

## Generalized flow and determinism in measurement-based quantum computation

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**Abstract.** We extend the notion of quantum information flow defined by Danos and Kashefi (2006 *Phys. Rev. A* **74** 052310) for the one-way model (Raussendorf and Briegel 2001 *Phys. Rev. Lett.* **86** 910) and present a necessary and sufficient condition for the stepwise uniformly deterministic computation in this model. The generalized flow also applied in the extended model with measurements in the  $(X, Y)$ ,  $(X, Z)$  and  $(Y, Z)$  planes. We apply both measurement calculus and the stabiliser formalism to derive our main theorem which for the first time gives a full characterization of the stepwise uniformly deterministic computation in the one-way model. We present several examples to show how our result improves over the traditional notion of flow, such as geometries (entanglement graph with input and output) with no flow but having generalized flow and we discuss how they lead to an optimal implementation of the unitaries. More importantly one can also obtain a better quantum computation depth with the generalized flow rather than with flow. We believe our characterization result is particularly valuable for the study of the algorithms and complexity in the one-way model.

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**1. Introduction**

Measurement-based quantum computation, and, more specifically, the one-way quantum computation model [1] provides both a new theoretical description and a novel prescription for implementations of quantum computing. In the standard model of quantum computation—the circuit model—a computation is described as a network of unitary single and two-qubit gates acting on a register of qubits, followed, at the end, by measurement of each qubit. In the one-way quantum computation [1], first a special entangled multi-qubit state called a *graph state* or *cluster state* [2] is prepared, then the qubits are measured in a specified order and in specified bases. An important aspect of this model is the way the inherent randomness of the measurement outcomes is accounted for, so that the overall computation remains deterministic. This is accomplished by conditioning the basis of certain measurements upon the outcome of others, and introducing a measurement order.

In [1, 3] a prescription is given to construct deterministic measurement-patterns equivalent to any gate network. Nevertheless, one of the potential advantages of this new model is the possibilities it opens up for the development and design of quantum algorithms in a completely new picture, without direct recourse to the circuit model. Thus an important question is, given a particular graph state and set of measurements, can these measurements be adapted in such a way that determinism of the computation is guaranteed? What is the structure of graph states and measurements which can be considered, such that the computation remains deterministic? In this paper, we provide a general framework to address such questions.

Previously, a geometric condition on a graph state known as *flow*, was developed which guaranteed that graph states satisfying a set of flow conditions would admit a deterministic computation, provided measurements were restricted to the  $(X, Y)$  plane of the Bloch sphere [4]. Nevertheless, this condition was sufficient but not necessary and took into account neither the possibility of measurements in other planes of the Bloch sphere nor the special way graph states transform under measurement of Pauli-operator observables [5]. In this paper, we provide a *generalized flow* condition. This is a set of geometric conditions on a graph, which are necessary and sufficient for that state to admit a stepwise uniformly deterministic one-way quantum computation under measurements in the  $(X, Y)$ ,  $(X, Z)$  and  $(Y, Z)$  planes.

The structure of this paper is as follows. We begin by reviewing the *Measurement Calculus*—an algebraic framework to represent and analyse measurement patterns in one-way

quantum computation and its extension [6]. After reviewing the flow condition introduced in [4], we shall then present definitions of generalized flow and prove its properties. We show how this can be further generalized in the cases where certain qubits are restricted to Pauli measurements alone before concluding with some examples of the application of these concepts.

## 2. A calculus for measurement-based quantum computation

In this section, we shall review the *Measurement Calculus* [6, 7], an algebraic means of representing measurement patterns that will provide us with the tools we shall employ in the remainder of this paper. We therefore restrict ourselves to variants of measurement-based quantum computing (MBQC) which can be described in this framework. This includes the dominant MBQC proposal, namely one-way quantum computation on graph states, but does not include probabilistic MBQC models such as [8]–[11].

We define an arbitrary pure single qubit state by

$$|+\theta,\phi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

and denote its orthogonal state (the opposite point in the Bloch sphere) with

$$|-\theta,\phi\rangle = \sin\left(\frac{\theta}{2}\right)|0\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|1\rangle,$$

where  $0 \leq \theta \leq \pi$  and  $0 \leq \phi \leq 2\pi$ .

A computation in the one-way model consists of the preparation of a suitable entangled resource state, called a graph state, and the measurement of individual qubits in a certain order, in bases which have a dependency on certain previous measurement outcomes. The graphical structure of the resource state and the measurement angles and their dependency all contribute. These ‘measurement patterns’ can be formalized as a sequence of the following commands:

1. 1-qubit preparations,  $N_i$ , to prepare the qubit  $i$  in state  $|+\rangle_i = |+\frac{\pi}{2},0\rangle_i$ ,
2. 2-qubit controlled- $Z$  entangling operators,  $E_{ij} := \Lambda Z_{ij}$ ,
3. 1-qubit destructive measurements,  $M_i^{\lambda,\alpha}$ , on plane  $\lambda \in \{(X, Y), (X, Z), (Y, Z)\}$ , defined by orthogonal projections into:

- $|\pm_{(X,Y),\alpha}\rangle := |\pm_{\frac{\pi}{2},\alpha}\rangle$  if  $\lambda = (X, Y)$ ,
- $|\pm_{(X,Z),\alpha}\rangle := |\pm_{\alpha,0}\rangle$  if  $\lambda = (X, Z)$ ,
- $|\pm_{(Y,Z),\alpha}\rangle := |\pm_{\alpha,\frac{\pi}{2}}\rangle$  if  $\lambda = (Y, Z)$ ,

with the convention that  $|+\theta,\phi\rangle\langle+\theta,\phi|_i$  corresponds to the outcome 0, while  $|-\theta,\phi\rangle\langle-\theta,\phi|_i$  corresponds to 1,

4. 1-qubit local Pauli corrections:  $X_i$  and  $Z_i$ ,

where  $i, j$  represent the qubits on which each of these operations apply, and  $0 \leq \alpha \leq \pi$ . Qubits are measured at most once, therefore we may represent unambiguously the outcome of the

measurement outcome for qubit  $j$  by  $s_j$ . To control the non-determinism of the measurement outcomes, certain local corrections will depend upon previous measurement outcomes. These dependent local corrections will be written as  $C_i^{s_j}$ , with  $C_i^0 = I$  and  $C_i^1 = C_i$ . It is convenient to employ some set-theoretic notations, for example,  $A^c$  denotes the complement of a subset  $A$ .

A *measurement pattern*  $\mathfrak{P} = (V, I, O, \lambda, S)$ , or simply a pattern, is defined by the choice of  $V$  a finite set of qubits, two possibly overlapping subsets  $I \subseteq V$  and  $O \subseteq V$  determining the pattern inputs and outputs, a finite sequence of commands acting on  $V$ , and a function  $\lambda : O^c \rightarrow \{(X, Y), (X, Z), (Y, Z)\}$  which specifies the plane of the measurement on each measured qubit (i.e. the non-output qubits  $O^c$ ). We call a pattern *runnable* where no command depends on an outcome not yet measured, no command acts on a qubit already measured or not yet prepared (except preparation commands), and a qubit  $i$  is measured (prepared) if and only if  $i$  is not an output (input).

We will see that it is useful to be able to *standardise* patterns: to put the commands sequence of the pattern in a particular order without changing the meaning of the computation. We choose an order where all the preparation commands appear first (i.e. right-most), then all the entanglement commands, followed by the measurements and then corrections. The standardisation procedure is the basis for the new physical architecture proposed by measurement-based quantum computing where one performs all the entanglement in the beginning followed by local operation and classical communications. Furthermore, the rewriting of a pattern to standard form allows one to check that a given pattern is runnable and it reveals parallelism in the pattern computation [6, 12]. Finally, the standard form provides us with tools to prove general theorems about patterns by exploring the geometry of the entanglement resource as we will see later in this paper.

To interchange the order of operations to bring them into the standard form we employ the following ‘commutation rules’:

$$\begin{aligned} M_i^{(X,Y),\alpha} X_i &= M_i^{(X,Y),-\alpha}, & M_i^{(X,Y),\alpha} Z_i &= M_i^{(X,Y),\alpha+\pi}, & M_i^{(X,Z),\alpha} X_i &= M_i^{(X,Z),-\alpha+\pi}, \\ M_i^{(X,Z),\alpha} Z_i &= M_i^{(X,Z),-\alpha}, & M_i^{(Y,Z),\alpha} X_i &= M_i^{(Y,Z),\alpha+\pi}, & M_i^{(Y,Z),\alpha} Z_i &= M_i^{(Y,Z),-\alpha+\pi}. \end{aligned}$$

Since the entanglement operations are to be performed first, when their order is interchanged with the correction operators they should not pick up the measurement-dependency in these corrections. This can be guaranteed by ensuring that the entanglement command is in the normalizer group of the group generated by the local Pauli correction groups. In order to commute the corrections to the end of the pattern we simply use the following equations, therefore we have the following simple observation.

**Proposition 1.** *Any one-way MBQC model admits a standardisation procedure if and only if the entanglement operator is the normalizer of all the correction operators.*

We write  $\mathfrak{H}_I$  ( $\mathfrak{H}_O$ ) for the Hilbert space spanned by the inputs (outputs). The *run* of a pattern consists simply in executing each command in sequence. If  $n$  is the number of measurements (which is also the number of ‘non-output’ qubits) then the run may follow  $2^n$  different branches. Each branch is associated with a unique binary string  $\mathbf{s}$  of length  $n$ , representing the classical outcomes of the measurements along that branch, and a unique *branch map*  $A_{\mathbf{s}}$  representing the linear transformation from  $\mathfrak{H}_I$  to  $\mathfrak{H}_O$  along that branch.

Branch maps decompose as  $A_{\mathbf{s}} = C_{\mathbf{s}} \Pi_{\mathbf{s}} U$ , where  $C_{\mathbf{s}}$  is a unitary map over  $\mathfrak{H}_O$  collecting all corrections on outputs,  $\Pi_{\mathbf{s}}$  is a projection from  $\mathfrak{H}_V$  to  $\mathfrak{H}_O$  (where  $\mathfrak{H}_V$  is the Hilbert space

spanned by all the qubits) representing the particular measurements performed along the branch, and  $U$  is a unitary embedding (or isometry) from  $\mathfrak{H}_1$  to  $\mathfrak{H}_V$  collecting the branch preparations, and entanglements. Therefore

$$\sum_s A_s^\dagger A_s = \sum_s U^\dagger \Pi_s U = I$$

and  $T(\rho) := \sum_s A_s \rho A_s^\dagger$  is a trace-preserving completely-positive map (cptp-map), explicitly given as a Kraus decomposition. One says that the pattern *realizes*  $T$ . We can now define various notions of determinism.

A pattern is said to be *deterministic* if it realizes a cptp-map that sends pure states to pure states, i.e. the branch maps are proportional. The class of deterministic patterns include projections, for example the pattern  $X_1^{s_2} M_2^0 E_{12} N_2$  implements  $|0\rangle\langle 0|$ . A more restricted class contains all the unitary and unitary embedding operators: a pattern is said to be *strongly deterministic* when branch maps are equal (up to a global phase), i.e. for all  $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}_2^n$ ,  $A_{\mathbf{s}_1} = e^{i\phi_{\mathbf{s}_1, \mathbf{s}_2}} A_{\mathbf{s}_2}$ . These are the patterns implementing quantum algorithms and hence understanding their structural properties are of particular interest. An important sub-class of deterministic patterns are those robust under the changes of the angles: a pattern is said to be *uniformly deterministic* if it is deterministic for all values of its measurement angles. In other words a uniformly deterministic pattern defines a class of quantum operators that can be performed given the same initial entanglement resources. Finally, a pattern is said to be *stepwise deterministic* if it is deterministic after performing each single measurement together with all the corrections depending on the result of that measurement. In other words, a pattern is stepwise deterministic if after each single measurement there exists a set of local corrections that will make the two branches equal (up to a global phase), which depends only on the result of this measurement and which has support of some or all of the non-measured qubits. It is easy to observe that stepwise determinism leads to a circuit type structure. This is illustrated further in the examples section.

The main result of this paper is a necessary and sufficient condition for strong uniform determinism based on the geometry of the entanglement structure which underlies a measurement pattern. Note that this class of patterns are universal for unitaries and unitary embeddings and captures interesting structural properties of them as described above.

Let us define an *open graph state*  $(G, I, O, \lambda)$  to be a state associated with an undirected graph  $G$  together with two subsets of nodes  $I$  and  $O$ , called inputs and outputs. We write  $V$  for the set of nodes in  $G$ ,  $I^c$ , and  $O^c$  for the complements of  $I$  and  $O$  in  $V$ ,  $N_G(i)$  for the set of neighbours of  $i$  in  $G$ ,  $i \sim j$  for  $(i, j) \in G$ , and  $E_G := \prod_{i \sim j} E_{ij}$  for the global entanglement operator associated to  $G$ . We first recall the definition of *flow*, under which one can construct a set of dependent corrections such that the obtained pattern is strongly and uniformly deterministic [4].

**Definition 2.** An open graph state  $(G, I, O, \lambda)$ , such that  $\forall i \in O^c, \lambda(i) = (X, Y)$ , has *flow* if there exists a map  $f : O^c \rightarrow I^c$  (from measured qubits to prepared qubits) together with a partial order  $>$  over  $V$  such that for all  $i \in O^c$ :

- (F1)  $i \sim f(i)$ ,
- (F2)  $i < f(i)$ ,
- (F3)  $\forall k \in N_G(f(i)) \setminus \{i\}$  we have  $i < k$ .

As one can see, a flow consists of two structures: a function  $f$  over vertices and a matching partial order over vertices. In order to obtain a deterministic pattern for an open graph state with flow, dependent corrections will be defined based on function  $f$ . The order of the execution of the commands is given by the partial order induced by the flow. The matching properties between the function  $f$  and the partial order  $>$  will make the obtained pattern runnable. Roughly speaking, flow is a collection of disjoint input–output paths with no loops, as shown in figure 6. This intuition is indeed the key concept in a flow finding algorithm [13, 14].

**Theorem 1 .** [1] *Suppose the open graph state  $(G, I, O, \lambda)$ , such that  $\forall i \in O^c, \lambda(i) = (X, Y)$ , has flow  $(f, >)$ , then the pattern:*

$$\mathfrak{P}_{f,G} := \prod_{i \in O^c}^{>} \left( X_{f(i)}^{S_i} Z_{N_G(f(i)) \setminus \{i\}}^{S_i} M_i^{(X,Y)\alpha_i} \right) E_G N_{I^c},$$

where the product follows the dependency order  $>$ , is runnable, uniformly and strongly deterministic, and realizes the unitary embedding:

$$U_G := \left( \prod_{i \in O^c} \langle +_{(X,Y),\alpha_i} | i \right) E_G N_{I^c}.$$

The above theorem provides a sufficient condition for determinism for the one-way model considering only measurements in the  $(X, Y)$  plane, which encompasses, for example, the measurement patterns proposed [1, 3]. Nevertheless, it can be useful to construct patterns which contain measurements in other planes [15], and this arises naturally when one uses the graph transformation rules associated with Pauli measurements [5] to reduce the size of a pattern. As we describe in this paper one can extend the notation of flow to obtain a necessary and sufficient condition considering measurements in all the  $(X, Y)$ ,  $(X, Z)$  and  $(Y, Z)$  planes. This will lead to a full characterization of deterministic computation in the MBQC models. As a result, we also obtain a tight bound on depth complexity that improves the presented results in [12].

### 3. Generalized flow

In order to describe the motivation behind our construction of the generalized flow, we first briefly explain the main idea behind the proof of the flow theorem (theorem 1). Recall that the *graph stabiliser* [5] at qubit  $i$  is defined as  $K_i = X_i (\prod_{j \in N_G(i)} Z_j)$  and one has the following relation for all  $i \in I^c$ :

$$K_i E_G N_{I^c} = E_G N_{I^c}. \quad (1)$$

Note that the above equation is slightly more general than the common graph stabiliser [5] as it can be applied to open graph states where input qubits are prepared in arbitrary states. Let  $i$  be a non-input qubit, prepared in  $|+\rangle$  state, and  $G'$  be the sub-graph obtained after removing the

vertex  $i$ , we have

$$\begin{aligned}
 K_i E_G N_{I^c} &= \prod_{j \in N_G(i)} Z_j X_i \prod_{j \in N_G(i)} E_{ij} E_{G'} N_{I^c} \\
 &= \prod_{j \in N_G(i)} Z_j \prod_{j \in N_G(i)} E_{ij} X_i \prod_{j \in N_G(i)} Z_j E_{G'} N_{I^c} \\
 &= \prod_{j \in N_G(i)} Z_j \prod_{j \in N_G(i)} Z_j \text{ graph states } E_G X_i N_i N_{I^c \setminus i} \\
 &= E_G N_{I^c}.
 \end{aligned}$$

The proof of theorem 1 is based on the following simple observation. We could make a measurement  $M_i^{(X,Y)\alpha}$  ‘deterministic’ (corrected) if it could be pre-composed by an acausal  $Z_i^{s_i}$  correction (i.e. conditioned on the outcome of a measurement which has not happened yet). This unphysical scenario is a useful starting point for our proof,

$$\langle +_{(X,Y),\alpha} | i \rangle = M_i^{(X,Y)\alpha} Z_i^{s_i}.$$

The flow construction guarantees that such a deterministic pattern with acausal corrections can be transformed into a runnable pattern, where all dependencies now do respect the proper causal ordering. It is easy to verify that, the pattern  $\mathfrak{P}_{f,G}$  in theorem 1 can be equivalently written in terms of acausal measurements as:

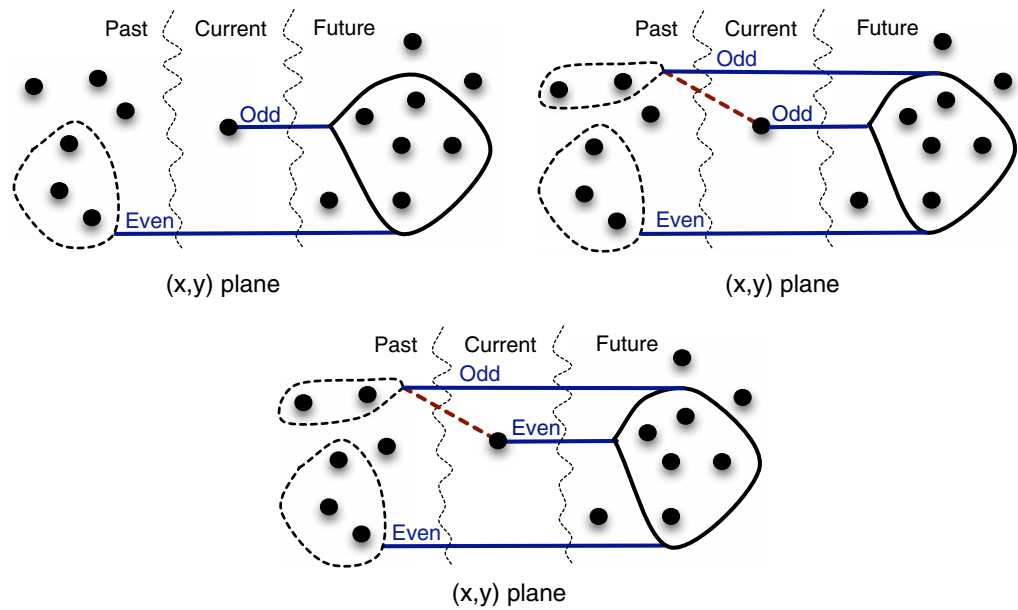
$$\mathfrak{P}_{f,G} = \prod_{i \in O^c}^> \left( M_i^{(X,Y)\alpha_i} Z_i^{s_i} K_{f(i)}^{s_i} \right) E_G N_{I^c}.$$

The key observation which allows us to transform this into a runnable pattern is that the flow conditions mean that there exists a stabiliser  $K_{f(i)}$  which when composed with the acausal correction, forms an operator which commutes with the measurement, and thus the pattern can be brought into runnable order.

A natural way to extend this idea is to consider a set of vertices as a *correcting set*. Hence instead of working with a function  $f : O^c \rightarrow I^c$  defining the correcting vertices, we will have a function  $g : O^c \rightarrow \mathcal{P}^{I^c}$  defining the correcting sets of vertices, where  $\mathcal{P}^{I^c}$  denotes the power set of all the subsets of vertices in  $I^c$ . It is important to note that the condition on these correcting sets will depend on the plane which the measurement will be performed, as measurement in different planes requires a different acausal correction. We define the odd neighborhood of a set of vertices  $K$  to be the set  $\text{Odd}(K) = \{u, |N_G(u) \cap K| = 1 \pmod{2}\}$ .

**Definition 3.** An open graph state  $(G, I, O, \lambda)$  has *generalized flow* if there exists a map  $g : O^c \rightarrow \mathcal{P}^{I^c}$  (from measured qubits to a subset of prepared qubits) and a partial order  $<$  over  $V$  such that for all  $i \in O^c$ ,

- (G1) if  $j \in g(i)$  and  $i \neq j$  then  $i < j$ ,
- (G2) if  $j \leq i$  and  $i \neq j$  then  $j \notin \text{Odd}(g(i))$ ,
- (G3) if  $\lambda(i) = (X, Y)$  then  $i \notin g(i)$  and  $i \in \text{Odd}(g(i))$ ,
- (G4) if  $\lambda(i) = (X, Z)$  then  $i \in g(i)$  and  $i \in \text{Odd}(g(i))$ ,
- (G5) if  $\lambda(i) = (Y, Z)$  then  $i \in g(i)$  and  $i \notin \text{Odd}(g(i))$ .

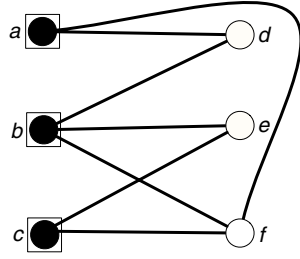


**Figure 1.** A pictorial presentation of the generalized flow conditions (G1–G5) for different measurement planes. The straight blue lines stand for multiple edges in the entanglement graph. The dotted red line represents single edges back from the current qubit to past qubits. The labels give the parity (odd or even) of the number of connections permitted between the qubits. The single qubit in the current layer denotes the qubit to be measured and its correcting set lies in the future layer (black closed curve). The neighbours of the correcting set which belong to the past layer are denoted by closed dotted curve.

One can consider the partial order  $<$  as a notion of time. Then, condition (G2) says that all the vertices with an odd number of connections to the correcting set  $g(i)$  should belong to the past of  $i$ . These conditions can be better understood pictorially, as illustrated in figure 1. Similar to the theorem 1, we will apply dependent stabiliser corrections on all the qubits in the correcting set. The evenness or oddness condition on the number of the connections between a vertex and its correcting set and neighbors will guarantee that the acausal correction on qubit  $i$  can be transformed to a correction with causal dependencies. In simple words, a generalized flow is also a collection of input–output paths (similar to flow), however, some particular type of loops are permitted, see figure 2.

#### 4. Determinism theorem

A necessary and sufficient condition for determinism in the extended one-way model is given in the following two theorems. It is important to note that this condition can be easily extended to any other MBQC models (e.g. teleportation-based models [8, 16]), since there exist compositional embeddings from the one-way model to all other MBQC models [6]. Recall that  $g(i)$ , where  $g$  is a generalized flow, is a subset of vertices.



**Figure 2.** A graph with generalized flow but no flow:  $g(a) = d$ ,  $g(b) = e$ ,  $g(c) = \{d, f\}$ . That means for example, following the proof of theorem 1 to correct the measurement at qubit  $a$ , one will apply the stabiliser of qubit  $d$  which equals to  $(X_d Z_a Z_b)^{s_a}$ .

**Theorem 2 .** Suppose the open graph state  $(G, I, O, \lambda)$  has generalized flow  $(g, >)$ , then the pattern:

$$\mathfrak{P}_{g,G} = \prod_{i \in O^c}^> \left( X_{g(i) \setminus \{i\}}^{s_i} Z_{\text{Odd}(g(i))}^{s_i} M_i^{\lambda(i)\alpha_i} \right) E_G N_{I^c},$$

where the product follows the dependency order  $>$ , is runnable, uniformly, strongly and stepwise deterministic, and realizes the unitary embedding:

$$U_G := \left( \prod_{i \in O^c} \langle +_{\lambda(i), \alpha_i} | i \rangle \right) E_G N_{I^c} 0.$$

**Theorem 3 .** Suppose the pattern  $\mathfrak{P}$  is uniformly, strongly and stepwise deterministic, then the underlying geometry of  $\mathfrak{P}$  has generalized flow and the pattern realizes the unitary embedding:

$$U_G := \left( \prod_{i \in O^c} \langle +_{\lambda(i), \alpha_i} | i \rangle \right) E_G N_{I^c} 0.$$

The next lemma will be used in the proof of theorem 3 and illustrates the role that the strong condition of uniformity will play. Denote by  $P_{|\psi\rangle}$  a projection to state  $|\psi\rangle$ .

**Lemma 4 .** If for all  $\alpha$  in the  $(X, Z), (X, Y)$  or  $(Y, Z)$  plane  $P_\alpha |\psi\rangle = e^{if(\alpha)} P_\alpha |\psi'\rangle$  then  $|\psi\rangle = e^{i\theta} |\psi'\rangle$ .

**Proof.** We write the proof for the case of a projection in  $(X, Z)$  plane as other cases are similar. It suffices to consider the angles of  $\alpha = \{0, \pi/2, \pi\}$ , or in other words, measurements of  $X$  and  $Z$  observables. First, we write the states in the basis of the eigenvectors of  $Z$ :

$$|\psi\rangle = a|0\rangle|\psi_0\rangle + a'|1\rangle|\psi_1\rangle, \quad |\psi'\rangle = b|0\rangle|\psi'_0\rangle + b'|1\rangle|\psi'_1\rangle.$$

The lemma condition with  $\alpha = 0$  and  $\alpha = \pi$  (i.e. projections on to  $|0\rangle$  and  $|1\rangle$ ) implies that

$$|\psi_0\rangle = |\psi'_0\rangle, \quad |\psi_1\rangle = |\psi'_1\rangle.$$

We also obtain  $a = e^{i\phi_0}b$  and  $a' = e^{i\phi_1}b'$  and therefore

$$|\psi'\rangle = e^{i\phi_0}(a|0\rangle|\psi_0\rangle + e^{i(\phi_1-\phi_0)}a'|1\rangle|\psi_1\rangle).$$

Now consider the projection on to  $|+\rangle$  state ( $\alpha = \pi/2$ ) which implies

$$a|\psi_0\rangle + a'|\psi_1\rangle = e^{i\theta}(a|\psi_0\rangle + a'e^{i(\phi_1-\phi_0)}|\psi_1\rangle).$$

Thus  $e^{i(\phi_1-\phi_0)} = 1$  or  $a' = 0$  which completes the proof.  $\square$

The following equations concerning acausal corrections play the central role in the proof of theorem 2:

$$\langle +_{(X,Y),\alpha} | i \rangle = M_i^{(X,Y)\alpha} Z_i^{s_i}, \quad (2)$$

$$\langle +_{(X,Z),\alpha} | i \rangle = M_i^{(X,Z)\alpha} X_i^{s_i} Z_i^{s_i}, \quad (3)$$

$$\langle +_{(Y,Z),\alpha} | i \rangle = M_i^{(Y,Z)\alpha} X_i^{s_i}, \quad (4)$$

**Proof of theorem 2.** We prove one case, where all the measurements are assumed to be in  $(Y, Z)$  plane, all other cases have a similar proof. Clearly, the generalized flow conditions make the following pattern a runnable one,

$$\mathfrak{P}_{g,G} := \prod_{i \in O^c}^> \left( X_{g(i)\setminus\{i\}}^{s_i} Z_{\text{Odd}(g(i))}^{s_i} M_i^{(Y,Z)\alpha_i} \right) E_G N_{I^c}.$$

By commuting the corrections and using the definition of the graph stabiliser,  $K_i$ , we have:

$$\mathfrak{P}_{g,G} = \prod_{i \in O^c}^> M_i^{(Y,Z)\alpha_i} X_i^{s_i} K_{g(i)}^{s_i} E_G N_{I^c}.$$

Note that to derive the above equality we have also used the trivial equations  $Z_j^{s_j} Z_j^{s_j} = I$  to complete the missing part of any stabiliser. Recall that condition (G2) for  $(Y, Z)$  measurement will guarantee to have the required even number of such missing  $Z$  operators. Finally from equations (1) and (4), we obtain the following uniformly, strongly and stepwise deterministic pattern

$$\mathfrak{P}_{g,G} = \left( \prod_{i \in O^c} \langle +_{(Y,Z),\alpha_i} | i \rangle \right) E_G N_{I^c}.$$

$\square$

**Proof of theorem 3.** We start from the end of the pattern computation (i.e. the last measurement commands). Let  $P_{O^c}$  be the projector over the state  $\Pi_{i \in O^c} | +_{\lambda(i),\alpha_i} \rangle$ . Suppose the last measurement is in the plane  $(X, Y)$  and is performed on qubit  $n$ , it creates then the following two branches

$$|\Psi\rangle \xrightarrow{M_n^{(X,Y),\alpha_n}} \begin{cases} P_{O^c} E_G N_{I^c} |\psi\rangle_I, & s_n = 0, \\ P_{O^c} Z_n E_G N_{I^c} |\psi\rangle_I, & s_n = 1, \end{cases}$$

where we have used the fact  $\langle -\alpha_n | Z_n = \langle +\alpha_n |$ . Now from the stepwise determinism there exists a collection of correction  $C_A$  on the remaining non-measured qubits, in this case being output qubits, such that

$$C_A P_{Oc} Z_n E_G N_{I^c} |\psi\rangle_I = P_{Oc} E_G N_{I^c} |\psi\rangle_I.$$

Since the corrections are performed on output qubits we can commute  $C_A$  with  $P_{Oc}$  and write

$$P_{Oc} C_A Z_n E_G N_{I^c} |\psi\rangle_I = P_{Oc} E_G N_{I^c} |\psi\rangle_I.$$

The above equation is valid for any value of  $\alpha_n$  (uniformity condition) and thus according to lemma 4,  $C_A Z_n E_G N_{I^c} |\psi\rangle_I = E_G N_{I^c} |\psi\rangle_I$ , so  $C_A Z_n$  stabilises the state  $E_G N_{I^c} |\psi\rangle_I$  and since it is a Pauli operator it can be written as a product of the Pauli group generators (graph stabiliser,  $K_i$ ), thus there exists a set  $S \subseteq V$  such that

$$C_A Z_n = \prod_{u \in S} X_u Z_{N(u)}.$$

It remains to prove that  $S$  is indeed the correcting set of qubit  $n$  and satisfies the condition of the generalized flow conditions. First we show that  $S \cap I = \emptyset$ , let  $|G\rangle = E_G N_{I^c} |+\rangle_I$ , then for any arbitrary subset  $K \subseteq I$  since  $A \cap I = \emptyset$  we have

$$C_A Z_n E_G N_{I^c} Z_K |+\rangle_I = C_A Z_n Z_K |G\rangle = Z_K |G\rangle.$$

In particular for  $K = \emptyset$  we obtain  $C_A Z_n |G\rangle = |G\rangle$ . Suppose now there exists  $e \in S \cap I$  and set  $K = \{e\}$ , since  $e \in S$  then  $Z_e$  anti-commutes with  $C_A$  and therefore

$$C_A Z_n Z_e |G\rangle = - Z_e C_A Z_n |G\rangle = - Z_e |G\rangle,$$

which leads to a contradiction and proves  $S \cap I = \emptyset$ .

On the other hand, since  $A$  does not intersect with the set of already measured qubits any  $u \in S$  cannot act on a measured qubit as it will not be simplified later and  $X_u$  on the other side will appear. Also  $N(u)$  should see measured qubits evenly so that  $Z_i$  on the previously measured qubit will cancel out each other as well. Therefore  $S$  is the correcting set for the qubit  $n$  in terms of definition 3. The presented argument can be similarly carried out for all the measurements performed in the previous stages which completes the proof.  $\square$

## 5. Pauli measurements

When measurement angles are set to integer multiples of  $(\pi/2)$ , the measurements made correspond to Pauli observables. These Pauli measurements play a central role in one-way quantum computing. In particular, it is known that the action of such a measurement on a graph state is to leave the remaining qubits in a graph state (up to a local Clifford-group correction) [5]. Definition 3 provides conditions for determinism when single qubits at any angle in specified Bloch-sphere planes are allowed. The special properties of Pauli measurements (for example, that they simultaneously lie in two measurement-planes) mean that if one restricts the measurement of certain qubits to certain specific Pauli measurements, one must extend the generalized flow conditions in order to account for these extra properties.

In this section, we introduce such an extension. We will use the convention that the labelling function  $\lambda(i)$  for any non-output qubit  $i$ , is either a plan— $(X, Y)$ ,  $(X, Z)$ , or  $(Y, Z)$ —or a vector— $X$ ,  $Y$ , or  $Z$  (i.e. Pauli measurements). Firstly, notice that a Pauli measurement, say  $X$ , can be interpreted as a  $(X, Y)$  or  $(X, Z)$  measurement and thus it may satisfy the conditions of either a  $(X, Y)$  or a  $(X, Z)$  measurement. Secondly, when a qubit is measured according to a Pauli operator, say  $X$ , then, after the measurement, the state of this qubit takes  $\pm X$  as its stabiliser. We use this property to allow already-measured qubits to be included in a correcting set. Finally, the following relation between Pauli correction and Pauli measurements will be used for the Pauli flow construction

$$M^X X = M^X, \quad (5)$$

$$M^Y Y = M^Y, \quad (6)$$

$$M^Z Z = M^Z. \quad (7)$$

**Defintion 5.** An open graph state  $(G, I, O, \lambda)$  has *Pauli flow* if there exists a map  $p : O^c \rightarrow \mathcal{P}^I$  (from measured qubits to a subset of prepared qubits) and a partial order  $<$  over  $V$  such that for all  $i \in O^c$ ,

- (P1) if  $j \in p(i)$ ,  $i \neq j$ , and  $\lambda(j) \notin \{X, Y\}$  then  $i < j$ ,
- (P2) if  $j \leq i$ ,  $i \neq j$ , and  $\lambda(j) \notin \{Y, Z\}$  then  $j \notin \text{Odd}(p(i))$ ,
- (P3) if  $j \leq i$ ,  $j \in p(i)$  and  $\lambda(j) = Y$  then  $j \in \text{Odd}(p(i))$ ,
- (P4) if  $\lambda(i) = (X, Y)$  then  $i \notin p(i)$  and  $i \in \text{Odd}(p(i))$ ,
- (P5) if  $\lambda(i) = (X, Z)$  then  $i \in p(i)$  and  $i \in \text{Odd}(p(i))$ ,
- (P6) if  $\lambda(i) = (Y, Z)$  then  $i \in p(i)$  and  $i \notin \text{Odd}(p(i))$ ,
- (P7) if  $\lambda(i) = X$  then  $i \in \text{Odd}(p(i))$ ,
- (P8) if  $\lambda(i) = Z$  then  $i \in p(i)$ ,
- (P9) if  $\lambda(i) = Y$  then either:  $i \notin p(i)$  and  $i \in \text{Odd}(p(i))$  or  $i \in p(i)$  and  $i \notin \text{Odd}(p(i))$ .

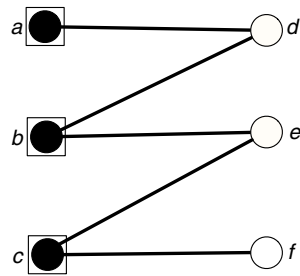
**Theorem 4.** Suppose the open graph state  $(G, I, O, \lambda)$  has Pauli flow  $(g, >)$ , then the pattern:

$$\mathfrak{P}_{g,G} := \prod_{i \in O^c}^> \left( X_{g(i) \cap \{j, j>i\}}^{s_i} Z_{\text{Odd}(g(i)) \cap \{j, j>i\}}^{s_i} M_i^{\lambda(i)\alpha_i} \right) E_G N_{I^c},$$

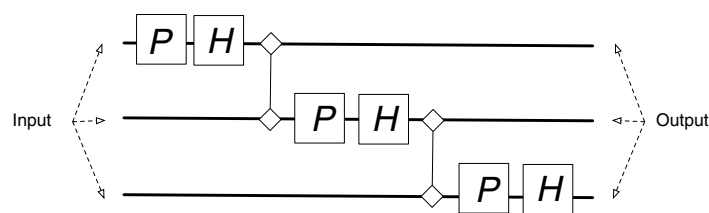
where the product follows the dependency order  $>$ , is deterministic and realizes the unitary embedding:

$$U_G := \left( \prod_{i \in O^c} \langle +_{\lambda(i), \alpha_i} | i \right) E_G N_{I^c} 0.$$

**Proof.** The proof is similar to the proof of theorem 2. In (P1), if  $\lambda(j) \in \{X, Y\}$ ,  $j$  may be in the  $p(i)$  even if  $j \leq i$  since  $M_i^X X_i = M_i^X$  and  $M_i^Y X_i Z_i = M_i^Y$ . Notice that if  $\lambda(j) = Y$ ,  $j \leq i$  and  $j \in p(i)$  then  $j$  must be in  $\text{Odd}(p(i))$  – (P3)—because of the  $Z_i$  command in  $M_i^Y X_i Z_i = M_i^Y$ . In (P2), if  $\lambda(j) = Z$ , then  $j$  may be in  $\text{Odd}(p(i))$  even if  $j \leq i$ , since  $M_i^Z Z_i = M_{iz}$ . The condition  $\lambda(j) \neq Y$  in (P2) is necessary because of (P3). Finally, (P7), (P8) and (P9) are obtained from (P4), (P5) and (P6) since an  $X$  measurement is both as a  $(X, Y)$  and as  $(X, Z)$  measurement, and so on.  $\square$



**Figure 3.** A graph with flow:  $f(a) = d$ ,  $f(b) = e$  and  $f(c) = f$ . The partial order over the vertices is  $a < \{b, d\} < \{e, c\} < f$ .



**Figure 4.** The circuit implementation of the pattern in figure 3, with controlled-Z, phase  $P(-\alpha)$  and Hadamard  $H$  gates.

## 6. Examples

Trivially, any open graph state with a flow also has a generalized flow, but the following set of examples show how the generalized flow can be beneficial. The open graph state in figure 2 has no flow (due the cyclic connections), but it admits a generalized flow. This example demonstrates the fact that having flow is not a necessary condition for uniform determinism, contrary to the existence of the generalized flow, as it is shown in theorem 3.

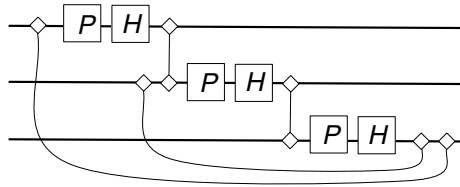
In relation to the circuit model, having a generalized flow leads to an optimal realization. It is known that given a pattern where the underlying geometry has flow, one can directly decompose the pattern into a circuit with no auxiliary qubits that implements the same unitary [4]. Consider for example the pattern given in figure 3 that implements the following pattern (all measurements are in  $(X, Y)$  plane)

$$X_f^{s_c} X_d^{s_c} M_c^\gamma \quad X_e^{s_b} Z_c^{s_b} M_b^\beta \quad X_d^{s_a} Z_b^{s_a} M_a^\alpha \quad E_{ad} E_{db} E_{be} E_{ec} E_{cf} \quad N_d N_e N_f$$

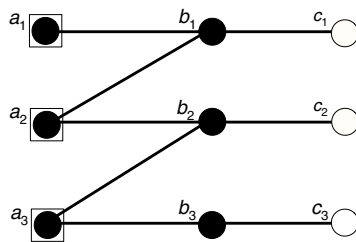
that can be decomposed to the circuit given in figure 4 using the construction of [4]. The base of this procedure is to replace the pattern  $X_{s_i}^j M_i^\alpha E_i j N_j$  with phase and Hadamard gates and the remaining edges of the graph with ctrl-Z gates.

Now if we follow the same construction for the pattern in figure 2 that has generalized flow but no flow, in order to remove all the auxiliary qubits, we obtain an acausal circuit (not runnable), in figure 5. Of course, there exists another causal circuit implementing the same unitary but it might need more gates. This suggests that patterns with generalized flow may implement more efficiently a given unitary.

Even if a graph has a flow, looking for a generalized flow is a way to decrease the depth of the computation, i.e. the parallel execution time. For instance, the open graph state in figure 6



**Figure 5.** The acausal circuit implementation of the pattern in figure 2.



**Figure 6.** Open graph state having a flow of depth 5 and a generalized flow of depth 2.

has a flow function  $f$  given below

$$f(a_i) = b_i \text{ and } f(b_i) = c_i,$$

with the corresponding partial order being  $a_1 < \{b_1, a_2\} < \{c_1, b_2, a_3\} < \{c_2, b_3\} < c_3$  and hence the depth of this flow is 5. We know the flow function  $f$  is unique [13] and since each  $b_i$  must be greater than  $a_i$  and each  $a_i$  is greater than  $b_{i-1}$ , thus the minimal depth is 5. However, this open graph state also has a generalized flow defined as

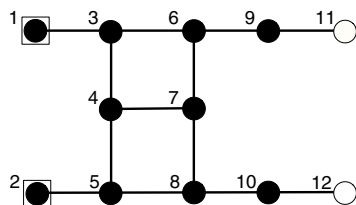
$$g(a_1) = \{b_1, b_2\}, \quad g(a_2) = \{b_2, b_3\}, \quad g(a_3) = b_3, \quad g(b_i) = c_i,$$

with partial order  $\{a_1, a_2, a_3\} < \{b_1, b_2, b_3\}$  with depth 2. One can easily extend this example (figure 6) to construct for any given  $n$ , an open graph state with  $3n$  vertices having no flow of depth less than  $n + 1$ , but a generalized flow of depth 2.

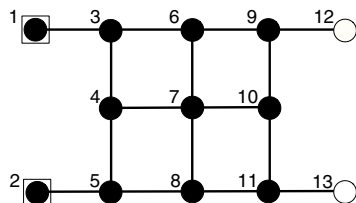
Our final examples deal with the case of Pauli flow. An open graph with no generalized flow is not uniformly deterministic, but it can be still deterministic if one restricts some of the angle of measurements to Pauli. For instance, the open graph state in figure 7, has no generalized flow, but it has the following Pauli flow with depth 1 when all the non-output qubits are  $X$ -measured (implementing the swap operator).

$$\begin{aligned} p(1) &= \{3, 7, 10\}, & p(2) &= \{5, 7, 9\}, & p(3) &= \{4, 8, 10\}, & p(4) &= \{7, 9, 10\}, \\ p(5) &= \{4, 6, 9\}, & p(6) &= \{9\}, & p(7) &= \{4, 6, 8, 9, 10\}, \\ p(8) &= \{10\}, & p(9) &= \{11\}, & p(10) &= \{12\}. \end{aligned}$$

Finally, Pauli flow is not necessary for determinism in a graph-state pattern whose measurements are solely Pauli—since these do not require any adaptive measurements. An example of this is the open graph state in figure 8, has no Pauli flow but is deterministic and realizes the swap operator [17].



**Figure 7.** Open graph state having no generalized flow but having a Pauli flow.



**Figure 8.** An open graph state with no Pauli or generalized flow implementing a deterministic pattern for the swap operator.

## 7. Conclusion

What makes the measurement-based quantum computing special is the fact that one can employ probabilistic measurement operators and yet perform a deterministic computation by imposing a causal-dependent structure over the measurements sequence. On the other hand the MBQC highlights the role of entanglement as a resource for quantum computing. Hence, a full understanding of the MBQC depends on gaining insight into the interplay of these two ingredients.

In this paper, we have extended the notion of flow [4] on the geometry of the entanglement graph required for a one-way computing to address the above questions. We have presented for the first time a full characterisation of stepwise uniformly deterministic computation in the one-way model independent of any reference to the circuit model. The class of stepwise uniformly deterministic patterns includes all the unitaries and unitary embeddings with key structural properties. For example if a given open graph state has generalized flow then one can specify a class of unitary (or unitary embedding) operators with a Clifford operator as the canonical element, leading to a characterization of unitaries based on the underlying entanglement resource that might become useful in a structural study of quantum operators. Having generalized flow is a necessary and sufficient condition, on the other hand if one is willing to restrict to a particular set of angles such as Pauli measurements then the Pauli flow criteria might be used.

One interesting consequence of patterns with generalized flow (but no flow) is that they can admit very compact implementations of given unitaries (as illustrated in our examples). In particular, the generalized flow admits a great deal of flexibility in the causal structure of the corrections which can have little in common with the structure of the associated quantum circuit. Further investigation of such features will be a line of future research. A further important open question is how one can design an efficient algorithm to find generalized flow given an open graph state, which would aid direct pattern design (as proposed in [18]). In the particular case where  $|I| = |O|$ , if a geometry has flow it is unique and can be found efficiently [13, 14, 18], using a combination of network flow algorithms, and Tarjan's algorithm to avoid acausal sequences of measurements (forming 'vicious circles'), as described in detail in [13, 14].

We conjecture that an algorithm for generalized flow will be similar to [13, 14]. First one would find a maximal collection of disjoint input–output paths to attempt to define the flow function. Then if the obtained partial order has a vicious circle one can attempt to cancel its effect using the additional vertices in the correcting set. We believe this work can form a basis for the development of novel quantum algorithms conceived solely in the language of measurement-based quantum computation.

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