

A NON-STANDARD SEMANTICS FOR KAHN NETWORKS IN CONTINUOUS TIME

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SÉMINAIRE PPS

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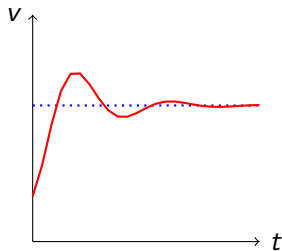
HYBRID SYSTEMS

A model for systems operating in **continuous time** $t \in \mathbb{R}^+$.

A controller to **cruise control** a car:

$$I(t) = K_p e(t)$$

- ▶ error: $e(t) = v_{\text{desired}} - v_{\text{actual}}$
- ▶ intensity of the engine: I
- ▶ parameters of the control: K_p



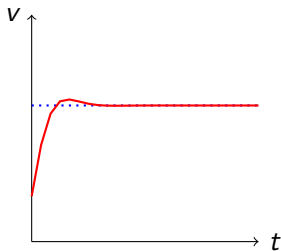
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A model for systems operating in **continuous time** $t \in \mathbb{R}^+$.

A PID-controller to **cruise control** a car:

$$I(t) = K_p e(t) + K_i \int_0^t e(t) dt + K_d \frac{de}{dt}$$

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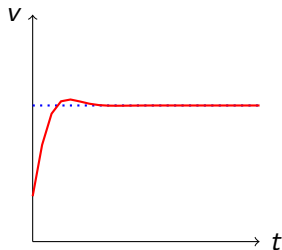
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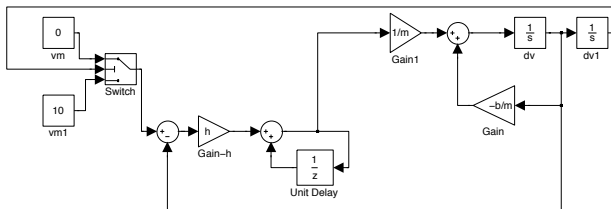
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We also want to have discontinuities!

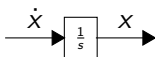
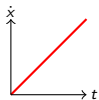
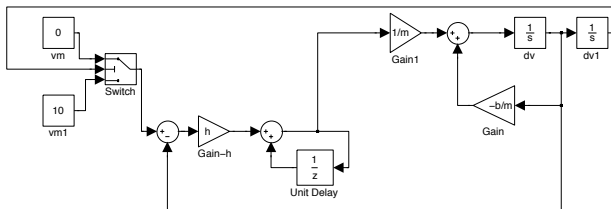
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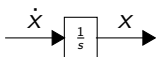
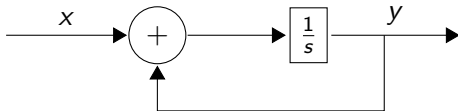
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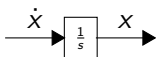
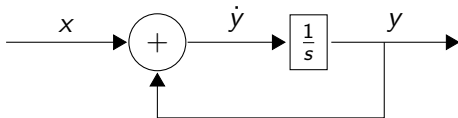
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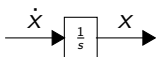
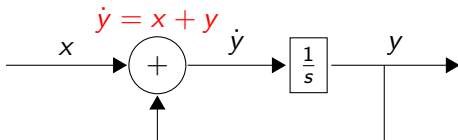
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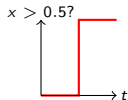
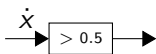
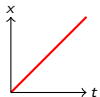
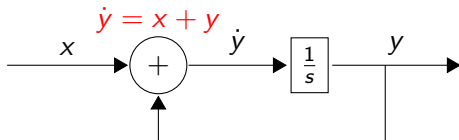
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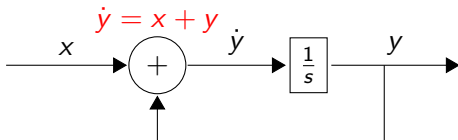
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How can we define a semantics for those systems?

The streams $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ on the wires could

- ▶ be integrable / derivable
- ▶ have discontinuities (zero-crossings)
- ▶ exhibit complex behaviors such as Zeno
- ▶ be approximated...

Let's take inspiration
from discrete time semantics.

KAHN PROCESS NETWORKS

A semantics for *distributed asynchronous* computations:
processes exchanging *sequences of data* on channels.

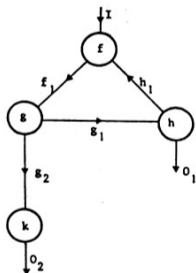


Fig.3. A parallel program schema.

G. Kahn. The semantics of a simple language for parallel programming. *Information processing*, 74:471–475, 1974.



KAHN NETWORKS IN CONTINUOUS TIME

It is difficult to define a semantics for hybrid systems.

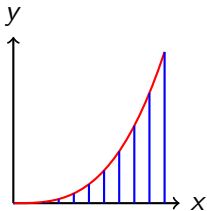
Can we adapt the works on Kahn networks?

KAHN NETWORKS IN CONTINUOUS TIME

It is difficult to define a semantics for hybrid systems.
Can we adapt the works on Kahn networks?

The sampling principle

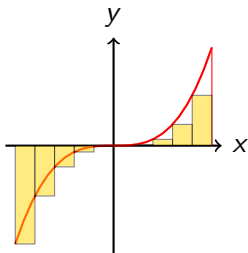
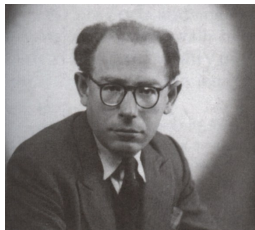
We can consider a continuous stream $t \mapsto x_t$ (with $t \in \mathbb{R}^+$)
as a discrete stream x_i (with $i \in \mathbb{N}$)
where the data x_i occurs at time $i\varepsilon$, with ε infinitesimal.



A continuous stream sampled at every ε seconds.

NON-STANDARD ANALYSIS

In the 60s, Robinson introduced an extension ${}^*\mathbb{R}$ of \mathbb{R}
(the *hyperreals*)
in which one can formally consider **infinitesimals**.



$$f'(x) = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

$$\int_{t=0}^T f(t)dt = \sum_{0 \leq i \leq T/\varepsilon} f(i)\varepsilon$$

with ε infinitesimal

THE PLAN

1. Define **Kahn networks** and their semantics
 - ▶ formalization of Kahn networks
 - ▶ Kahn networks form a free fixpoint category

2. A **non-standard semantics** for Kahn networks
 - ▶ non-standard semantics using internal cpo

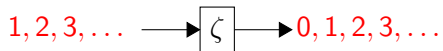
THE PLAN

1. Define **Kahn networks** and their semantics
 - ▶ formalization of Kahn networks
 - ▶ Kahn networks form a free fixpoint category
 - ▶ *related works*:
 - ▶ Kahn
 - ▶ categorical structure of the models:
Hildebrandt, Panangaden, Winskel, Stark, ...
2. A **non-standard semantics** for Kahn networks
 - ▶ non-standard semantics using internal cpo
 - ▶ *related works*:
 - ▶ Bliudze, Krob
 - ▶ Benveniste, Caillaud, Pouzet

**KAHN
NETS**

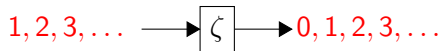
EXAMPLES OF KAHN NETWORKS

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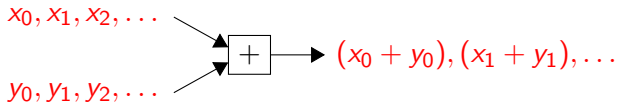


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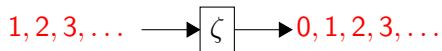


- ▶ Add two discrete streams:

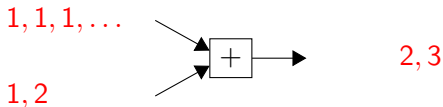


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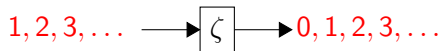


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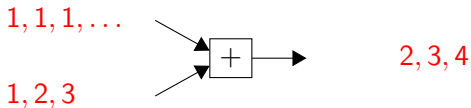


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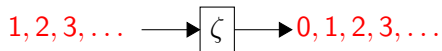


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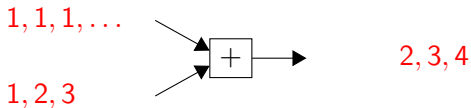


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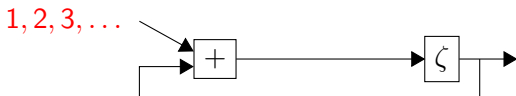
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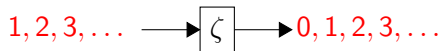


- ▶ A net composed of **generators**:

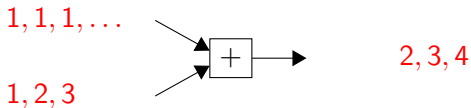


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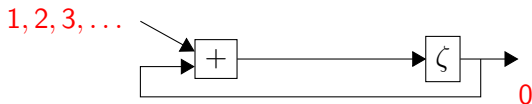
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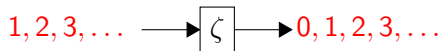


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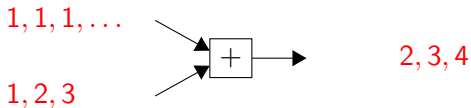


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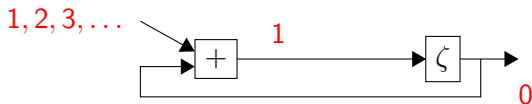
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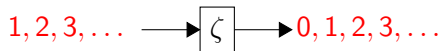


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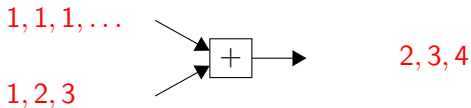


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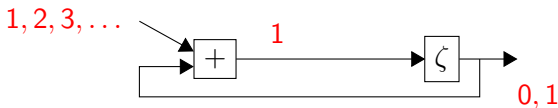
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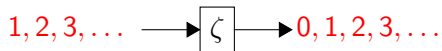


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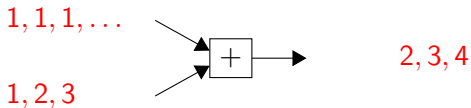


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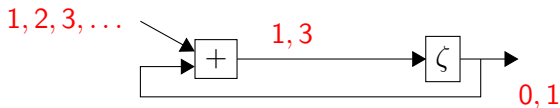
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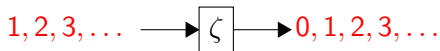


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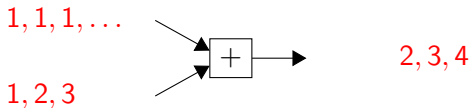


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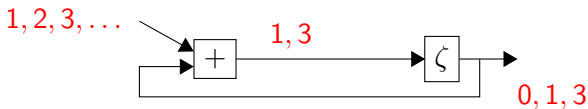
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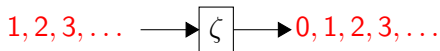


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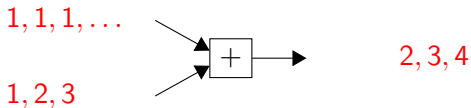


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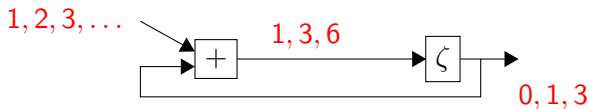
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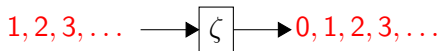


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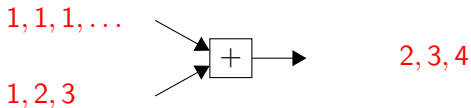


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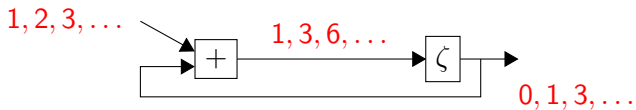
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SEMANTICS OF KAHN NETWORKS

Definition

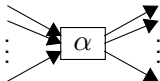
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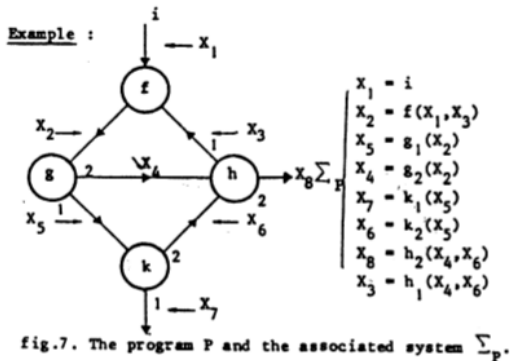
To each generator $\alpha : m \rightarrow n$



we associate a *Scott-continuous function* $K^m \rightarrow K^n$.

SEMANTICS OF KAHN NETWORKS

The semantics of a composed net is given by associating a set of equations to the network



and taking the unique *minimal solution* (which exists).

A semantics of what?

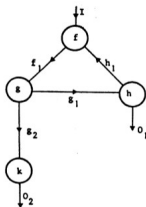


Fig.3. A parallel program schema.

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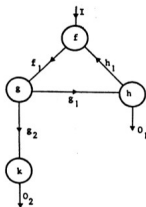
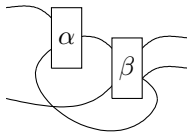
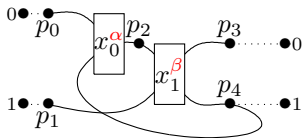


Fig.3. A parallel program schema.



is formalized by



SIGNATURES

The “basic building blocks” of Kahn networks are the elements of a signature:

Definition

A **signature** $\Sigma = \{\alpha_i : m_i \rightarrow n_i\}$ consists of a set Σ of **symbols** (or **generators**) α_i with **arity** m_i and **coarity** n_i .

Example

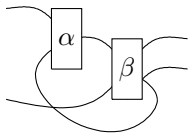
For instance,



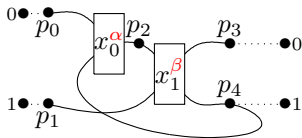
is formalized by the signature

$$\Sigma = \{\zeta : 1 \rightarrow 1, + : 2 \rightarrow 1\}$$

A FORMAL DEFINITION OF NETS



is formalized by



Definition (Net)

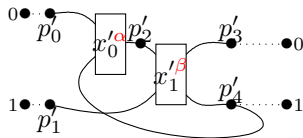
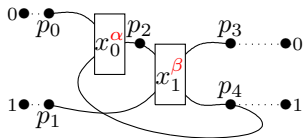
A **net** $N = (P, O, \lambda, s, t)$ with m inputs and n outputs consists of

- ▶ P : a finite set of *ports*,
- ▶ O : a finite set of *operators*,
- ▶ $\lambda : O \rightarrow \Sigma$: a *labeling function* ,
- ▶ $s : S_N \rightarrow P$ and $t : T_N \rightarrow P$: *source and inj. target functions*

$$S_N = \{(x, i) \mid x \in O, i \in \{0 \dots \sigma \circ \lambda(x) - 1\}\} \uplus \{0 \dots n - 1\}$$

$$T_N = \{(x, i) \mid x \in O, i \in \{0 \dots \tau \circ \lambda(x) - 1\}\} \uplus \{0 \dots m - 1\}$$

α -CONVERSION OF NETS



Definition

Two nets $N = (P, O, \lambda, s, t)$ and $N' = (P', O', \lambda', s', t')$ are α -convertible when there exists a pair of bijective functions

$$\varphi_P : P \rightarrow P' \quad \text{and} \quad \varphi_O : O \rightarrow O'$$

such that

- ▶ $\forall x \in O, \quad \lambda'(\varphi_O(x)) = \lambda(x)$
- ▶ $\forall (x, i) \in S_N, \quad \varphi_P(s(x, i)) = s'(\varphi(x), i)$

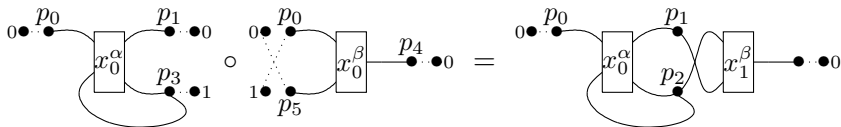
THE CATEGORY OF NETS

We form a category \mathbf{Net}_Σ whose

- ▶ objects are natural integers $n \in \mathbb{N}$
- ▶ morphisms $N : m \rightarrow n$ are nets with m inputs and n outputs

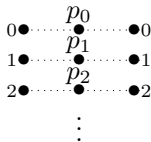
Categorical structure:

- ▶ **Composition:** juxtaposing and “linking the wires”



(up to α -conversion, otherwise we get a bicategory)

- ▶ **Identities:**



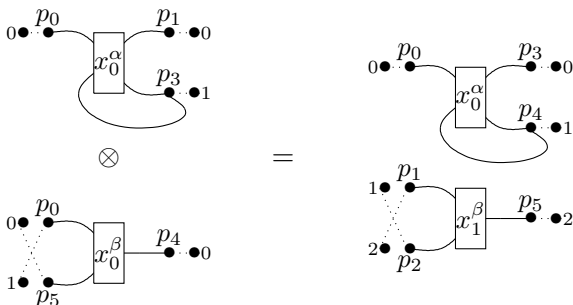
THE CATEGORY OF NETS

We form a **monoidal** category \mathbf{Net}_Σ whose

- ▶ objects are natural integers $n \in \mathbb{N}$
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Categorical structure:

- ▶ **Tensor product:** from $N : m \rightarrow n$ and $N' : m' \rightarrow n'$ we can define $N \otimes N' : (m + m') \rightarrow (n + n')$



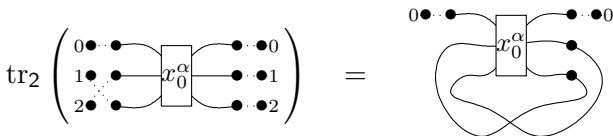
THE CATEGORY OF NETS

We form a **traced monoidal** category \mathbf{Net}_Σ whose

- ▶ objects are natural integers $n \in \mathbb{N}$
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Categorical structure:

- ▶ **Trace:** from $N : (m + k) \rightarrow (n + k)$
we can define $\text{tr}_k(N) : m \rightarrow n$
satisfying suitable axioms



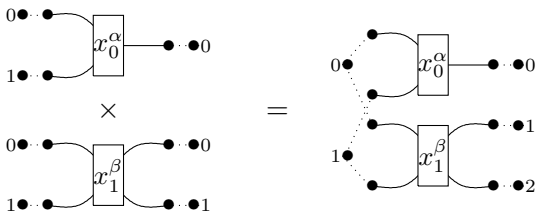
THE CATEGORY OF NETS

We form a **traced cartesian(?)** category \mathbf{Net}_Σ whose

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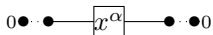
Categorical structure:

- ▶ The tensor is almost a **cartesian product**:
from $N_1 : m \rightarrow n_1$ and $N_2 : m \rightarrow n_2$
we can define $N_1 \times N_2 : m \rightarrow (n_1 + n_2)$



CARTESIAN PRODUCT OF NETS

The “cartesian product” we defined is not one: given $N : m \rightarrow n$



it does *not* satisfy

- ▶ **sharing:** $N \times N = (\text{id}_n \times \text{id}_n) \circ N$

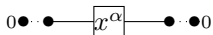


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We consider nets up to the equivalence relation generated by the above equations!

Suppose given a signature (Σ, σ, τ) .

Definition

A Σ -**object** in a monoidal category \mathcal{C} consists of

- ▶ an object A
- ▶ for every symbol $\alpha \in \Sigma$, a morphism

$$\alpha \quad : \quad A^{\otimes \sigma(\alpha)} \rightarrow A^{\otimes \tau(\alpha)}$$

where

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \dots \otimes A}_{k \text{ times}}$$

Σ -OBJECTS

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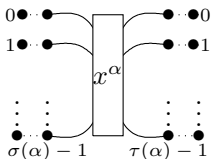
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Remark

In the category \mathbf{Net}_Σ , 1 is a Σ object, where to every symbol $\alpha \in \Sigma$ is associated the morphism



A UNIVERSAL DESCRIPTION OF NETS

Definition

A **fixpoint category** is a category with

- ▶ cartesian products
- ▶ a trace.

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The category \mathbf{Net}_Σ is the *free fixpoint category* containing a Σ -object (i.e. an interpretation of the generators).

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an interpretation of all Kahn nets.**

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The category \mathbf{Net}_Σ is the *free fixpoint category* containing a Σ -object (i.e. an interpretation of the generators).

Any interpretation of the generators in a fixpoint category canonically induces an interpretation of all Kahn nets.

- ▶ We can give semantics of KN in other fixpoint categories.
- ▶ Avoids some technical details (solving systems of equations).

A SEMANTICS OF HYBRID SYSTEMS?

The Kahn model

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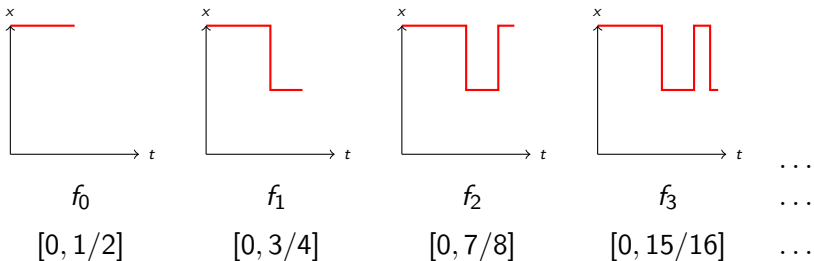
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A SEMANTICS OF HYBRID SYSTEMS?

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These do not form a cpo!



... because of the Zeno effect.

**NON-STANDARD
ANALYSIS
A CRASH COURSE**

NON-STANDARD ANALYSIS

In order to give a meaning to the infinitesimals, we replace reals by **hyperreals** which are sequences $(x_i)_{i \in \mathbb{N}}$ of reals.

- ▶ A real x is seen as the constant sequence (x) .
- ▶ An infinitesimal number is a sequence converging towards 0.
- ▶ An “infinite” number is a sequence converging towards $\pm\infty$.
- ▶ The usual operations are extended pointwise on sequences:
$$(x_i) \times (y_i) = (x_i \times y_i)$$

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- ▶ An “infinite” number is a sequence converging towards $\pm\infty$.
- ▶ The usual operations are extended pointwise on sequences:
 $(x_i) \times (y_i) = (x_i \times y_i)$
- ▶ What is the inverse of $(0, 1, 0, 1, 0, 1, \dots)$?
- ▶ In order to recover usual properties one has to consider *equivalence classes* of sequences.

HYPERREALS

We will define a collection \mathcal{F} of subsets (called **large**) of \mathbb{N} and consider the equivalence relation such that

$$(x_i) \equiv (y_i) \quad \text{whenever} \quad \{i \in \mathbb{N} \mid x_i = y_i\} \in \mathcal{F}$$

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The set \mathcal{F} should satisfy properties:

- ▶ two sequences equal at every index excepting a finite number should be equal: \mathcal{F} should contain all cofinite sets
- ▶ two sequences are either equal (equivalent) or different:

$$\forall U \subseteq \mathbb{N}, \quad U \in \mathcal{F} \quad \text{or} \quad \mathbb{N} \setminus U \in \mathcal{F}$$

- ▶ ...

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The set \mathcal{F} should be a *non-principal ultrafilter* on \mathbb{N} .

Definition

An **ultrafilter** \mathcal{F} on \mathbb{N} is a collection of subsets of \mathbb{N} such that

1. intersection: $\forall U, V \in \mathcal{F}, \quad U \cap V \in \mathcal{F}$
2. supersets: $\forall U \in \mathcal{F}, \forall V \subseteq \mathbb{N}, \quad U \subseteq V \Rightarrow V \in \mathcal{F}$
3. proper: $\emptyset \notin \mathcal{F}$
4. complement: $\forall U \subseteq \mathbb{N}, \quad U \in \mathcal{F} \text{ or } \mathbb{N} \setminus U \in \mathcal{F}$

(with AC such an \mathcal{F} exists).

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We write $\langle x_i \rangle$ for the equivalence class of (x_i) .

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Definition

- ▶ The field of **hyperreals** ${}^*\mathbb{R}$ is $\mathbb{R}^{\mathbb{N}} / \equiv$.
- ▶ The ring of **hyperintegers** ${}^*\mathbb{N}$ is $\mathbb{N}^{\mathbb{N}} / \equiv$.

INFINITESIMAL AND UNLIMITED

Definition

An hyperreal $x \in {}^*\mathbb{R}$ is

- ▶ **infinitesimal**: if $x \neq 0$ and $\forall r \in \mathbb{R}, |x| < r$
- ▶ **unlimited**: if $\forall r \in \mathbb{R}, |x| > r$

Example

- ▶ infinitesimal: $x = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$
- ▶ unlimited: $x = \langle 1, 2, 3, 4, \dots \rangle$

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Intuition

- ▶ ${}^*\mathbb{R}$ is \mathbb{R} completed with infinitesimals and unlimited.
- ▶ ${}^*\mathbb{N}$ is \mathbb{N} completed with unlimited

NON-STANDARD ANALYSIS

Continuity, derivation, integration, etc.
have the “expected” formulations
in this framework.

THE STANDARD PART

Notation

Given $x, y \in {}^*\mathbb{R}$, we write $x \approx y$ whenever $x - y$ is infinitesimal.

Lemma

Given a limited $x \in {}^\mathbb{R}$, there exists a unique real $\text{st}(x)$, the **standard part** of x , such that $x \approx \text{st}(x)$.*

NON-STANDARD ANALYSIS – CONTINUITY

Proposition

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **continuous** at $x \in \mathbb{R}$ iff

$$\forall y \in {}^*\mathbb{R}, \quad y \approx x \quad \Rightarrow \quad f(y) \approx f(x)$$

or equivalently, for every infinitesimal ε there exists an infinitesimal δ such that

$$f(x + \varepsilon) = f(x) + \delta$$

NON-STANDARD ANALYSIS – DERIVATIVE

Proposition

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ admits $y \in \mathbb{R}$ as **derivative** at $x \in \mathbb{R}$ iff for every infinitesimal ε ,

$$\frac{f(x + \varepsilon) - f(x)}{\varepsilon} \approx y$$

In this case, for any ε infinitesimal,

$$f'(x) = \text{st} \left(\frac{f(x + \varepsilon) - f(x)}{\varepsilon} \right)$$

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Proposition

Integrals can be reformulated similarly as Riemann sums...

**NON-STANDARD
SEMANTICS
OF KAHN NETWORKS**

TOWARDS A NON-STANDARD SEMANTICS

- ▶ The elements of the Kahn cpo are the streams:

$$\mathbb{R}^{\leq \mathbb{N}}$$

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- ▶ We will see those functions as sequences

$$f(0), f(\varepsilon), f(2\varepsilon), f(3\varepsilon), \dots$$

for some infinitesimal ε

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We have to take in account infinitesimal variations!

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But for every $n \in \mathbb{N}$, $n\varepsilon$ is an infinitesimal!

TOWARDS A NON-STANDARD SEMANTICS

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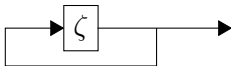
for some infinitesimal ε

$$*\mathbb{R} \leq *N$$

We consider hypersequences of hyperreals.

FIXPOINTS IN ${}^*\mathbb{R} \leq {}^*\mathbb{N}$

The semantics of the following net should be the constant stream f such that $\forall n \in {}^*\mathbb{N}, f(n\varepsilon) = 0$:



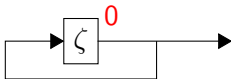
However, if we compute its semantics using the fixpoint construction we get the stream f such that

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The value $f(n\varepsilon)$ is not defined for unlimited hyperintegers $n \in {}^*\mathbb{N}$!

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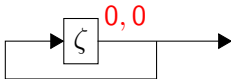
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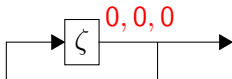
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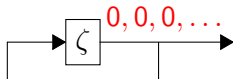
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INTERNAL THINGS

From $(D_i \subseteq \mathbb{R})_{i \in \mathbb{N}}$ we can define a subset $D \subseteq {}^*\mathbb{R}$:

- ▶ $D_0 = \{x_0, y_0, z_0, \dots\}$
 - ▶ $D_1 = \{x_1, y_1, z_1, \dots\}$
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Definition

An **internal set** $D \subseteq {}^*\mathbb{R}$ is a set such that there exists a family $(D_i \subseteq \mathbb{R})_{i \in \mathbb{N}}$ for which

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Internal functions $f = \langle f_i \rangle$, *internal relations*, etc. are defined similarly.

THE TRANSFER PRINCIPLE

Proposition

The transfer principle:

a first-order formula is satisfied for \mathbb{R}

iff

it is satisfied for ${}^\mathbb{R}$,*

if we suppose that all the sets, etc in the formula to be internal

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Lemma

Internal induction principle: if D is an internal subset of ${}^*\mathbb{N}$ s.t.

- ▶ $0 \in D$
- ▶ $\forall n \in D, n + 1 \in D$

then $D = {}^*\mathbb{N}$.

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Proposition

An internal Scott-continuous function f between two internal cpo admits a least (internal) fixpoint

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Remark

An internal cpo (D, \leq) is not necessarily a cpo!

THE INFINITESIMAL-TIME DOMAIN

Definition

- ▶ The category **ICPO**:
internal cpo and internal Scott-continuous functions
- ▶ The **infinitesimal-time domain** $IT \in \mathbf{ICPO}$:
the internal cpo of internal functions in ${}^*\mathbb{R}^{\leq} {}^*\mathbb{N}$

THE INFINITESIMAL-TIME DOMAIN

Definition

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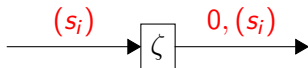
Proposition

*The category **ICPO** is a fixpoint category.*

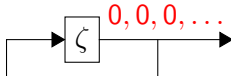
A Σ -object in this category canonically induces
a semantics of Kahn networks

EXAMPLE – THE CONSTANT STREAM

If we interpret



then the net

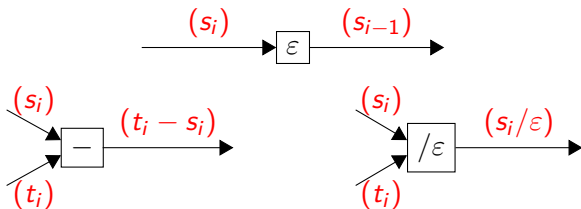


is interpreted as the constant stream $s : {}^*\mathbb{R}^{{}^*\mathbb{N}}$ such that

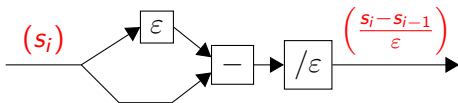
$$\forall i \in {}^*\mathbb{N}, \quad s_i = 0$$

EXAMPLE – DERIVATION

If we interpret



(with ϵ infinitesimal) then the net



is interpreted as the function

$$D : {}^*\mathbb{R}^{*N} \rightarrow {}^*\mathbb{R}^{*N}$$

$$(s_i) \mapsto \left(\frac{s_i - s_{i-1}}{\epsilon} \right)$$

EXAMPLE – DERIVATION

The function $D : {}^*\mathbb{R}^{*\mathbb{N}} \rightarrow {}^*\mathbb{R}^{*\mathbb{N}}$ acts as a derivation operator for streams.

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In order to compare

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we introduce

sampling	standardisation
$S : CT \rightarrow IT$	$T : IT \rightarrow CT$
$s \mapsto (s(i\varepsilon))_{i \in {}^*\mathbb{N}}$	$(s_j)_{j \in {}^*\mathbb{N}} \mapsto t \mapsto \text{st}(s_{\lfloor t/\varepsilon \rfloor})$

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Proposition

For any continuously differentiable $s \in CT$, $T(D(S(s))) = s'$.

NON-STANDARD SEMANTICS GOING FURTHER

In this way, we give a non-standard **semantics** for hybrid systems:

- ▶ we can interpret all the common building blocks
(derivation, integration, zero-crossing, etc.)
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- ▶ we have a built-in notion of approximation
- ▶ NSA enables us to use discrete techniques for continuous: continuous-time bisimulations, game semantics, etc.?

THANKS!

Questions?