Lectures on Proof Theory, Linear Logic and Proof Search

Lecture 3: Gödel's completeness theorem for LK

4th march 2010

LK Rules (1)

***Identity Rules** (Axiom and Cut))*

$$\frac{}{A \vdash A}\ \textit{Axiom} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textit{Cut}$$

***Structural Rules** (Exchange, Weakening and Contraction)*

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ \textit{LEx} \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ \textit{REx}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LW} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \textit{RW}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LC} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \textit{RC}$$

LK Rules (2)

Logical Rules (¬, ∧, ∨, ⇒, ∀, ∃)

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge 1 \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge 2 \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ L\vee \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee 1 \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee 2$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ L\Rightarrow \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ R\Rightarrow$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ L\forall \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta}\ R\forall \quad (*)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ L\exists \quad (*) \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ R\exists$$

(*) For these rules, $x \notin FV(\Gamma, \Delta)$.

Example of a proof in LK

$\vdash \exists x \forall y (P(x) \Rightarrow P(y))$ The Drinker Property

$$\frac{}{P(y) \vdash P(y)}\ Ax$$

$$\frac{P(y) \vdash P(y)}{P(y) \vdash P(y), P(z)}\ RW$$

$$\frac{P(y) \vdash P(y), P(z)}{P(t), P(y) \vdash P(y), P(z)}\ LW$$

$$\frac{P(t), P(y) \vdash P(y), P(z)}{P(t) \vdash P(y), P(y) \Rightarrow P(z)}\ R\Rightarrow$$

$$\frac{P(t) \vdash P(y), P(y) \Rightarrow P(z)}{P(t) \vdash P(y), \forall z.(P(y) \Rightarrow P(z))}\ R\forall$$

$$\frac{P(t) \vdash P(y), \forall z.(P(y) \Rightarrow P(z))}{P(t) \vdash P(y), \exists x.\forall y.(P(x) \Rightarrow P(y))}\ R\exists$$

$$\frac{P(t) \vdash P(y), \exists x.\forall y.(P(x) \Rightarrow P(y))}{\vdash P(t) \Rightarrow P(y), \exists x.\forall y.(P(x) \Rightarrow P(y))}\ R\Rightarrow$$

$$\frac{\vdash P(t) \Rightarrow P(y), \exists x.\forall y.(P(x) \Rightarrow P(y))}{\vdash \forall y.(P(t) \Rightarrow P(y)), \exists x.\forall y.(P(x) \Rightarrow P(y))}\ R\forall$$

$$\frac{\vdash \forall y.(P(t) \Rightarrow P(y)), \exists x.\forall y.(P(x) \Rightarrow P(y))}{\vdash \exists x.\forall y.(P(x) \Rightarrow P(y)), \exists x.\forall y.(P(x) \Rightarrow P(y))}\ R\exists$$

$$\frac{\vdash \exists x.\forall y.(P(x) \Rightarrow P(y)), \exists x.\forall y.(P(x) \Rightarrow P(y))}{\vdash \exists x.\forall y.(P(x) \Rightarrow P(y))}\ RC$$

Alternative Rules for $\wedge$ and $\vee$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge' \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ R\wedge'$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}\ L\vee' \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee'$$

These new rules are called *multiplicative rules* while the original rules of LK are called *additive rules*.

Both sets of rules are equivalent thanks to the structural rules.

Symmetry of LK (1)

Sequents are now of the form: $\vdash' \Gamma$.

Implication is a defined connective: $A \Rightarrow B \equiv \neg A \vee B$

Negation only appears on atomic formulas, thanks to de Morgan's laws:

$$\begin{array}{ll} \neg(A \vee B) \equiv (\neg A \wedge \neg B) & \neg \forall x A \equiv \exists x \neg A \\ \neg(A \wedge B) \equiv (\neg A \vee \neg B) & \neg \exists x A \equiv \forall x \neg A \end{array}$$

More precisely, when writing $\neg A$, we will always mean the *negation normal form* of this formula for the obviously terminating and confluent rewriting system:

$$\begin{array}{l|l} \neg(A \vee B) \longrightarrow (\neg A \wedge \neg B) & \neg \forall x A \longrightarrow \exists x \neg A \\ \neg(A \wedge B) \longrightarrow (\neg A \vee \neg B) & \neg \exists x A \longrightarrow \forall x \neg A \\ \neg\neg A \longrightarrow A & \end{array}$$

Symmetry of LK (2)

Identity Rules

$$\frac{}{\vdash' A, \neg A}\ \textit{Axiom} \qquad \frac{\vdash' A, \Gamma \quad \vdash' \neg A, \Delta}{\vdash' \Gamma, \Delta}\ \textit{Cut}$$

Structural Rules

$$\frac{\vdash' \Gamma, B, A, \Delta}{\vdash' \Gamma, A, B, \Delta}\ \textit{Ex} \qquad \frac{\vdash' \Gamma}{\vdash' A, \Gamma}\ \textit{W} \qquad \frac{\vdash' A, A, \Gamma}{\vdash' A, \Gamma}\ \textit{C}$$

Logical Rules

$$\frac{\vdash' A, \Gamma \quad \vdash' B, \Gamma}{\vdash' A \wedge B, \Gamma}\ \wedge \qquad \frac{\vdash' A, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee 1 \qquad \frac{\vdash' B, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee 2$$

$$\frac{\vdash' A, \Gamma}{\vdash' \forall x A, \Gamma}\ \forall \quad (*) \qquad \frac{\vdash' A[t/x], \Gamma}{\vdash' \exists x A, \Gamma}\ \exists$$

Symmetry of LK (2)

Identity Rules

$$\frac{}{\vdash' A, \neg A}\ \textit{Axiom} \qquad \frac{\vdash' A, \Gamma \quad \vdash' \neg A, \Delta}{\vdash' \Gamma, \Delta}\ \textit{Cut}$$

Structural Rules

$$\frac{\vdash' \Gamma, B, A, \Delta}{\vdash' \Gamma, A, B, \Delta}\ \textit{Ex} \qquad \frac{\vdash' \Gamma}{\vdash' A, \Gamma}\ \textit{W} \qquad \frac{\vdash' A, A, \Gamma}{\vdash' A, \Gamma}\ \textit{C}$$

Logical Rules

$$\frac{\vdash' A, \Gamma \quad \vdash' B, \Gamma}{\vdash' A \wedge B, \Gamma}\ \wedge \qquad \frac{\vdash' A, B, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee$$

$$\frac{\vdash' A, \Gamma}{\vdash' \forall x A, \Gamma}\ \forall \quad (*) \qquad \frac{\vdash' A[t/x], \Gamma}{\vdash' \exists x A, \Gamma}\ \exists$$

Completeness and Soundness

Two important questions:

- Are all theorems true? *Soundness*
- Are all the true formulas provable? *Completeness*

What do we mean by "*true*"?

- Soundness is easily proved by induction on the deduction rules.
- Completeness is more complicated.

Intuition of the proof

Actually, we will prove the contrapositive: if $\vdash F$ is not provable, then we can find a model of the language in which F is not satisfied, that is F is not valid.

The proof scheme will actually be the following: we will design a proof search procedure that will search for a proof of $\vdash F$. Since there is no such proof, we cannot end up with an object which is a proof: the resulting object will actually be a failure from which we will build a counter-model for F by correctly choosing the truth values for the predicates in order to falsify all formulas in the base sequent.

Derivation Trees

Improper Rules

$$\frac{\boxed{\text{Open}}}{L;\Gamma;\Phi}\ \textit{Open} \qquad \frac{L;\Gamma;\Phi}{\boxed{\text{Root}}}\ \textit{Root} \qquad \frac{\boxed{\text{Failure}}}{L;\emptyset;\emptyset}\ \textit{Failure}$$

Classification Rules

$$\frac{l, L;\Gamma;\Phi}{L; l,\Gamma;\Phi}\ \textit{Literal} \qquad \frac{L;\Gamma;\Phi,\exists xA}{L;\exists xA,\Gamma;\Phi}\ \textit{Existential}$$

Logical Rules

$$\frac{}{A,\neg A, L;\Gamma;\Phi}\ \textit{axiom} \qquad \frac{}{L;\top,\Gamma;\Phi}\ \top \qquad \frac{L;A,\Gamma;\Phi \quad L;B,\Gamma;\Phi}{L;A\wedge B,\Gamma;\Phi}\ \wedge$$

$$\frac{L;A,B,\Gamma,\Phi}{L;A\vee B,\Gamma,\Phi}\ \vee \qquad \frac{L;A[y/x],\Gamma;\Phi}{L;\forall xA,\Gamma;\Phi}\ \forall \qquad \frac{L;A[t/x];\Phi,\exists xA}{L;\emptyset;\exists xA,\Phi}\ \exists$$

Example: the Return of the Drinker

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           \dfrac{}{P(y_1), \neg P(y_0), P(y_0), \neg P(t_0) \;;\; \emptyset \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\text{axiom}
          }{\neg P(y_0), P(y_0), \neg P(t_0) \;;\; P(y_1) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\text{literal}
         }{P(y_0), \neg P(t_0) \;;\; \neg P(y_0), P(y_1) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\text{literal}
        }{P(y_0), \neg P(x_0) \;;\; (\neg P(y_0) \vee P(y_1)) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\vee
       }{P(y_0), \neg P(x_0) \;;\; \forall y(\neg P(y_0) \vee P(y)) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\forall
      }{P(y_0), \neg P(x_0) \;;\; \emptyset \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\exists
     }{\neg P(x_0) \;;\; P(y_0) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\text{literal}
    }{\emptyset \;;\; \neg P(x_0), P(y_0) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\text{literal}
   }{\emptyset \;;\; (\neg P(x_0) \vee P(y_0)) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\vee
  }{\emptyset \;;\; \forall y(\neg P(x_0) \vee P(y)) \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\forall
 }{\emptyset \;;\; \emptyset \;;\; \exists x \forall y(\neg P(x) \vee P(y))}\,\exists
}{\emptyset \;;\; \exists x \forall y(\neg P(x) \vee P(y)) \;;\; \emptyset}\,\text{existential}
$$

Systematic Derivation Trees

Definition: Systematic Derivation Tree

Let us consider ϵ an enumeration of all terms in the language.
A ***systematic derivation tree*** *for a sequent $L; \Gamma; \Phi$ is a derivation tree such that:*

- *the axiom rule is applied as soon as it is possible. That means that in a branch of a systematic derivation tree, there cannot be two opposite literals in the left component of the sequent except in the upmost sequent which must be followed by an axiom rule;*
- *when the $\exists$ rule is used, the bound variable must be instantiated with the first term (according to ϵ that has not yet been used in the instantiation of this formula lower in the derivation tree.*

Example II

$$\neg P(t_k), P(y_{k-1}), \ldots, \neg P(t_1), P(y_0), \neg P(t_0) \;;\; P(y_k) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

axiom

$$P(y_{k-1}), \ldots, \neg P(t_1), P(y_0), \neg P(t_0) \;;\; \neg P(t_k), P(y_k) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

literal

$$\vdots$$

$$P(y_1), \neg P(t_1), P(y_0), \neg P(t_0) \;;\; \emptyset \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\exists$

$$\neg P(t_1), P(y_0), \neg P(t_0) \;;\; P(y_1) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

literal

$$P(y_0), \neg P(t_0) \;;\; \neg P(t_1), P(y_1) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

literal

$$P(y_0), \neg P(t_0) \;;\; (\neg P(t_1) \vee P(y_1)) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\vee$

$$P(y_0), \neg P(t_0) \;;\; \forall y (\neg P(t_1) \vee P(y)) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\forall$

$$P(y_0), \neg P(t_0) \;;\; \emptyset \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\exists$

$$\neg P(t_0) \;;\; P(y_0) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

literal

$$\emptyset \;;\; \neg P(t_0), P(y_0) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

literal

$$\emptyset \;;\; (\neg P(t_0) \vee P(y_0)) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\vee$

$$\emptyset \;;\; \forall y (\neg P(t_0) \vee P(y)) \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\forall$

$$\emptyset \;;\; \emptyset \;;\; \exists x \forall y (\neg P(x) \vee P(y))$$

$\exists$

$$\emptyset \;;\; \exists x \forall y (\neg P(x) \vee P(y)) \;;\; \emptyset$$

existential

Corollaries and Extensions

Definition: Systematic ω-Derivation Tree

We add the following rule:

$$\frac{L; \neg F; \Phi}{L; \emptyset; \Phi}\ \omega \qquad \text{for} \quad F \in \mathfrak{T}$$

Compactness Theorem

If a set $\mathcal{S}$ of formulas is such that all its finite subsets are satisfiable then $\mathcal{S}$ itself is satisfiable.

Disjunction and Existence Properties

Thanks to cut-elimination we have:

Disjunction Property

If $\vdash_{LJ} A \vee B$*, then* $\vdash_{LJ} A$ *or* $\vdash_{LJ} B$

Existence Property

If $\vdash_{LJ} \exists x A$*, then there exists a term* t *such that* $\vdash_{LJ} A[t/x]$

In LJ, we have constructive proofs for an empty theory:

$$\vdash F$$

So that we have such a result for purely logical proofs, not for mathematical proofs.

How to extend this?

Harrop theories

Definition: Harrop theories

- *A(n occurrence of a) subformula B of A is said to be strictly positive iff it does not appear on the left of an implication or under a negation;*
- *a Harrop formula is a formula A such that the $\vee$ and $\exists$ connectives never appear as principal connectives of a strictly positive subformula of A;*
- *a Harrop theory is a set of Harrop formulas.*

Theorem/ Exercise: Harrop theories are constructive.

If Γ is a Harrop theory, the following properties hold:

- *if $\Gamma \vdash_{LJ} A \vee B$ then $\Gamma \vdash_{LJ} A$ or $\Gamma \vdash_{LJ} A$;*
- *if $\Gamma \vdash_{LJ} \exists x A$ then there is a term t such that $\Gamma \vdash_{LJ} A[t/x]$.*

We say that Harrop theories are constructive.

Strong Normalization

$$\cfrac{\cfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,RC \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut$$

$$\begin{array}{c}\cfrac{\cfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,cut \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,cut \\ \vdots\; RC\&LC \\ \Gamma, \Gamma' \vdash \Delta, \Delta'\end{array}$$

Strong Normalization (2)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{}{A \vdash A}\,ax}{A \vee A \vdash A, A}\,L\vee}{A \vee A \vdash A}\,RC \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{}{A \vdash A}\,ax}{A, A \vdash A \wedge A}\,R\wedge}{A \vdash A \wedge A}\,LC}{A \vee A \vdash A \wedge A}\,cut$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A \vee A \vdash A, A}}{A \vee A \vdash A}\,RC \qquad \dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{A \vdash A}}{A, A \vdash A \wedge A}}{A \vee A, A \vdash A \wedge A}\,cut \qquad \dfrac{\dfrac{\vdots}{A \vee A \vdash A, A}}{A \vee A \vdash A}\,RC}{A \vee A, A \vee A \vdash A \wedge A}\,cut}{A \vee A \vdash A \wedge A}\,LC$$

Size of the cut-free proof

Proposition/Exercise: Size of the Cut Free Proofs

- $S(0, h) = h$
- $S(d + 1, h) = 4^{S(d,h)}$

LL Sequent Calculus

Identity Rules:

$$\dfrac{}{\vdash A^{\perp}, A}\ \textit{ax} \qquad \dfrac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \textit{cut}$$

Structural Rule:

$$\dfrac{\vdash \Gamma, B, A, \Delta}{\vdash \Gamma, A, B, \Delta}\ \textit{Ex}$$

LL Sequent Calculus

Logical Rules:

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ \parr \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ \otimes$$

$$\frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ \& \qquad \frac{\vdash F, \Gamma}{\vdash F \oplus G, \Gamma}\ \oplus 1 \qquad \frac{\vdash G, \Gamma}{\vdash F \oplus G, \Gamma}\ \oplus 2$$

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ \bot \qquad \frac{}{\vdash \top, \Gamma}\ \top$$

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ ? \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ !$$

$$\frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ ?W \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ ?C$$

Important Characteristics of Linear Logic

- Thanks to the additional structure put in the sequents themselves, we can capture more things at the logical level and not at the term level like in classical logic;
- The control on structural rules allows a careful study of cut-elimination, which via Curry-Howard corresponds to execution of a functional program;
- Thanks to the richness of the sequents it is possible to consider the sequents as storing a state of the computation in a process of proof-search (logic programming paradigm).
- Lots of other directions...

Conclusion

The rules that at first seemed to be the less significant in logic (at such an extent that they are missing in Natural Deduction) are eventually crucial in the proof theoretic analysis of logic. Indeed, it is by controlling these rules that we can choose the focus we want to put on logic and the level of detail we desire: Controlling the structural rules, we can *zoom* and catch more details of the proofs.

From the computer science point of view, the very structured object that a LL proof is allows for various uses and applications.

Less formally and more informally, an interest of this study of structure is that we came from a logical study driven by the notion of *truth* and that now we can do logic as study of geometrical properties of proofs, the logical character being assured by some formal requirement such as cut-elimination, symmetrical properties...