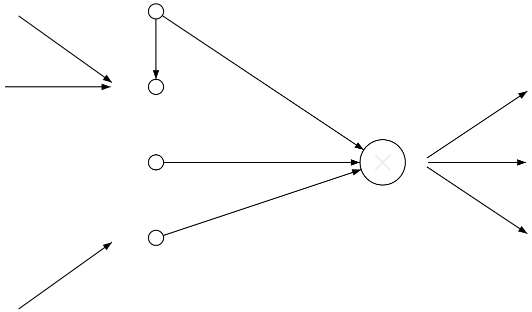
- perform a W-edge-mutation = replace an edge by a marked W-graph F and share orphan edges
- F is called the new piece of the edge-mutation
- an edge-mutation is cut-free if each new edge has the marked vertex in its boundary

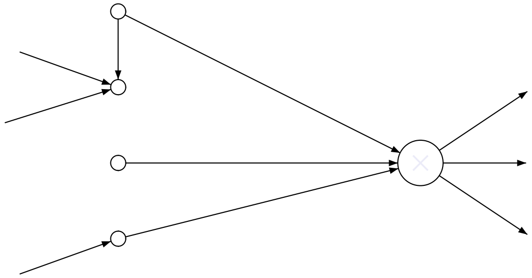












T-moves

- T-moves are special T-mutations,
- those which push the theme:
- Any T-vertex-mutation is a T-move
- A T-label-mutation is a T-move if
- the old label is a family, and the new one is a member of this family
- A T-edge-mutation is a T-move if
- its new piece is the label of the deleted edge.

Connectors for themes: negation

¬ is the tautological involution on T

Logical constants as themes: 1, $\perp$

- 1 is the positive theme defined by the T-graph with a single (pointed) vertex
- can be pushed only if no orphan edges on source side!
- $\perp$ is -1

Logical constants as themes: 0, $\top$

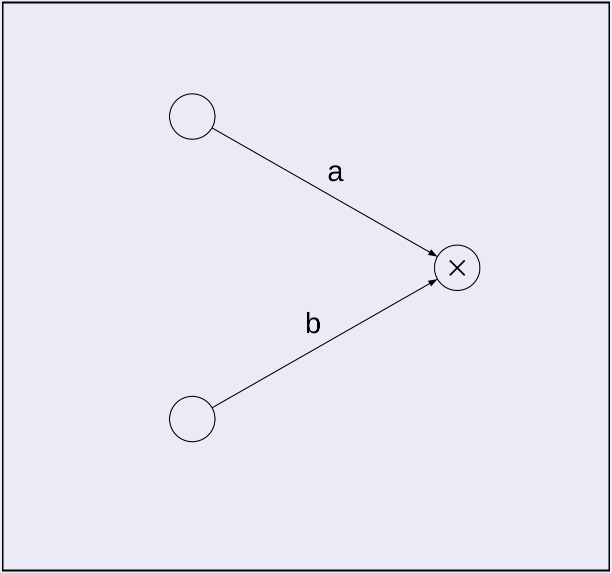
The empty choice!

- 0 is the positive theme defined by the empty family (of themes)
- $\top$ is -0

Connectors for themes: $\otimes$, $⅋$

- given two themes m, n,
- $m \otimes n$ is the edge-theme with two edges
- labelled respectively with m and n,
- and converging to the marked vertex.
- it also denotes the corresonding positive theme
- $⅋$ is defined by De Morgan's law

tensor product



Connectors for mutations: $\oplus$, &

- given two mutations m, n,
- $m \oplus n$ is the label-theme defined by the family (m, n)
- & is defined by De Morgan's law
- $m \,\&\, n = \neg(\neg m \oplus \neg n)$

Connectors for mutations: ?, !

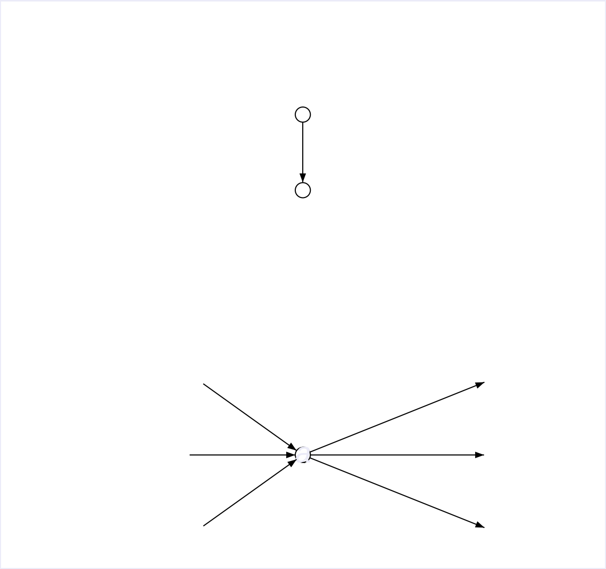
- given a theme m,
- $?m$ would be the label-theme defined by the family $(\neg 1, m, ?m \parr ?m)$
- Warning: something should be coinductive somewhere
- ! is defined by De Morgan's law

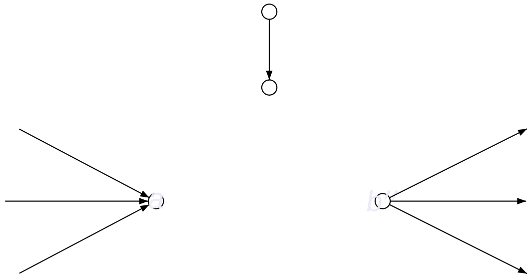
Quantification for mutations: ∃, ∀

- given a family $m := (m_i)_{i \in I}$ of themes indexed by a set I,
- $\exists m$ is the label-theme defined by m
- $\forall$ is defined by De Morgan's law

Cut as a vertex-mutation

- Given a theme c, $cut c$ is the vertex-mutation with one edge labelled by c
- Daimon could be here disguised as $cut \top$ (put the token at target and no orphan edge there)

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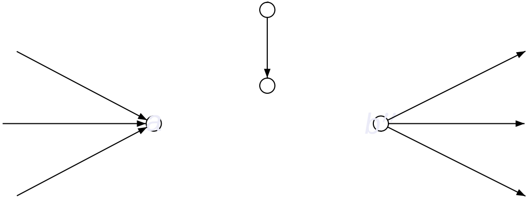
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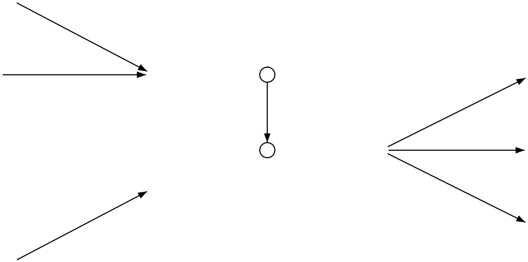
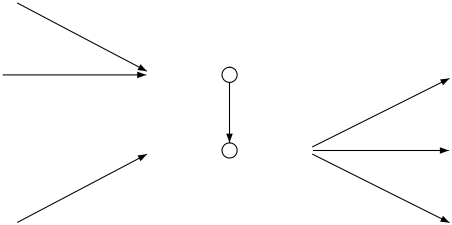
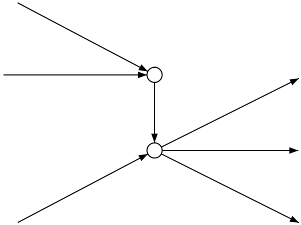
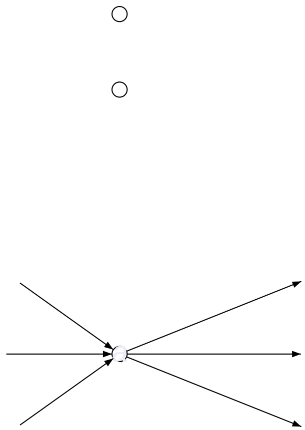


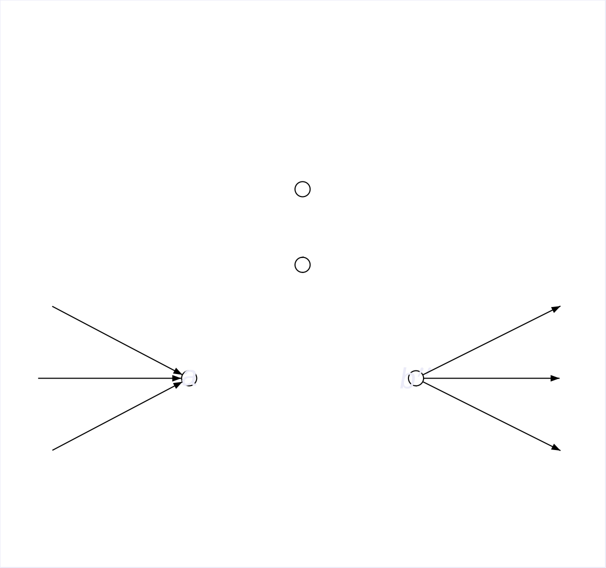


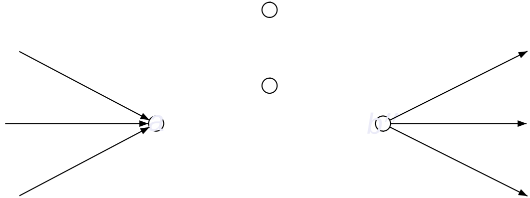


Mix/W as a vertex-mutation

- Mix is the vertex-mutation with two vertices (and no edge)
- Daimon could be here again: put the token at a fresh isolated vertex!



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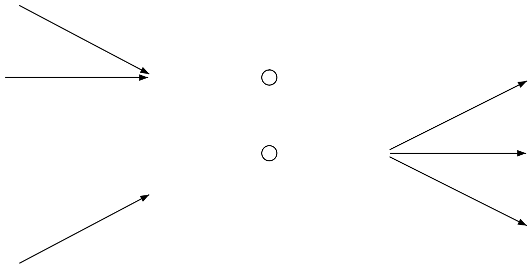
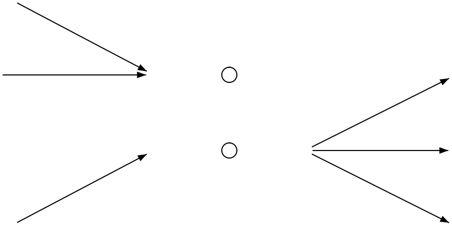
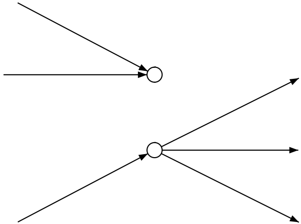






Mistigri

Let us play on a mutable graph, with a token

- At the beginning; put the token somewhere
- Passive moves: push the token along a negative edge
- Internal moves: perform a vertex-mutation (at the token, and lift the token)
- Active move: perform an edge-mutation or a label mutation along a positive edge at the token
- the token crosses the edge (just before processing)
- At the end, the player/team with the token loses
- the token cannot make a passive loop (acyclicity)

Sequential plays

- Sequence of composable moves by the owner of the token
- Winner: escapes the graph
- Loser: keeps the token
- Noetherianity: no infinite plays (but no exponentials yet)

So what?

- You can do some game theory but
- You can restrict plays to open subset
- and raise the question:
- do you get a sheaf? or a stack?
- and the answer is
- No! Try something else.

n-spaces

A small class of topological spaces:

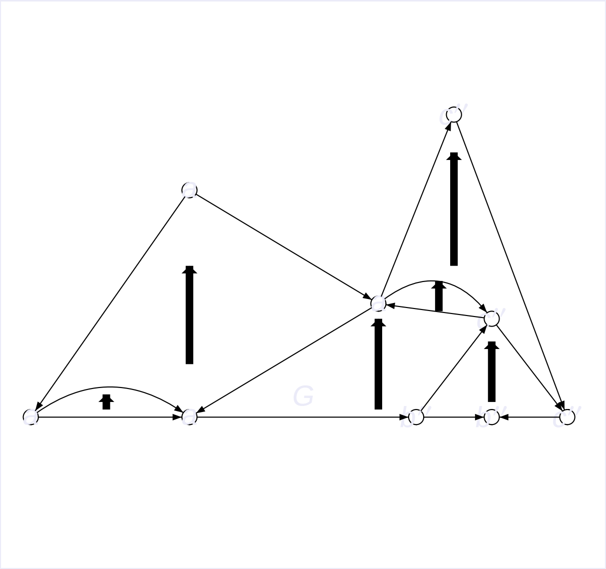
- a n-space is a topological space
- equipped with a dimension function d
- with values in $[0, n]$, satisfying
- $d(x) < d(y)$ for x in the boundary of y
- a point of dimension d is called a d-cell.

Our graphs are naturally 1-spaces. We will glue 1-spaces with 2-cells.

Glued mutable graphs

A history on a 2-space assigns

- to each 1-cell a theme
- so that the 1-skeleton becomes a (non-acyclic) mutable graph
- to each 2-cell, the description of a mutation
- from one half of its boundary to the other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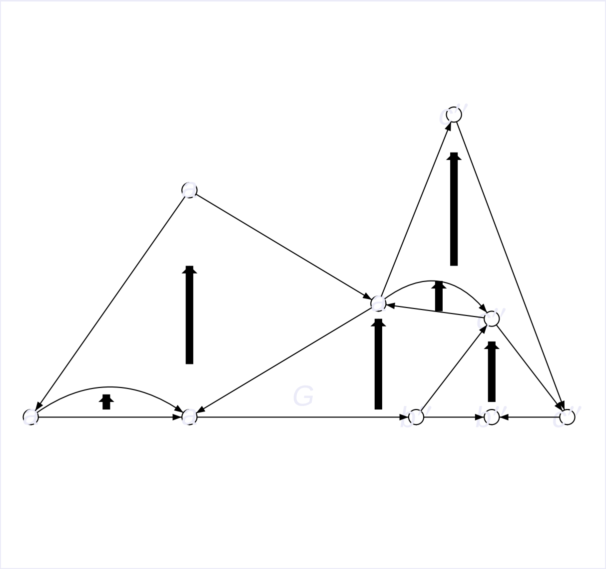
The stack of stories

Given a mutable graph G

- a story on G is a 2-space
- equipped with a history
- containing (a copy of) G
- and obtained from it by adding successive 2-cells

A story has a final state, which is a mutable graph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The story of a vertex-mutation

For a vertex-mutation concerning an open sequent s

- $m : G \to H$
- glue G and H along $G - s$
- add a 2-cell with orphan edges and their twins in its boundary
- with the suitable history

The story of a label-mutation

For a label-mutation concerning an edge e

- $m : G \to H$
- glue G and H along $G - e$
- add a 2-cell with e and the new edge in its boundary
- with the corresponding history.

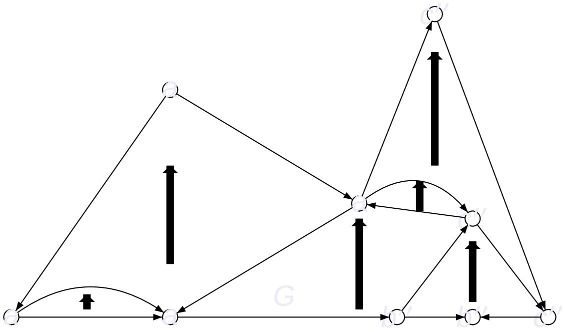
The story of an edge-mutation

For an edge-mutation concerning an open sequent s

- $m : G \to H$
- glue G and H along $G - s$
- add a 2-cell with orphan edges and the new piece in its boundary
- with the corresponding history.

The story of a sequential play

is obtained by glueing successive moves as before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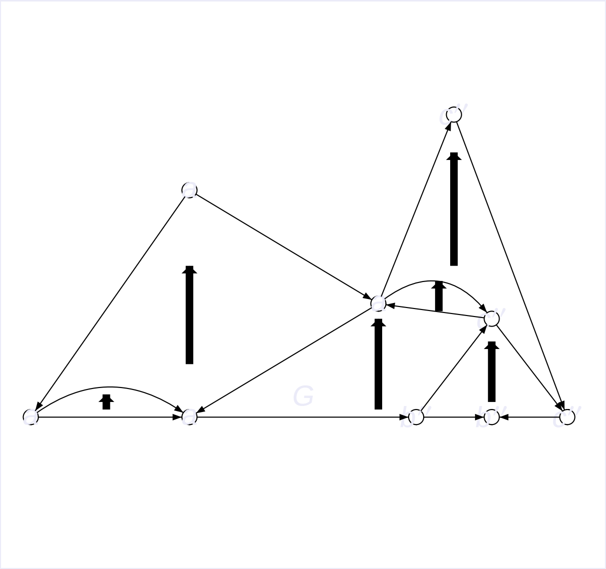
The stack of stories on G

Given a mutable graph

- stories on G come with a projection ot G, thus
- stories on G can be restricted to open subsets, thus
- stories form a prestack on G
- which is a stack $Sto(G)$.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Special substacks of $Sto(G)$

Given a mutable graph G

- $Sto_{cf}(G)$ is the substack of cut-free stories
- $Sto_{w}(G)$ is the stack of winning stories

Strategies on G will be substacks of $Sto(G)$.

The sheaf of substacks

Given a stack S on X

substacks of S form a sheaf.

Properties of substacks

Given a substack S' of $Sto(G)$

- S' is full if with any local story, it contains isomorphic ones
- S' is prefix-closed if with any local story, it contains the prefixes for some sequentialization 'not all!)
- S' is welcoming if with any local story, it contains its extensions by opponent moves
- S' is noetherian if it contains no infinite local story
- S' is complete if any local story there can be extended at any vertex of the final state where ther is no incoming edge
- S' is deterministic if a local story there can be extended by two different player's moves, then it can be extended by both together

These properties are required locally, and pass to the global case.

Strategies

Given a mutable graph G

A strategy for G is a full, prefix-closed and welcoming substack of $Sto(G)$

The strategy associated to a MALL proof

Given a T-sequent S with weights in MALL and a proof of S

Exercise: build the corresponding strategy

Special sheaves of strategies

Given a mutable graph G

- Denote by $Strat(G)$ the sheaf of strategies on G
- $Strat_{cf}(G)$ is the subsheaf of cut-free strategies
- $Strat_{det}(G)$ is the subsheaf of deterministic strategies
- $Strat_{w}(G)$ is the subsheaf of winning strategies

Cut-elimination

Two aspects of cut-elimination

- Descent; given a vertex-mutation $m : H \to G$, cut-elimination yields a stack morphism $ce : m_* Sto_{cf}(H) \to Sto_{cf}(G)$ sending winning stories to winning stories
- Stein factorization: given a mutable graph G, cut-elimination yields a sheaf morphism $ce : Strat(G) \to Strat_{cf}(G)$ sending winning strategies to winning strategies

Future work

- Find good classes of strategies
- Find good equivalence of strategies (full abstraction)
- Understand exponentials
- Understand existence