# Resources for measurement-based quantum computation: a unifying view

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Alternative quantum computing models to the standard unitary-based model have been independently introduced and developed. One of them, introduced by Nielsen [11] is based on projective measurements only. Another, the one-way quantum computation introduced by Briegel and Raussendorf [13] consists in applying local measurements on an offline-prepared entangled state, called cluster state.

The model via measurements only has been successively improved [12], decreasing the resources (in terms of number of different projective measurements) used for a universal quantum computation. The last improvement is based on state transfer [15]. The resources used by the one-way quantum computation are not limited to local measurements, since the preparation of the initial cluster state has to be taken into account. It turns out that the resources used by the one-way quantum computation if the cluster state is prepared by means of projective measurements [9] are similar to the resources used by the quantum computation via measurements only based on state transfer. Moreover, the measurement-based preparation of the initial cluster state leads to a natural unification of these models.

If, instead of a measurement-based preparation, a unitary-based preparation of the cluster state is considered, a natural unification between the one-way quantum computation and an hybrid model allowing both unitary transformation and projective measurements is obtained [3].

## I. INTRODUCTION

Quantum measurement is universal for quantum computation (Nielsen [11], Raussendorf [13, 14]). Two models for performing measurement-based quantum computation exist: the one-way quantum computer was introduced by Briegel and Raussendorf [13], then this model has been developed, providing a formal framework, the *measurement calculus* [3] which is based on rewriting terms. The second model is based on projective measurements only, introduced by Nielsen [11]. The more recent development of this second model is based on *state transfers* [15] instead of teleportation. From this development, a finite but approximate quantum universal family of observables is exhibited, which includes only one two-qubit observable, while others are one-qubit observables [15]. Moreover an infinite but exact quantum universal family of observables has been exhibited [7], including also only one two-qubit observable.

Even if one-way quantum computation consists in applying local measurements on a cluster state, there is no universal family composed of one-qubit observable only since the preparation of the cluster state has to be considered.

Considering the preparation of the initial cluster state, these two models of measurement-based

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quantum computation, *i.e.* one-way quantum computation and measurement-based quantum computation via measurements-only, can be compared.

The comparison of these two models of measurement-based quantum computation, which was initiated in [18], then purchased in [1, 3, 7], closer and more natural connections appear between these two models.

## II. QUANTUM COMPUTATION VIA MEASUREMENTS ONLY BASED ON TELEPORTATION

Universality of quantum computation via measurements only, consists in proving that any quantum circuit can be simulated by means of projective measurements only. This proof can be decomposed into 3 levels:

- Probabilistic *steps of simulation* of some basic unitary transformation $U$, where a sequence of projective measurements leads to the simulation of $U$ up to a Pauli operator (see fig. 2);
- *full simulation of* $U$, which consists in combining steps of simulations in order to obtain a non-probabilistic simulation of some basic unitary transformation $U$. This step is described by a classical strategy (see fig. 3) called *classical control.*
- Finally, full simulations of some basic unitary transformations can be composed, leading to the simulation of quantum circuits.

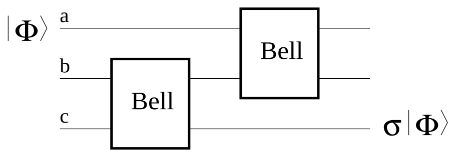



FIG. 1: Teleportation with measurements only

In the original paper by Nielsen, the first step is achieved by means of generalized teleportation. First, notice that a state $|\phi\rangle$ can be teleported from a qubit $a$ to a qubit $c$ using Bell measurements only (fig. 1). Generalized teleportation consists in modifying the basis of the measurement performed during teleportation (fig. 2) so that a one-qubit unitary transformation $U$ is simulated up to a Pauli operator: the probabilistic step of simulation of $U$ is obtained. This step of simulation needs two ancillary qubits. An extension to the simulation of 2-qubit unitary transformations (e.g. $CNot$) can be done using only 2-qubit measurements[12]. These simulations require four ancillary qubits.

A step of simulation of $U$ can be abstracted into a black box (fig. 3) with one input and four outputs according to the four Pauli operators, for use at the next higher level which is the full simulation of $U$.

For a given step of simulation of $U$, the *full simulation of* $U$ is given by an automaton where each state encapsulates a step of simulation (fig. 3). This automaton is interpreted as follows:

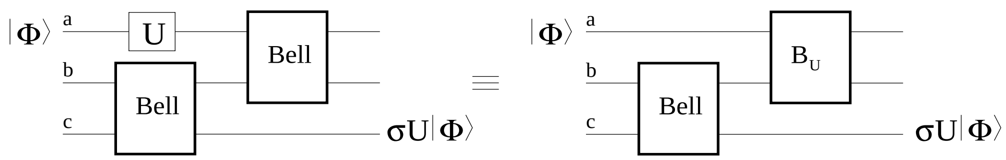



FIG. 2: Left: Teleportation of $U|\phi\rangle$; Right: a step of simulation of U

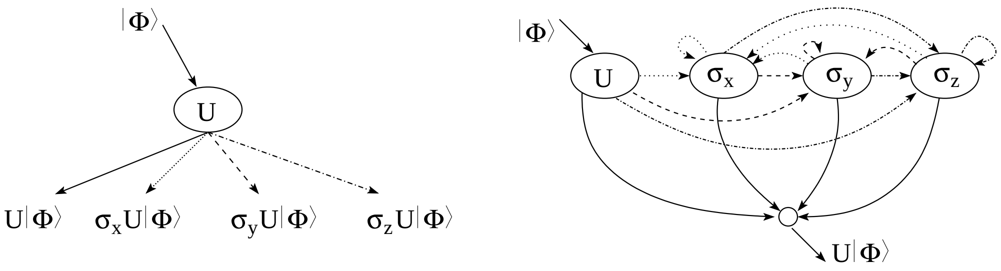



FIG. 3: Left: A step of simulation of $U$; Right: Full simulation of $U$

$U$ is simulated on a quantum state $|\phi\rangle$ so $\sigma U|\phi\rangle$ is obtained, where $\sigma$ is a Pauli operator. If $\sigma = I$ then the simulation is terminated, otherwise $\sigma$ is simulated. From this step of simulation, the state $\sigma'\sigma\sigma U|\phi\rangle = \sigma' U|\phi\rangle$ is obtained. If $\sigma' = I$ the simulation is terminated, otherwise $\sigma'$ is simulated, and so on. A similar automaton, leading to the full simulation of $CNot$ can be done.

Since any one qubit unitary transformations and the $CNot$ transformation can be fully simulated by means of projective measurements only, composition of these full simulation leads to the last level of the proof of universality, i.e. the simulation of quantum circuits composed of $CNOT$ and one-qubit unitary transformations.

## III. QUANTUM COMPUTATION VIA MEASUREMENTS ONLY BASED ON STATE TRANSFER

Whereas the scheme of computation by Nielsen and improved by Leung is based on teleportation, we introduced [15] a measurement-based quantum computation based on *state transfer*, which is an alternative to teleportation for purpose of computation. State transfer needs less measurements and less ancillary qubits than teleportation, but on the other hand, state transfer cannot replace teleportation in *non-local* applications.

**Lemma 1.** *For a given qubit a and an ancillary qubit b, the sequence of measurements*

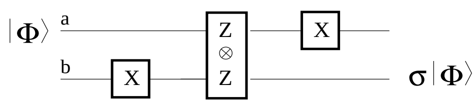



FIG. 4: State transfer

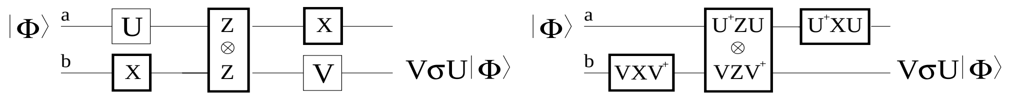



FIG. 5: Left: State Transfer with additional unitary transformations $U$ and $V$; Right: Generalized State Transfer.

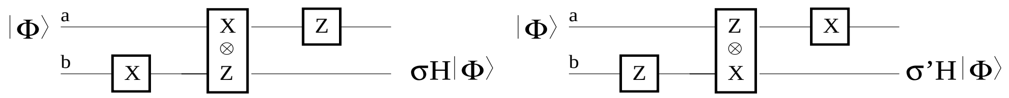



FIG. 6: Steps of simulation of H - Left: $U = H$ and $V = Id$; Right: $U = Id$ and $V = H$ (note that for all $\sigma$, there exists $\sigma'$ such that $H\sigma = \sigma' H$)

$\{X^{(b)}, Z^{(a)} \otimes Z^{(b)}, X^{(a)}\}$ *(see fig. 4), transfers the state* $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ *from a to b up to a Pauli operator which depends on the classical outcomes of the measurements.*

The proof of *Lemma 1* can be found in Ref. 7.

By modifying the measurements performed during state transfer, all 1-qubit unitary transformations $U$ can be simulated up to a Pauli operator using generalized state transfers. For use in later developments, a general scheme with two unitary transformations $U$ and $V$ is introduced, see fig. 5.

For instance, the generalized state transfers which simulate $H$, $T$ and $H.T$ are given in fig. 6 and 7.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}, CNot = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

**Lemma 2.** *For a given* 2*-qubit register* $a, b$ *and one ancillary qubit* $c$*, the sequence of measurements* $\{Z^{(c)}, Z^{(a)} \otimes X^{(c)}, Z^{(c)} \otimes X^{(b)}, X^{(c)}\}$ *(see fig. 8), simulates the* 2*-qubit unitary transformation* $CNot$ *on the state* $|\phi\rangle$ *of* $a, b$ *up to a* 2*-qubit Pauli operator which depends on the classical outcomes of the measurements.*

The proof of *Lemma 2* can be found in Ref. 7.

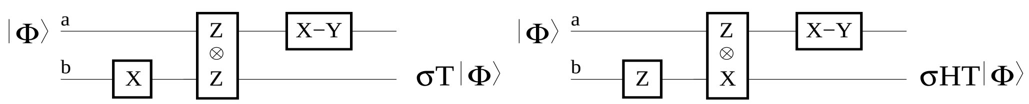



FIG. 7: Left: Step of simulation of $T$: $U = T$ and $V = Id$; Right: Step of simulation of $HT$: $U = T$ and $V = H$.

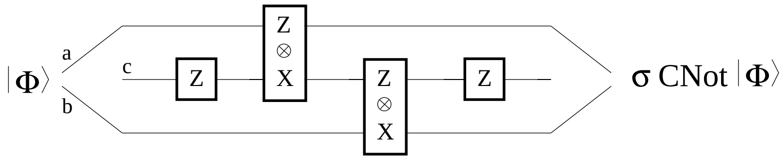



FIG. 8: Step of simulation of $CNot$

## IV. UNIVERSAL FAMILIES OF OBSERVABLES

In this section, universal families of observables are presented. Notice that there exist two types of quantum computation universalities:

- A family $\mathcal{F}$ of operators (unitary transformations or measurements) is *quantum universal* iff for all operator $O$, there exists a combination (compositions and tensor products) of some elements of $\mathcal{F}$ which simulates $O$.
- A family $\mathcal{F}$ of operators is *approximatively quantum universal* iff for all operator $O_1$ and for all $\epsilon > 0$, there exists an operator $O_2$ and a combination of some elements of $\mathcal{F}$ which simulates $O_2$, such that $||O_1 - O_2|| \leq \epsilon$ (where $||.||$ is a usual distance over matrices).

**Theorem 1.** *The family of observables* $\mathcal{O}_1 = \{Z, X \otimes X, Z \otimes Z, X \otimes Z, \frac{1}{\sqrt{2}}(X - Y) \otimes X\}$ *is approximatively quantum universal.*

Proof of theorem 1 can be found in [12]. Starting from the model of quantum computation via projective measurement based on teleportation, Leung proved the above theorem. The proof is based on the approximative universality of the family of unitary transformations is $U_1 = \{H, T, CNot\}$ [10].

**Theorem 2.** *The family of observables* $\mathcal{O}_2 = \{Z \otimes X, X, Z, \frac{X-Y}{\sqrt{2}}\}$ *is approximatively quantum universal.*

Proof of theorem 2 can be found in [15]. The proof is based on generalized state transfer. Figures reffig:sim2, **??** and 8 show steps of simulation of the elements of the approximate universal family $\mathcal{U}_2 = \{H, HT, CNot\}$ using the elements of $\mathcal{O}_2$.

**Theorem 3.** *The family of observables* $\mathcal{O}_3 = \{Z \otimes X, Z, cos(\theta)X + sin(\theta)Y, \theta \in [0\ 2\pi]\}$ *is quantum universal.*

Proof of theorem 3 can be found in [7]. This proof can be decomposed into two steps. First step consists in proving that the family of unitary transformations $\mathcal{U}_3 = \{CNot, H.R_z(\varphi), \varphi \in [0, 2\pi]\}$ is universal. The proof of this universality has been independently given in [4, 7, 13]. The simulation of $CNot$ is given in fig. 8, and the simulation of $H.R_z(\varphi)$ is presented in figure 9.

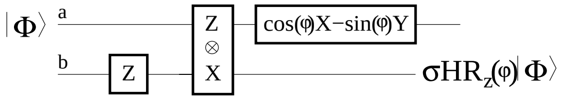



FIG. 9: Step of simulation of $HR_{\hat{z}}(\varphi)$: $V_1 = R_{\hat{z}}(\varphi)$ and $V_2 = H$.)

one-bit teleportation

FIG. 10: One-bit Teleportation

## V. THE SECRET OF THE ONE-WAY QUANTUM COMPUTER IS HIDDEN IN THE INITIAL CLUSTER STATE

One-way quantum computation consists in measuring qubit after qubit a lattice of qubits, initially prepared in an entangled state: the *cluster state*. This is a *one-way* computation because the entanglement is consumed step by step. Therefore the creation of the intial cluster state is a crutial point.

The initial cluster state is a special instance of a more general family of quantum states called *graph states* [5, 17]. The preparation of such quantum states has been studied in [9]. A preparation consists in associating with any mathematical graph, the corresponding quantum graph state.

Any graph states can be prepared by means of unitary transformations, or by means of projective measurements. Unifications proposed in [1, 3] are based on the unitary based preparation of the cluster state, unifying the one-way model with a model based on one-bit teleportation [6]. This last model, based on one-bit teleportation is not a model of quantum computation based on projective measurements only, but an hybrid model composed of unitary transformations and projective measurements. A measurement-based version of the one-bit teleportation, which is similar to a state transfer, is presented in [1].

The unification proposed in [7] is based on a measurement-based preparation of the initial cluster state [9], leading to a direct unification between the one-way model and the model based on state transfer, which is a model of quantum computation based on projective measurements only. Notice that this unification is partial since only one-dimensional cluster states are taken into account.

### A. Unitary-based preparation of the initial cluster state

In order to create the initial cluster state on a given lattice of qubits, the following preparation is performed (see fig. **??**):

- Some qubits of the lattice are input qubits, i.e. qubits which are in an unknown state $|\phi\rangle$, others are ancillary qubits. Each ancillary qubit is initialized in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- An Ising transformation is applied on the whole lattice. This Ising transformation is

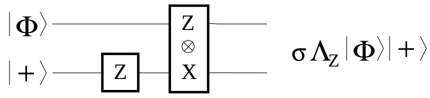



FIG. 11: Simulation of $\Lambda_Z$ on $|\phi\rangle \otimes |+\rangle$ without ancillary qubit

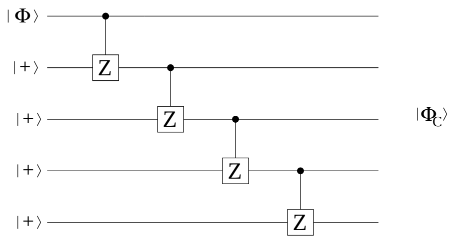



FIG. 12: Cascade of $\Lambda_Z$ for creating the initial cluster state $|\phi_C\rangle$

equivalent to the application of the 2-qubit unitary transformation Controlled-$Z$ ($\Lambda_Z$) on each pair of neighboring qubits, where:

$$\Lambda_Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

### B. Measurement-based preparation of the initial cluster state

**Lemma 3.** *For a given qubit $a$ in an unknown state $|\phi\rangle$ and a given qubit $b$ in the state $|+\rangle$, the sequence of measurements $\{Z^{(b)}, Z^{(a)} \otimes X^{(b)}\}$ (see fig. 11) simulates the unitary transformation $\Lambda_Z$ on $|\phi\rangle \otimes |+\rangle$ up to a two-qubit Pauli operator.*

Proof can be found in [7].

### C. Creation of a one-dimensional Cluster State

In order to create the initial cluster state on a one-dimensional $n$-qubit lattice composed of a unique input qubit, a *cascade* of $\Lambda_Z$ can be performed, see fig. **??**.

For each $\Lambda_Z$ of the previous cascade the state of the second input qubit is $|+\rangle$, thus, according to *Lemma 1*, the cascade of $\Lambda_Z$ of fig. **??** can be simulated by a cascade of measurements see fig. 12. Note that this simulation requires no additional ancillary qubit.

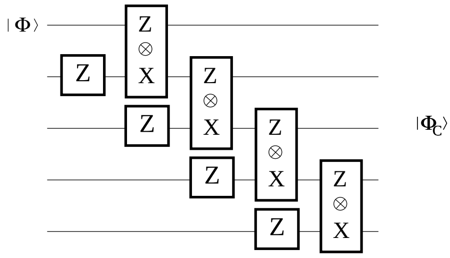



FIG. 13: Cascade of measurements for creating the initial cluster state $|\phi_C\rangle$ up to a Pauli operator

| | | | | |
|---|---|---|---|---|
| X | Y | Y | Y | |

FIG. 14: Simulation of the Hadamard transformation

## VI. EXECUTIONS ON A ONE-WAY QUANTUM COMPUTER

An execution on a one-way quantum computer is a sequence of one-qubit measurements on a cluster state. For instance, for a given five-qubit cluster state, if the first qubit is considered as an input qubit $|\phi\rangle$ and if the sequence of measurements $\{X, Y, Y, Y\}$ is performed on the first four qubits (see fig 12), then the state of the last qubit is $\sigma H |\phi\rangle$. Thus the one-way quantum computer of figure **??** simulates the Hadamard transformation.

If the phase of preparation of the cluster state and the phase of execution are both represented (see fig 13*a*), then the measurements implied in the preparation of the cluster state and those implied in the execution can be decomposed into a succession of generalized state transfers (see fig 13*b*).

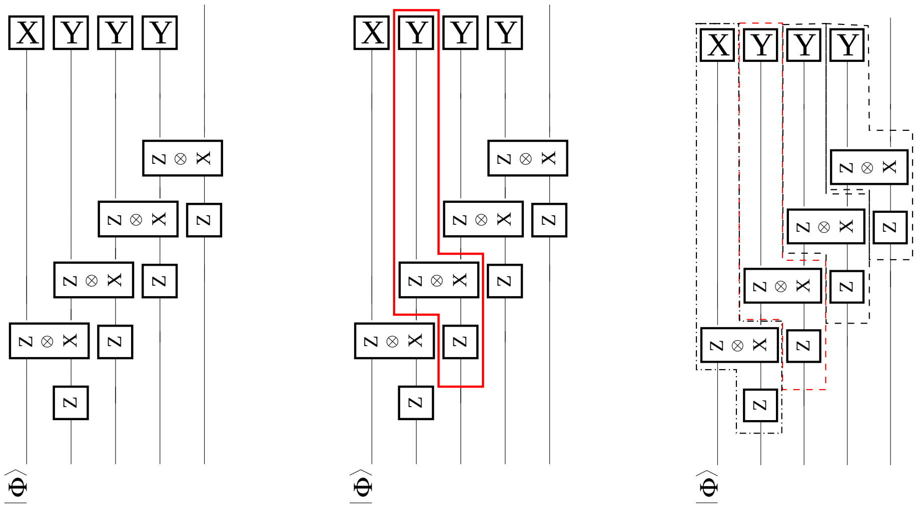
This decomposition offers a natural translation from any one-dimensional one-way quantum computer to quantum computation via projective measurements only. Moreover a straightforward interpretation of the action of any one-dimensional one-way quantum computer is obtained. For instance, the one-way quantum computer of figure **??** can be decomposed (see fig 13*c*) into a step of simulation of $H$ (fig 4), and three steps of simulation of $HS^\dagger$ (fig 5), thus the action $U$ of this one-way quantum computer is $U = (HS^\dagger)(HS^\dagger)(HS^\dagger)(H) = H$.

Similarly, the action $U$ of the one-way quantum computer presented in figure **??** is $U = (H)(HS^\dagger)(H)(H) = S^\dagger$.

More generally, the measurements allowed in a one-way quantum computation are in the basis $\mathcal{B}(\varphi) = \{\frac{|0\rangle + e^{i\varphi}|1\rangle}{\sqrt{2}}, \frac{|0\rangle - e^{i\varphi}|1\rangle}{\sqrt{2}}\}$ for all $\varphi$. The observable associated with $\mathcal{B}(\varphi)$ is $\mathcal{O}(\varphi) = cos(\varphi)X + sin(\varphi)Y$. Each $\mathcal{O}(\varphi)$-measurement is associated with a generalized state transfer with $V_1 = R_{\hat{z}}(-\varphi)$ and $V_2 = H$ (see fig 8).

Thus the action $U$ of the one-way quantum computer of figure **??** is $U = HR_{\hat{z}}(-\zeta)HR_{\hat{z}}(-\eta)HR_{\hat{z}}(-\xi)H$.

The connexions between the one-way quantum computer and generalized state transfer can also be used for designing new one-way quantum computers. For instance the one-way quantum




*(a)* *(b)* *(c)*

FIG. 15: Execution on a one-way quantum computer

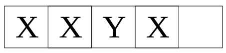



FIG. 16: Simulation of $S^\dagger$

computers introduced in figure **??** simulate respectively $H$ and $S$ while requiring less qubits than those introduced by Briegel and Raussendorf [13, 14].

## VII. CONCLUSION

The connections established between models of measurement-based quantum computation, permit a natural translation of each one-dimensional one-way quantum computer into a sequence of state transfers. Therefore, by these close connections, the measurement-based quantum computer is unified: a one-way quantum computer is nothing but a quantum computer based on state transfers, in which a large part of the measurements (those independent on the program we want to perform) are grouped in a stage of *initialization*. Note that this initialization can be performed using an Ising transformation, which is unitary. The initialization produces the *cluster state*, on which the rest of the measurements (composed of one-qubit measurements only) are performed in order to complete the computation.

Moreover this unification has been obtained by decreasing the resources required by each of these models independently, leading to a natural unification of the measurement-based quantum

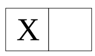



FIG. 17: Simulation of $H$

| | | | |
|---|---|---|---|
| X | Y | Y | |

FIG. 18: Simulation of $(HS^{\dagger})(HS^{\dagger})(H) = S$

| | | | | |
|---|---|---|---|---|
| X | ξ | η | ζ | |

FIG. 19: Simulation of a general one-qubit unitary transformation

computing.

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