Term Algebras and Equational Reasoning

Signatures and finite terms

Σ: Set of function symbols having an arity $n \in \mathbb{N}$.
$\mathcal{X}$: Set of variables.

$\mathcal{T}(\mathcal{X}, \Sigma)$: Set of terms over $\mathcal{X}$ and Σ:

$$\frac{x \in \mathcal{X}}{x \in \mathcal{T}(\mathcal{X}, \Sigma)} \qquad \frac{t_1, \ldots, t_n \in \mathcal{T}(\mathcal{X}, \Sigma) \qquad f \text{ has arity } n \in \Sigma}{f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{X}, \Sigma)}$$

We note $Var(t)$ the set of variables of the term t. A term t is closed if $Var(t) = \emptyset$.

Example

Signature:

$b:$	arity 0	$s:$	arity 1
$+:$	arity 2	$eq:$	arity 2
$true:$	arity 0	$false:$	arity 0

Terms:

$$eq(s(x+b),b) \quad eq(true,false) \quad eq(s(x+b),false)$$

closed

Positions of terms

$\mathbb{N}^*$: positions over $\mathbb{N}$

$$\frac{}{\Lambda \in \mathbb{N}^*} \qquad \frac{i \in \mathbb{N} \text{ and } p \in \mathbb{N}^*}{ip \in \mathbb{N}^*}$$

To simplify the notation, we usually write a position by omitting the symbol Λ, *e.g.* 2 denotes the position 2Λ.

$Pos(t)$: positions of a term t

$$\frac{}{\Lambda \in Pos(t)} \qquad \frac{p \in Pos(t_i) \text{ and } 1 \leq i \leq n}{ip \in Pos(f(t_1, \ldots, t_n))}$$

Example

$$Pos(f(g(a, h(b)), x, c)) = \{\Lambda, 1, 2, 3, 11, 12, 121\}$$

The relation $\leq_{pref}$ over positions

Concatenation of positions: $\begin{cases} \Lambda.q & = & q \\ (ip).q & = & i(p.q) \end{cases}$

Comparing positions: $p \leq_{pref} q$ iff $\exists r \in \mathbb{N}^* \; p.r = q$

Example

1	$\leq_{pref}$	1211	"1 is smaller than 1211"
231	$\geq_{pref}$	23	"231 is greater than 23"
12	$\bowtie$	2	"12 is parallel to 2"$(12 \not\leq_{pref} 2 \;\&\; 2 \not\leq_{pref} 12)$

Sub-terms

$v \trianglelefteq t$: v is a subterm/subtree of t:

$$\frac{}{t \trianglelefteq t} \qquad \frac{v \trianglelefteq t_i}{v \trianglelefteq f(t_1, \ldots, t_n)}$$

$v \trianglelefteq t$: v is a subterm of t.
$v \triangleleft t$: v is a strict subterm of t.
$ST(t)$: All the subterms of t.

Example

$g(x, y) \triangleleft f(g(x, y), a)$ and $a \triangleleft f(g(x, y), a)$ but $f(x, a) \not\trianglelefteq f(g(x, y), a)$.

Sub-term X at position Y

$t|_p$: subterm of t at position p

$$\frac{}{t|_\Lambda = t} \qquad \frac{t_i|_q = v}{f(t_1, \ldots, t_n)|_{iq} = v}$$

Example

$f(g(a, h(b)), x, c)|_{11} = a$ but $f(g(a, h(b)), x, c)|_{21}$ is not defined.

Exercise : Show that $p.q \in Pos(t)$ implies $t|_{p.q} = (t|_p)|_q$.

Replacement

$t[p/\!\!/v]$: replacement of the subterm $t|_p$ by the term v

- $t[\Lambda/\!\!/v] = v$
- $f(t_1, \ldots, t_n)[ip/\!\!/v] = f(t_1, \ldots, t_i[p/\!\!/v], \ldots, t_n)$

Other notations: $t[v]_p$ or $t[v]$ if p is clear from the context.

Example

$f(h(x, y), a)[12/\!\!/b] = f(h(x, b), a)$ and $f(h(x, y), a)[1/\!\!/b] = f(b, a)$.

Exercise : Show the following properties:

- If $p \in Pos(s)$ and $q \in Pos(t)$, then $(s[t]_p)|_{p.q} = t|_q$ and $(s[t]_p)[r]_{p.q} = s[t[r]_q]_p$.
- If $p.q \in Pos(s)$, then $(s[t]_{p.q})|_p = (s|_p)[t]_q$ and $(s[t]_{p.q})[r]_p = s[r]_p$.
- If $p, q \in Pos(s)$ and $p \bowtie q$, then $(s[t]_p)|_q = s|_q$ and $(s[t]_p)[r]_q = (s[r]_q)[t]_p$.

Σ-algebras

A Σ-algebra $\mathcal{A}$ is defined by two ingredients:

- A non-empty domain $\mathbf{A}$.
- An interpretation function $f^{\mathcal{A}} : \mathbf{A}^n \mapsto \mathbf{A}$, for each $f/n \in \Sigma$.
- We write $\mathcal{A} = \langle \mathbf{A}, \{f^{\mathcal{A}} : \mathbf{A}^n \mapsto \mathbf{A} \mid f/n \in \Sigma\} \rangle$.

Example

Let $\Sigma = \{b/0, s/1, p/2\}$. We give three different Σ-algebras:

1. $\mathbf{A}$ is the set $\mathbb{N}$, $b^{\mathcal{A}} = 0$, $s^{\mathcal{A}} : n \mapsto n + 1$ and $p^{\mathcal{A}} : (n, m) \mapsto n + m$.
2. $\mathbf{A}$ is the set $\mathbb{Z}$, $b^{\mathcal{A}} = -5$, $s^{\mathcal{A}} : n \mapsto n * 13$ and $p^{\mathcal{A}} : (n, m) \mapsto n * m$.
3. Syntactic Algebra: $\mathbf{A}$ is the set of all the terms over $\mathcal{X}$ and Σ such that $b^{\mathcal{A}} = b$, $s^{\mathcal{A}} : t \mapsto s(t)$ and $p^{\mathcal{A}} : (t, u) \mapsto p(t, u)$.

Valuations

We use valuations to interpret terms in a Σ-algebra.

Let $\mathcal{A}$ be a Σ-algebra and let $\mathcal{X}$ be a set of variables.

A $\mathcal{A}$-valuation is an application $\sigma : \mathcal{X} \to \mathcal{A}$.

Definition

Given a $\mathcal{A}$-valuation $\sigma : \mathcal{X} \to \mathcal{A}$, we define a function $\widehat{\sigma} : \mathcal{T}(\mathcal{X}, \Sigma) \to \mathcal{A}$ interpreting arbitrary terms into the Σ-algebra $\mathcal{A}$ as follows:

$$\begin{array}{lcl} \widehat{\sigma}(x) & = & \sigma(x) \\ \widehat{\sigma}(f(t_1, \ldots, t_n)) & = & f^{\mathcal{A}}(\widehat{\sigma}(t_1), \ldots, \widehat{\sigma}(t_n)) \end{array}$$

Remark: By abuse of notation, we usually do not distinguish the valuation σ from the morphism $\widehat{\sigma}$.

Substitutions are valuations

A substitution θ is valuation from $\mathcal{X}$ to the syntactic algebra (the set of all the terms).
Finite substitutions (having a finite domain) are denoted $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$.
A renaming is an isomorphic substitution.

Example

- $\theta_1 = \{x/y, y/x\}$ and $\theta_2 = \{x/y, y/z, z/w\}$ are renamings.
- Given $t = f(x, g(y))$ and $\theta = \{x/g(a), y/f(x, x)\}$ we have $\theta(t) = f(g(a), g(f(x, x)))$.

Congruence

The symbol $f \in \Sigma$ is monotonic w.r.t the relation R iff
$a_i \; R \; b_i$ implies $f^{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \; R \; f^{\mathcal{A}}(a_1, \ldots, b_i, \ldots, a_n)$.

A congruence $\sim$ is an equivalence relation (reflexive, symmetric, transitive) for a Σ-algebra $\mathcal{A}$ iff every symbol $f \in \Sigma$ is monotonic w.r.t. $\sim$.

Example

$\sim = \{(x, y) \mid 4 \text{ divides } x - y\}$ is a congruence.

Notation : $\mathcal{A}/_{\sim}$ is the set of equivalence classes of a $\mathcal{A}$ modulo the congruence $\sim$. We write $[e]_{\sim}$ to denote the equivalent class of the element $e \in \mathcal{A}$ w.r.t. the congruence $\sim$.

The quotient algebra

Given a Σ-algebra $\mathcal{A}$ with domain $\mathbf{A}$, the quotient algebra $\mathcal{A}^{\sim}$ over $\mathcal{A}$ is defined by:

- The domain is $\mathbf{A}/_{\sim}$
- The Interpretations are $f^{\mathcal{A}^{\sim}}([a_1], \ldots, [a_n]) = [f^{\mathcal{A}}(a_1, \ldots, a_n)]$.

Example

Let $\Sigma = \{b/0, suc/1, pred/1, add/2\}$ and let $\mathcal{A}$ be the Σ-algebra given by $\mathbf{A} = \mathbb{Z}$, $b^{\mathcal{A}} = 0$, $suc^{\mathcal{A}}(n) = n + 1$, $pred^{\mathcal{A}}(n) = n - 1$ and $add^{\mathcal{A}}(n, m) = n + m$.
Let $\sim = \{(x, y) \mid 4 \text{ divides } x - y\}$.
We have $\mathbf{A}/_{\sim} = \{[0], [1], [2], [3]\}$, where

$$\begin{aligned} [0] &= \{\ldots, -4, 0, 4, 8, \ldots\} \\ [1] &= \{\ldots, -3, 1, 5, 9, \ldots\} \\ [2] &= \{\ldots, -6, -2, 2, 6, 10, \ldots\} \\ [3] &= \{\ldots, -5, -1, 3, 7, \ldots\} \end{aligned}$$

We have $b^{\mathcal{A}^{\sim}} = [0]$, $suc^{\mathcal{A}^{\sim}}([n]) = [n + 1]$, $pred^{\mathcal{A}^{\sim}}([n]) = [n - 1]$ and $add^{\mathcal{A}^{\sim}}([n], [m]) = [n + m]$.

$\mathcal{A}$-valuations vs $\mathcal{A}^{\sim}$-valuations

- Given a $\mathcal{A}$-valuation σ, we can always construct the $\mathcal{A}^{\sim}$-valuation τ given by $\tau(x) = [\sigma(x)]_{\sim}$. We usually write $\tau = \sigma_{\sim}$.
- Given a $\mathcal{A}^{\sim}$-valuation τ, we can always construct a $\mathcal{A}$-valuation σ s.t. $\tau(x) = [\sigma(x)]_{\sim}$. This means that we choose an element e in the equivalence class $\tau(x)$ and then we define $\sigma(x) = e$.

Homomorphism, endomorphism, isomorphism

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-algebras. A morphism is a function $\Phi : \mathcal{A} \to \mathcal{B}$ s.t. for all $n \geq 0$, for all $f/n \in \Sigma$ and for all $a_1, \ldots, a_n \in \mathbf{A}$ we have

$$\Phi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\Phi(a_1), \ldots, \Phi(a_n))$$

An endomorphism over a Σ-algebra $\mathcal{A}$ is a morphism from $\mathcal{A}$ to itself. An isomorphism is a bijective morphism.

Theorem

For every $\mathcal{A}$-valuation $\sigma : \mathcal{X} \to \mathcal{A}$, there is a unique morphism $\widehat{\sigma} : \mathcal{T}(\mathcal{X}, \Sigma) \to \mathcal{A}$, s.t.

$$\begin{array}{lcl} \widehat{\sigma}(x) & = & \sigma(x) \\ \widehat{\sigma}(f(t_1, \ldots, t_n)) & = & f^{\mathcal{A}}(\widehat{\sigma}(t_1), \ldots, \widehat{\sigma}(t_n)) \end{array}$$

$\widehat{\sigma}$ is the function that we have defined so far to interpret arbitrary terms in a Σ-algebra.

Exercise : Let $\Sigma = \{a/0, s/1, g/1, h/2\}$. Define a Σ-algebra $\mathcal{B}$ and a morphism Φ between the syntactic Σ-algebra and $\mathcal{B}$.

Let $\mathcal{B}$ be the Σ-algebra defined by:

- The domain $\mathbf{B} = \{n \in \mathbb{N} \mid n \geq 2\}$
- The interpretations:

$$\begin{array}{lcllcl} a^{\mathcal{B}} & = & 2 & g^{\mathcal{B}}(t) & = & t+1 \\ s^{\mathcal{B}}(t) & = & t & h^{\mathcal{B}}(u,t) & = & u+t \end{array}$$

Let Φ be the following function:

$$\begin{array}{lcllcl} \Phi(a) & = & 2 & \Phi(g(t)) & = & \Phi(t)+1 \\ \Phi(s(t)) & = & \Phi(t) & \Phi(h(u,t)) & = & \Phi(u)+\Phi(t) \end{array}$$

One verifies $\Phi(f^{\mathcal{A}}(t_1, \ldots, t_n)) = f^{\mathcal{B}}(\Phi(t_1), \ldots, \Phi(t_n))$ for every f/n in Σ.

Semantical equational reasoning

A Σ-equation is a pair of terms denoted $s \doteq t$.

Definition

Let $\mathcal{A}$ be a Σ-algebra.

- $\mathcal{A}$ is a model of an equation $s \doteq t$, written $\mathcal{A} \models s \doteq t$, iff the equality $\widehat{\sigma}(s) = \widehat{\sigma}(t)$ holds for every $\mathcal{A}$-valuation σ.
- $\mathcal{A}$ is a model of a set of Σ-equations $\mathcal{E}$, written $\mathcal{A} \models \mathcal{E}$, iff $\mathcal{A}$ is a model of every equation in $\mathcal{E}$.

Example

The first Σ-algebra on slide (Σ-algebras) is a model of the equation $p(x, y) \doteq p(y, x)$.

What about the second and the third?

Properties of semantical equational reasoning

- $\mathcal{A} \models s \doteq s$.
- If $\mathcal{A} \models s \doteq t$, then $\mathcal{A} \models t \doteq s$.
- If $\mathcal{A} \models s \doteq t$ and $\mathcal{A} \models t \doteq u$, then $\mathcal{A} \models s \doteq u$.
- If $\mathcal{A} \models s \doteq t$, then $\forall u\ \forall p \in Pos(u)$, $\mathcal{A} \models u[s]_p \doteq u[t]_p$.

Proof.

By induction on u. □

- If $\mathcal{A} \models s \doteq t$, then $\mathcal{A} \models \theta(s) \doteq \theta(t)$, for every substitution θ.

Proof.

We want to prove $\widehat{\sigma}(\theta(s)) = \widehat{\sigma}(\theta(t))$ for every $\mathcal{A}$-valuation σ, so that let us take an arbitrary $\mathcal{A}$-valuation σ. Let τ_σ be the $\mathcal{A}$-valuation given by $\tau_\sigma(x) = \widehat{\sigma}(\theta(x))$. We first show by induction on u that $\widehat{\tau_\sigma}(u) = \widehat{\sigma}(\theta(u))$ (easy). Now, $\mathcal{A} \models s \doteq t$ implies $\widehat{\tau}(s) = \widehat{\tau}(t)$ for every $\mathcal{A}$-valuation τ, so that in particular for τ_σ. Thus, $\widehat{\tau_\sigma}(s) = \widehat{\sigma}(\theta(s)) = \widehat{\sigma}(\theta(t)) = \widehat{\tau_\sigma}(t)$. □

Semantic equational consequence

Definition

The equation $s \doteq t$ is a semantic consequence of a set of equations $\mathcal{E}$, written $\mathcal{E} \models s \doteq t$, iff every model of $\mathcal{E}$ is also a model of $s \doteq t$ iff for every Σ-algebra $\mathcal{A}$, $\mathcal{A} \models \mathcal{E}$ implies $\mathcal{A} \models s \doteq t$.

Syntactic rules for equational reasoning

$$\frac{s \doteq t \in \mathcal{E}}{s \doteq t}\ \text{(axiom)} \qquad \frac{}{s \doteq s}\ \text{(reflexivity)}$$

$$\frac{s \doteq t}{t \doteq s}\ \text{(symmetry)} \qquad \frac{s \doteq t \quad t \doteq u}{s \doteq u}\ \text{(transitivity)}$$

$$\frac{s \doteq t}{\theta(s) \doteq \theta(t)}\ \text{(substitution)} \qquad \frac{s \doteq t}{u[s]_p \doteq u[t]_p}\ \text{(context)}$$

Derivation of $s \doteq t$ from the set $\mathcal{E}$ (which gives a tree) is denoted by $\mathcal{E} \vdash s \doteq t$.

Example

Let $\mathcal{E} = \{0 + z \doteq z, s(y) + x \doteq s(y + x)\}$. Let us use the notations $\underline{3} = s(s(s(0)))$ and $\underline{4} = s(s(s(s(0))))$.

We derive $s(0) + \underline{3} \doteq \underline{4}$ from $\mathcal{E}$ as follows:

$$\dfrac{\dfrac{\dfrac{s(y) + x \doteq s(y + x) \in \mathcal{E}}{s(y) + x \doteq s(y + x)}\ \text{(axiom)}}{s(0) + \underline{3} \doteq s(0 + \underline{3})}\ \text{(substitution)} \qquad \dfrac{\dfrac{\dfrac{0 + z \doteq z \in \mathcal{E}}{0 + z \doteq z}\ \text{(axiom)}}{0 + \underline{3} \doteq \underline{3}}\ \text{(substitution)}}{s(0 + \underline{3}) \doteq s(\underline{3})}\ \text{(context)}}{s(0) + \underline{3} \doteq s(\underline{3})}\ \text{(transitivity)}$$

The relation $\leftrightarrow_{\mathcal{E}}^*$

$$\frac{s \doteq t \in \mathcal{E}}{\theta(s) \leftrightarrow_{\mathcal{E}} \theta(t)} \qquad \frac{s \doteq t \in \mathcal{E}}{\theta(t) \leftrightarrow_{\mathcal{E}} \theta(s)} \qquad \frac{s \leftrightarrow_{\mathcal{E}} t}{u[s]_p \leftrightarrow_{\mathcal{E}} u[t]_p}$$

$\leftrightarrow_{\mathcal{E}}^*$ is the reflexive-transitive closure of $\leftrightarrow_{\mathcal{E}}$.

Exercise :

1. $\leftrightarrow_{\mathcal{E}}^*$ is stable by substitution.
2. $\leftrightarrow_{\mathcal{E}}^*$ is an equivalence relation.
3. $\leftrightarrow_{\mathcal{E}}^*$ is a congruence over $\mathcal{T}(\mathcal{X}, \Sigma)$.

Properties of $\leftrightarrow_{\mathcal{E}}^{*}$

Exercise : $\mathcal{E} \vdash s \doteq t$ if and only if $s \leftrightarrow_{\mathcal{E}}^{*} t$.

Proof.

Left-Right: by induction on derivation of $\mathcal{E} \vdash s \doteq t$.
Right-Left: by induction on the length of $\leftrightarrow_{\mathcal{E}}^{*}$.

□

Substitution Lemma

Lemma

Let σ be any $\mathcal{A}$-valuation, and let τ be the $\mathcal{A}^{\sim}$-valuation defined by $\tau : x \mapsto [\sigma(x)]_{\sim}$. Then $\widehat{\tau}(t) = [\widehat{\sigma}(t)]_{\sim}$ for every term t.

Proof.

By induction on t.

- For $t = x$ the property holds by definition.
- For $t = f(t_1, .., t_n)$, we have

$$\begin{array}{llll} \widehat{\tau}(f(t_1, .., t_n)) & = & f^{\mathcal{A}^{\sim}}(\widehat{\tau}(t_1), \ldots, \widehat{\tau}(t_n)) & =_{h.r} \\ f^{\mathcal{A}^{\sim}}([\widehat{\sigma}(t_1)]_{\sim}, \ldots, [\widehat{\sigma}(t_n)]_{\sim}) & =_{def} & [f^{\mathcal{A}}(\widehat{\sigma}(t_1), \ldots, \widehat{\sigma}(t_n))]_{\sim} & = \\ [\widehat{\sigma}(t)]_{\sim} & = & & \end{array}$$

□

A particular case of this Lemma is given when σ is a substitution and τ a $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\mathcal{E}}}$-valuation.
An even more particular case is when σ is the *id* substitution so that $\widehat{\tau}(t) = [t]_{\leftrightarrow^*_{\mathcal{E}}}$.

$\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$ as a model

Theorem

$\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$ as a model of $\mathcal{E}$.

Proof.

Take any equation $s \doteq t \in \mathcal{E}$. Take any $\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$-valuation τ. By previous remark we can construt a $\mathcal{T}(\mathcal{X},\Sigma)$-valuation σ (which is a substitution in this case) such that $\tau(x) = [\sigma(x)]_{\leftrightarrow^*_\mathcal{E}}$. The substitution lemma guarantees that $\widehat{\tau}(t) = [\widehat{\sigma}(t)]_{\leftrightarrow^*_\mathcal{E}}$ for every term t.
Now, $s \doteq t \in \mathcal{E}$ implies (by def)
$\mathcal{E} \vdash s \doteq t$ iff (previous exercise) $s \leftrightarrow^*_\mathcal{E} t$ implies ($\leftrightarrow^*_\mathcal{E}$ is stable by substitution)
$\widehat{\sigma}(s) \leftrightarrow^*_\mathcal{E} \widehat{\sigma}(t)$ iff (by def) $[\widehat{\sigma}(s)]_{\leftrightarrow^*_\mathcal{E}} = [\widehat{\sigma}(t)]_{\leftrightarrow^*_\mathcal{E}}$ iff (Subst. Lemma) $\widehat{\tau}(s) = \widehat{\tau}(t)$.
Since any $\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$-valuation τ satifies the equations $\mathcal{E}$: $\mathcal{T}(\mathcal{X},\Sigma)^{\leftrightarrow^*_\mathcal{E}}$ is a model of $\mathcal{E}$. □

Birkhoff's Theorem (1933)

Let $\mathcal{E}$ be a set of Σ-equations.

- **(Soundness)** If $\mathcal{E} \vdash s \doteq t$, then $\mathcal{E} \models s \doteq t$.

Proof.

By induction on $\mathcal{E} \vdash s \doteq t$ using the properties of semantical equational reasoning. □

- **(Completeness)** If $\mathcal{E} \models s \doteq t$, then $\mathcal{E} \vdash s \doteq t$.

Proof.

If $\mathcal{E} \models s \doteq t$, then every model of $\mathcal{E}$ is a a model of $s \doteq t$. Since $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\mathcal{E}}}$ is a model of $\mathcal{E}$ (previous theorem), then in particular $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\mathcal{E}}} \models s \doteq t$.
This means that $\widehat{\tau}(s) = \widehat{\tau}(t)$ for every $\mathcal{T}(\mathcal{X}, \Sigma)^{\leftrightarrow^*_{\mathcal{E}}}$-valuation τ, so in particular for $\tau_{id} : x \mapsto [id(x)]_{\leftrightarrow^*_{\mathcal{E}}}$. Thus, $\widehat{\tau_{id}}(s) = \widehat{\tau_{id}}(t)$.
By the Substitution Lemma $\widehat{\tau_{id}}(s) = [\widehat{id}(s)]_{\leftrightarrow^*_{\mathcal{E}}}$ and $\widehat{\tau_{id}}(t) = [\widehat{id}(t)]_{\leftrightarrow^*_{\mathcal{E}}}$, thus $[\widehat{id}(s)]_{\leftrightarrow^*_{\mathcal{E}}} = [s]_{\leftrightarrow^*_{\mathcal{E}}} = [t]_{\leftrightarrow^*_{\mathcal{E}}} = [\widehat{id}(t)]_{\leftrightarrow^*_{\mathcal{E}}}$ which means $s \leftrightarrow^*_{\mathcal{E}} t$. We conclude $\mathcal{E} \vdash s \doteq t$.

□