New Journal of Physics
The open-access journal for physics

Towards minimal resources of measurement-based quantum computation

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New Journal of Physics **9** (2007) 206
Received 15 February 2007
Published 29 June 2007
Online at http://www.njp.org/
doi:10.1088/1367-2630/9/6/206

Abstract. We improve the upper bound on the minimal resources required for measurement-only quantum computation (M A Nielsen 2003 *Phys. Rev.* A **308** 96–100; D W Leung 2004 *Int. J. Quantum Inform.* **2** 33; S Perdrix 2005 *Int. J. Quantum Inform.* **3** 219–23). Minimizing the resources required for this model is a key issue for experimental realization of a quantum computer based on projective measurements. This new upper bound also allows one to reply in the negative to the open question presented by Perdrix (2004 *Proc. Quantum Communication Measurement and Computing*) about the existence of a trade-off between observable and ancillary qubits in measurement-only QC.

Contents

PII: S1367-2630(07)44240-9
1367-2630/07/010206+10$30.00
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1. Introduction

The discovery of new models of quantum computation (QC), such as the one-way quantum computer [1] and the projective measurement-based model [2], have opened up new experimental avenues toward the realization of a quantum computer in laboratories. At the same time they have challenged the traditional view about the nature of QC.

Since the introduction of the quantum Turing machine by Deutsch [3], unitary transformations have played a central role in QC. However, it is known that the action of unitary gates can be simulated using the process of quantum teleportation based on projective measurements only [2]. Characterizing the minimal resources that are sufficient for this model to be universal, is a key issue.

Resources of measurement-only QCs are composed of two ingredients: (i) a universal family of observables, which describes the measurements that can be performed; (ii) the number of ancillary qubits used to simulate any unitary transformation.

Successive improvements of the upper bounds on these minimal resources have been made by Leung and others [4, 5]. These bounds have been significantly reduced by the introduction of the state transfer (which is a restricted form of teleportation): one two-qubit observable ($Z \otimes X$) and three one-qubit observables (X, Z and $(X + Y)/\sqrt{2}$), associated with only one ancillary qubit, are sufficient for simulating any unitary-based QC [6]. Are these resources minimal? In [7], a sub-family of observables ($Z \otimes X$, Z, and $(X - Y)/\sqrt{2}$) is proved to be universal, however two ancillary qubits are used to make this sub-family universal.

These two results lead to an open question: is there a trade-off between observables and ancillary qubits in measurement-only QC? In this paper, we reply in the negative to this open question by proving that the sub-family $\{Z \otimes X, Z, (X - Y)/\sqrt{2}\}$ is universal using only one ancillary qubit, improving the upper bound on the minimal resources required for measurement-only QC.

Questions of universal resources can also be considered in the context of the one-way QC [1]. In the one-way model, a computation consists of applying local measurements over a large entangled state called a graph state [8]. The resources of the one-way model are composed of two ingredients: (i) a universal family of one-qubit observables, which describes the local measurements that can be performed, (ii) a family of graph states on which the measurements are applied.

In their seminal work, Briegel and Raussendorf [1] proved that the ability to apply measurements according to Z and in the (X, Y) plane (i.e. measurements according to observables of the form $\cos(\alpha)X + \sin(\alpha)Y$) over cluster states (i.e. graph states described by a rectangular grid) are universal resources for QC. In [9], (X, Z)-measurements over triangular grids have been proved to be universal resources for one-way QC too. In [10], several families of graph states are proved to be universal, moreover, necessary conditions for a family of graph states to be universal are stated.

Since graph states, and more generally quantum states, are non standard resources of QC, the resources of the one-way model can be alternatively expressed in terms of one- and two-qubit operations: the preparation of the initial graph state can be described in terms of unitary transformations or projective measurements [11]. The preparation of the initial graph state is the basis of several unifications between the one-way and the measurement-only QC [12]–[14].

Experimentally, measurement-based QC can be implemented for systems such as linear optics [15, 16]. The number of experimental proposals is now growing across the full spectrum

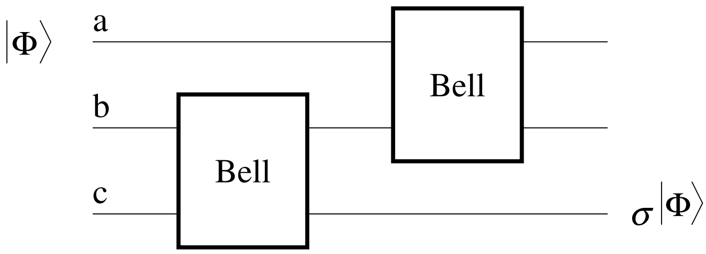

Figure 1. Bell measurement-based teleportation. Ancillary qubits b and c start in an arbitrary state.

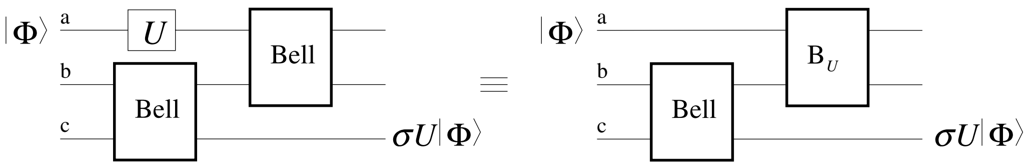

Figure 2. Simulation of U up to a Pauli operator. Bold boxes stand for measurements, and thin boxes for unitary gates.

of quantum information processing technologies. Moreover, the use of projective measurements instead of unitary transformations opens new theoretical perspectives for the parallelization of quantum algorithms [17].

2. Measurement-only QC

The simulation of a given unitary transformation U by means of projective measurements can be decomposed into the following.

1. (*Simulation step*) first, U is probabilistically simulated up to a Pauli operator, leading to σU, where σ is either an identity or a Pauli operator σ_x, σ_y, or σ_z. Each σ occurs with the same probability $1/4$ and is known by the classical outcome of the measurements. Since Pauli operators are used to denote both observables and unitary gates, a σ-notation is used to denote the unitary transformations, whereas capital letters are used to denote observables.
2. (*Correction*) then, a corrective strategy which consists of conditionally combining steps of simulation, is used to obtain a non-probabilistic simulation of U.

Teleportation can be realized by two successive Bell measurements (figure 1), where a Bell measurement is a projective measurement in the basis of the Bell states $\{|B_{ij}\rangle\}_{i,j\in\{0,1\}}$, where $|B_{ij}\rangle = \frac{1}{\sqrt{2}}(\sigma_z^i \otimes \sigma_x^j)(|00\rangle + |11\rangle)$. A step of simulation of U is obtained by replacing the second measurement by a measurement in the basis $\{(U^\dagger \otimes I)|B_{ij}\rangle\}_{i,j\in\{0,1\}}$ (figure 2).

If a step of simulation is represented as a probabilistic black box (figure 3, left panel), there exists a corrective strategy (figure 3, right panel) which leads to a full simulation of U. This strategy consists of conditionally composing steps of simulation of U, but also of each Pauli operator. A similar simulation step and strategy are given for the two-qubit

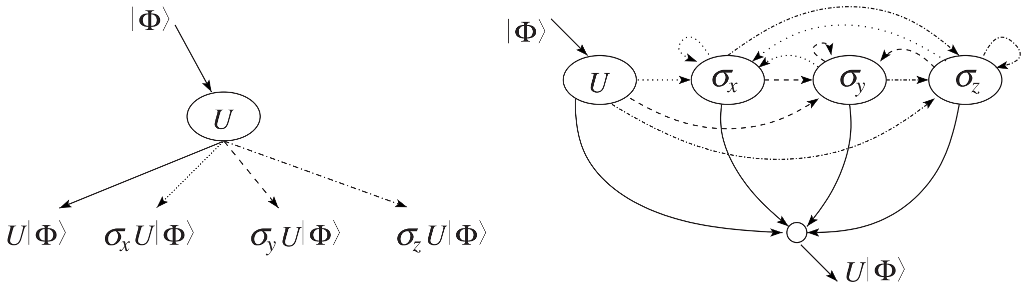

Figure 3. Left panel: step of simulation abstracted into a probabilistic black box representation. Each type of arrow represents a possible classical outcome. Right panel: conditional composition of steps of simulation.

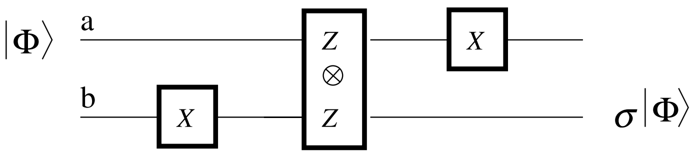

Figure 4. State transfer.

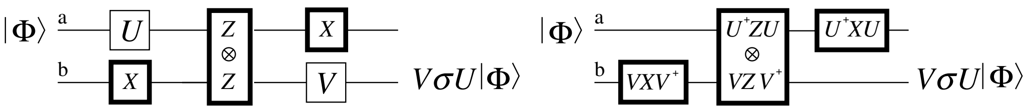

Figure 5. Step of simulation based on state transfer.

unitary transformation ΛX (*controlled-X*) in [2]. Notice that this simulation uses four ancillary qubits.

As a consequence, since any unitary transformation can be decomposed into ΛX and one-qubit unitary transformations, any unitary transformation can be simulated by means of projective measurements only. More precisely, for any circuit C of size n—with basis ΛX and all one-qubit unitary transformations—and for any $\epsilon > 0$, $O(n \log(n/\epsilon))$ projective measurements are enough to simulate C with probability greater than $1 - \epsilon$.

Approximative universality, based on a finite family of projective measurements, can also be considered. Leung [5] has shown that a family composed of five observables $\mathcal{F}_0 = \{Z, X \otimes X, Z \otimes Z, X \otimes Z, \frac{1}{\sqrt{2}}(X - Y) \otimes X\}$ is approximatively universal, using four ancillary qubits. It means that for any unitary transformation U, any $\epsilon > 0$ and any $\delta > 0$, there exists a conditional composition of projective measurements from $\mathcal{F}_0$ leading to the simulation of a unitary transformation $\tilde{U}$ with probability greater than $1 - \epsilon$ and such that $\|U - \tilde{U}\| < \delta$.

In order to decrease the number of two-qubit measurements—four in $\mathcal{F}_0$—and the number of ancillary qubits, a new scheme called *state transfer* has been introduced [6]. The state transfer (figure 4) replaces the teleportation scheme for realizing a step of simulation. Composed of one two-qubit measurement, two one-qubit measurements, and using only one ancillary qubit, the state transfer can be used to simulate any one-qubit unitary transformation up to a Pauli operator (figure 5). For instance, simulations of H and HT—see section 3 for definitions of H and T—are

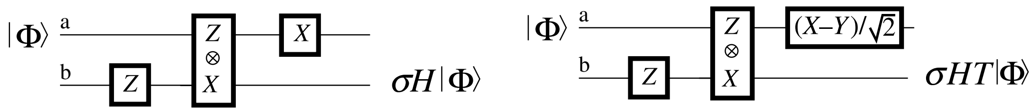

Figure 6. Simulation of H and HT up to a Pauli operator.

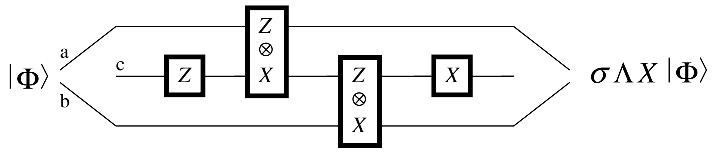

Figure 7. Simulation of ΛX up to a Pauli operator.

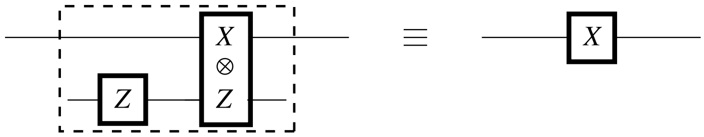

Figure 8. X-measurement simulation.

represented in figure 6. Moreover a scheme composed of two two-qubit measurements, two one-qubit measurements, and using only one ancillary qubit can be used to simulated ΛX up to a Pauli operator (figure 7). Since $\{H, T, \Lambda X\}$ is a universal family of unitary transformations, the family $\mathcal{F}_1 = \{Z \otimes X, X, Z, \frac{1}{\sqrt{2}}(X - Y)\}$ of observables is approximatively universal, using one ancillary qubit [6]. This result improves the result by Leung, since only one two-qubit measurement and one ancillary qubit are used, instead of four two-qubit measurements and four ancillary qubits. Moreover, one can prove that at least one two-qubit measurement and one ancillary qubit are required for approximative universality. Thus, it turns out that the upper bound on the minimal resources for measurement-only QC differs from the lower bound, on the number of one-qubit measurements only.

In [7], it has been shown that the number of these one-qubit measurements can be decreased, since the family $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{1}{\sqrt{2}}(X - Y)\}$, composed of one two-qubit and only two one-qubit measurements, is also approximatively universal, using *two* ancillary qubits. The proof is based on the simulation of X-measurements by means of Z and $Z \otimes X$ measurements (figure 8). This result leads to a possible *trade-off* between the number of one-qubit measurements and the number of ancillary qubits required for approximative universality.

In this paper, we mean to prove that the family $\mathcal{F}_2$ is approximatively universal, using only one ancillary qubit. Thus, the upper bound on the minimal resources required for approximative universality is improved, and moreover we answer the open question of the trade-off between observables and ancillary qubits. Notice that we prove that the trade-off conjectured in [7] does not exist, but another trade-off between observables and ancillary qubits may exist since the bounds on the minimal resources for measurement-only QC are not tight.

3. Universal family of unitary transformations

There exist several universal families of unitary transformations, for instance $\{H, T, \Lambda X\}$ is one of them:

$$h = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad T = \begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{\mathrm{i}\pi/4}\end{pmatrix}, \quad \Lambda X = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \quad \Lambda Z = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}.$$

We prove that the family $\{HT, \sigma_y, \Lambda Z\}$ is also approximatively universal.

Theorem 1. *$\mathcal{U} = \{HT, \sigma_y, \Lambda Z\}$ is approximatively universal.*

The proof is based on the following properties (see [18] for details). Let $R_{\mathbf{n}}(\alpha)$ be the rotation of the Bloch sphere about the axis $\mathbf{n}$ through an angle α.

Proposition 1. *If $\mathbf{n} = (a, b, c)$ is a real unit vector, then for any α,* $R_{\mathbf{n}}(\alpha) = \cos(\alpha/2)I - \mathrm{i}\sin(\alpha/2)(a\sigma_x + b\sigma_y + c\sigma_z)$.

Proposition 2. *For a given vector $\mathbf{n}$ of the Bloch sphere, if θ is an irrational multiple of π, then for any α and any $\epsilon > 0$, there exists k such that*

$$\| R_{\mathbf{n}}(\alpha) - R_{\mathbf{n}}(\theta)^k) \| < \epsilon/3.$$

Proposition 3. *If $\mathbf{n}$ and $\mathbf{m}$ are non parallel vectors of the Bloch sphere, then for any one-qubit unitary transformation U, there exists $\alpha, \beta, \gamma, \delta$ such that:*

$$U = \mathrm{e}^{\mathrm{i}\alpha} R_{\mathbf{n}}(\beta) R_{\mathbf{m}}(\gamma) R_{\mathbf{n}}(\delta).$$

Proposition 4 (Włodarski [19]). *If α is not an integer multiple of $\pi/4$ and $\cos\beta = \cos^2\alpha$, then either α or β is an irrational multiple of π.*

Proof of theorem 1. First we prove that any one-qubit unitary transformation can be approximated by HT and $\sigma_y HT$. Consider the unitary transformations $U_1 = T$, $U_2 = HTH$ and $U_3 = \sigma_y HTH\sigma_y$. Notice that T is, up to an unimportant global phase, a rotation by $\pi/4$ radians around z-axis on the Bloch sphere:

$$U_1 = T = \mathrm{e}^{-\mathrm{i}\pi/8}(\cos(\pi/8)I - \mathrm{i}\sin(\pi/8)\sigma_z),$$
$$U_2 = HTH = \mathrm{e}^{-\mathrm{i}\pi/8}(\cos(\pi/8)I - \mathrm{i}\sin(\pi/8)\sigma_x),$$
$$U_3 = \sigma_y HTH\sigma_y = \mathrm{e}^{-\mathrm{i}\pi/8}(\cos(\pi/8)I + \mathrm{i}\sin(\pi/8)\sigma_x).$$

Composing U_1 and U_2 gives, up to a global phase:

$$U_2U_1 = (\cos(\pi/8)I - \mathrm{i}\sin(\pi/8)\sigma_x)(\cos(\pi/8)I - \mathrm{i}\sin(\pi/8)\sigma_z),$$
$$= \cos^2(\pi/8)I - \mathrm{i}[\cos(\pi/8)(\sigma_x + \sigma_z) - \sin(\pi/8)\sigma_y]\sin(\pi/8).$$

According to proposition 1, U_2U_1 is a rotation of the Bloch sphere about an axis along $\mathbf{n} = (\cos(\pi/8), -\sin(\pi/8), \cos(\pi/8))$ and through an angle θ defined as a solution of

$\cos(\theta/2) = \cos^2(\pi/8)$. Since $\pi/8$ is not an integer multiple of $\pi/4$ but a rational multiple of π, according to proposition 4, such a θ is an irrational multiple of π. This irrationality implies that for any angle α, the rotation around $\mathbf{n}$ about angle α can be approximated to arbitrary accuracy by repeating rotations around $\mathbf{n}$ about angle θ (see proposition 3). For any α and any $\epsilon > 0$, there exists k such that

$$\| R_{\mathbf{n}}(\alpha) - R_{\mathbf{n}}(\theta)^k \| < \epsilon/3.$$

Moreover, composing U_1 and U_3 gives, up to a global phase:

$$\begin{aligned} U_3 U_1 &= (\cos(\pi/8) I + \mathrm{i} \sin(\pi/8)\sigma_x)(\cos(\pi/8) I - \mathrm{i} \sin(\pi/8)\sigma_z), \\ &= \cos^2(\pi/8) I - \mathrm{i}[\cos(\pi/8)(-\sigma_x + \sigma_z) + \sin(\pi/8)\sigma_y] \sin(\pi/8). \end{aligned}$$

U_3U_1 is a rotation of the Bloch sphere about an axis along $\mathbf{m} = (-\cos(\pi/8), \sin(\pi/8), \cos(\pi/8))$ and through the angle θ. Thus, for any α and any $\epsilon > 0$, there exists k such that

$$\| R_{\mathbf{m}}(\alpha) - R_{\mathbf{m}}(\theta)^k) \| < \epsilon/3.$$

Since $\mathbf{n}$ and $\mathbf{m}$ are non-parallel, any one-qubit unitary transformation U, according to proposition 2, can be decomposed into rotations around $\mathbf{n}$ and $\mathbf{m}$: there exist $\alpha, \beta, \gamma, \delta$ such that

$$U = \mathrm{e}^{\mathrm{i}\alpha} R_{\mathbf{n}}(\beta) R_{\mathbf{m}}(\gamma) R_{\mathbf{n}}(\delta).$$

Finally, for any U and $\epsilon > 0$, there exist k_1, k_2, k_3 such that

$$\| U - R_{\mathbf{n}}(\theta)^{k_1} R_{\mathbf{m}}(\theta)^{k_2} R_{\mathbf{n}}(\theta)^{k_3} \| < \epsilon.$$

Thus, any one-qubit unitary transformation can be approximated by means of U_2U_1, and U_3U_1. Since $U_2U_1 = (HT)(HT)$and $U_3U_1 = \sigma_y HTH\sigma_y T = -(\sigma_y HT)(\sigma_y HT)$, the family $\{HT, \sigma_y\}$ approximates any one-qubit unitary transformation.

With the additional $\wedge Z$ gate, the family $\mathcal{U}$ is approximatively universal. □

4. Universal family of projective measurements

In [7], the family of observables $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is proved to be approximatively universal using two ancillary qubits. We prove that this family requires only one ancillary qubit to be universal.

Theorem 2. *$\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is approximatively universal, using one ancillary qubit only.*

The proof consists of simulating the unitary transformations of the universal family $\mathcal{U}$. First, one can notice that HT can be simulated up to a Pauli operator, using measurements of $\mathcal{F}_2$, as depicted in figure 6. So, the universality is reduced to the ability to simulate $\wedge Z$ and the Pauli operators—Pauli operators are needed to simulate $\sigma_y \in \mathcal{U}$, but also to perform the corrections required by the corrective strategy (figure 3).

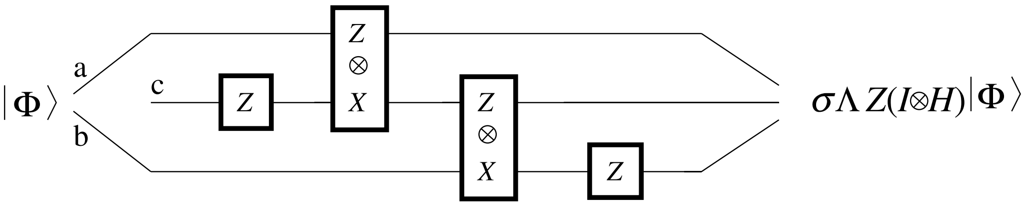

Figure 9. Simulation of $\Lambda Z(I \otimes H)$.

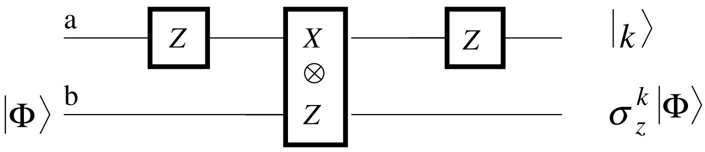

Figure 10. Simulation of σ_z.

Lemma 5. *For a given two-qubit register* a, b *and one ancillary qubit* c*, the sequence of measurements according to* Z_c, $Z_a \otimes X_c$, $Z_c \otimes X_b$, *and* Z_b *(see figure 9) simulates* $\Lambda Z(I \otimes H)$ *on qubits* a, b, *up to a Pauli operator. The resulting state is located on qubits* a *and* c.

Proof. One can show that, if the state of the register a, b is $|\Phi\rangle$ before the sequence of measurements, the state of the register a, b after the measurements is $\sigma\Lambda Z(I \otimes H)|\Phi\rangle$, where $\sigma = \sigma_z^{s_1} \otimes \sigma_x^{s_3}\sigma_z^{s_2+s_4}$ and s_is are the classical outcomes of the sequence of measurements. □

In order to simulate Pauli operators, a new scheme, different from the state transfer, is introduced.

Lemma 6. *For a given qubit* b *and one ancillary qubit* a*, the sequence of measurements* Z_a, $X_a \otimes Z_b$ *and* Z_a *(figure 10) simulates, on qubit* b*, the application of* σ_z *with probability* $1/2$ *and* I *with probability* $1/2$.

Proof. Let $|\Phi\rangle = \alpha|0\rangle + \beta|1\rangle$ be the state of qubit b. After the first measurement, the state of the register a, b is $|\psi_1\rangle = (\sigma_x^{s_1} \otimes I)|0\rangle \otimes |\Phi\rangle$ where $s_1 \in \{0, 1\}$ is the classical outcome of the measurement.

Since $\langle\psi_1|X \otimes Z|\psi_1\rangle = 0$, the state of the register a, b is:

$$|\psi_2\rangle = \frac{\sqrt{2}}{2}(\sigma_x^{s_1} \otimes I)(I + (-1)^{s_2} X \otimes Z)|0\rangle \otimes |\Phi\rangle,$$

$$= \frac{\sqrt{2}}{2}(\sigma_x^{s_1}\sigma_z^{s_2} \otimes I)(|0\rangle \otimes |\Phi\rangle + |1\rangle \otimes \sigma_z|\Phi\rangle).$$

Let $s_3 \in \{0, 1\}$ be the outcome of the last measurement, on qubit a. If $s_1 = s_3$ then state of the qubit b is $|\Phi\rangle$, and $\sigma_z|\Phi\rangle$ otherwise. One can prove that these two possibilities occur with equal probabilities. □

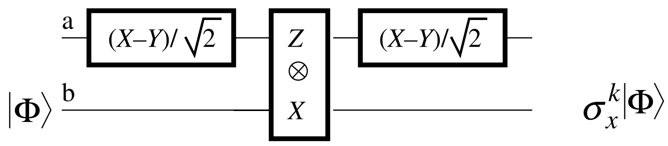

Figure 11. Simulation of σ_x.

Lemma 7. *For a given qubit b and one ancillary qubit a, the sequence of measurements* $\left(\frac{X-Y}{\sqrt{2}}\right)_a$*,* $Z_a \otimes X_b$ *and* $\left(\frac{X-Y}{\sqrt{2}}\right)_a$ *(figure 11) simulates, on qubit b, the application of* σ_x *with probability* $1/2$ *and I with probability* $1/2$.

The proof of lemma 7 is similar to the proof of lemma 6.

Proof of theorem 2. First notice that the family of unitary transformations $\mathcal{U}' = \{HT, \sigma_y, \wedge Z(I \otimes H)\}$ is approximatively universal since $\mathcal{U} = \{HT, \sigma_y, \wedge Z\}$ is universal.

HT and $\wedge Z(I \otimes H)$ can be simulated up to a Pauli operator (lemma 5). The universality of the family of observables $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is reduced to the ability to simulate any Pauli operators. Lemma 7 (resp. lemma 6), shows that σ_x (σ_z) can be simulated with probability $1/2$, moreover if the simulation fails, the resulting state is same as the original one. Thus, this simulation can be repeated until a full simulation of σ_x (σ_z). Finally, $\sigma_y = i\sigma_z\sigma_x$ can be simulated, up to a global phase, by combining simulations of σ_x and σ_z. Thus, $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is approximatively universal using only one ancillary qubit. □

5. Conclusions

We have proved a new upper bound on the minimal resources required for measurement-only QC: one two-qubit, and two one-qubit observables are universal, using one ancillary qubit only. This new upper bound has experimental applications, but also allows one to prove that the trade-off between observables and ancillary qubits, conjectured in [7], does not exist. This new upper bound is not tight since the lower bound on the minimal resources for this model is one two-qubit observable and one ancillary qubit.

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