First-Order Rewriting

Rewriting System

A rewriting rule is a pair of terms $l \mapsto r$ such that l is not a variable and $Var(r) \subseteq Var(l)$.
A rewriting system is a set of rewriting rules.
Given a rewriting system $\mathcal{R}$, the rewriting relation $\rightarrow_{\mathcal{R}}$ is generated by the following rules:

$$\frac{l \mapsto r \in \mathcal{R}}{\theta(l) \rightarrow_{\mathcal{R}} \theta(r)} \qquad \frac{s \rightarrow_{\mathcal{R}} t}{u[s]_p \rightarrow_{\mathcal{R}} u[t]_p}$$

The notation $\rightarrow^{+}_{\mathcal{R}}$ is used for the transitive closure of $\rightarrow_{\mathcal{R}}$ and $\rightarrow^{*}_{\mathcal{R}}$ for the reflexive-transitive closure of $\rightarrow_{\mathcal{R}}$.

Remark This definition implies that if $t \rightarrow_{\mathcal{R}} u$, then there is a rewriting rule $l \mapsto r \in \mathcal{R}$, a substitution σ, a term v, and a position $p \in Pos(v)$ such that $t = v[\sigma(l)]_p$ and $u = v[\sigma(r)]_p$.

Example

Rewriting system for Peano arithmetic:

$$\begin{array}{lcl} 0 + y & \mapsto & y \\ s(x) + y & \mapsto & s(x + y) \\ 0 * y & \mapsto & 0 \\ s(x) * y & \mapsto & (x * y) + y \end{array}$$

Reduction sequence:

$$\begin{array}{ll} \underline{2} * \underline{3} = \underline{s(s(0)) * s(s(s(0)))} & \to \\ [\underline{s(0) * s(s(s(0)))}] + s(s(s(0))) & \to \\ [[\underline{0 * s(s(s(0)))}] + s(s(s(0)))] + s(s(s(0))) & \to \\ [\underline{0 + s(s(s(0)))}] + s(s(s(0))) & \to \\ \underline{s(s(s(0))) + s(s(s(0)))} & \to \\ s(\underline{s(s(0)) + s(s(s(0)))}) & \to \\ s(s(\underline{s(0) + s(s(s(0)))})) & \to \\ s(s(s(\underline{0 + s(s(s(0)))}))) & \to \\ s(s(s(s(s(s(0)))))) = \underline{6} & \end{array}$$

Confluence

Confluence is an undecidable property.

Decision procedures

- Sound methods
- Terminating methods

Some techniques to show confluence

- Confluence by strong confluence
- Confluence by diamond property
- Confluence by equivalence
- Confluence by commutation
- Confluence by interpretation
- Confluence by critical pairs
- Confluence by orthogonality
- Confluence by decreasing diagrams

Confluence by strong confluence

Recall that $\mathcal{R}$ is strongly confluent (SC) iff

$$\begin{array}{ccc} s & \rightarrow_{\mathcal{R}} & t \\ \downarrow_{\mathcal{R}} & & \downarrow_{*\mathcal{R}} \\ u & \rightarrow^{=}_{\mathcal{R}} & v \end{array}$$

Example

$$\mathcal{R} = \left\{ \begin{array}{lcl} f(x,x) & \mapsto & g(x) \\ f(x,y) & \mapsto & g(y) \\ g(x) & \mapsto & f(x,a) \end{array} \right.$$

We only check one-step diagrams.

Theorem

If $\mathcal{R}$ is strongly confluent, then $\mathcal{R}$ is confluent.

Proof.

Let $s \to^n t$ and $s \to^m u$. We reason by induction on $\langle n, m \rangle$ using the lexicographic order.

- If $n = 0$ or $m = 0$, then the property is trivial.
- If $n > 0$ and $m > 0$, we have $s \to^1 t' \to^{n-1} t$ and $s \to^1 u' \to^{m-1} u$.
  By using the hypothesis we have $t' \to^* v$ et $u' \to^= v$. We thus consider $u' \to^= v$ and $u' \to^{m-1} u$, we remark that $\langle 1/0, m-1 \rangle <_{lex} \langle n, m \rangle$, and thus by the i.h. there is v' such that $v \to^* v'$ et $u \to^* v'$.
  Now, we consider $t' \to^{n-1} t$ and $t' \to^* v \to^* v'$, we remark that $\langle n-1, k \rangle <_{lex} \langle n, m \rangle$ for any k, and thus by the i.h. t and v' are joinable, and this close the whole diagram.

□

Confluence by diamond property

Recall that $\mathcal{R}$ has the diamond property (DP) iff

$$\begin{array}{ccc} s & \rightarrow_{\mathcal{R}} & t \\ \downarrow_{\mathcal{R}} & & \downarrow_{\mathcal{R}} \\ u & \rightarrow_{\mathcal{R}} & v \end{array}$$

This is a particular case of strongly confluence.

Corollary

If $\mathcal{R}$ has the diamond property, then $\mathcal{R}$ is confluent.

Confluence by equivalence

Theorem

Let $\mathcal{R}$ and $\mathcal{S}$ two rewrite systems such that $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{S}} \subseteq \rightarrow^*_{\mathcal{R}}$ and $\mathcal{S}$ is strongly confluent. Then $\mathcal{R}$ is confluent.

Proof.

- If $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{S}} \subseteq \rightarrow^*_{\mathcal{R}}$, then $\rightarrow^*_{\mathcal{R}} = \rightarrow^*_{\mathcal{S}}$.
- If $\mathcal{S}$ is strongly confluent, then $\mathcal{S}$ is confluent.
- If $\rightarrow^*_{\mathcal{R}} = \rightarrow^*_{\mathcal{S}}$, then $\mathcal{R}$ is confluent iff $\mathcal{S}$ is confluent.

□

(Famous) Example: confluence of β

Show that $(\lambda x.t)u \mapsto_\beta t\{x\backslash u\}$ is confluent.

Define a relation $\gg$ as follows:

$$\frac{}{x \gg x} \qquad \frac{t \gg t'}{\lambda x.t \gg \lambda x.t'}$$

$$\frac{t \gg t' \text{ and } u \gg u'}{t\ u \gg t'\ u'} \qquad \frac{t \gg t' \text{ and } u \gg u'}{(\lambda x.t)\ u \gg t'\{x\backslash u'\}}$$

Let $\mathcal{R} = \beta$ and let $\mathcal{S} = \gg$. Now,

1. Show that $\beta \subseteq \gg \subseteq \beta^*$.
2. Show that $\gg$ is strongly confluent.
3. Conclude that β is confluent (on all the terms).

Confluence by commutation [Hindley-Rosen]

Two systems $\mathcal{R}$ and $\mathcal{S}$ are said to commute iff

$$\begin{array}{ccc} s & \rightarrow^*_{\mathcal{R}} & t \\ \downarrow_{*\mathcal{S}} & & \downarrow_{*\mathcal{S}} \\ u & \rightarrow^*_{\mathcal{R}} & v \end{array}$$

Two systems $\mathcal{R}$ and $\mathcal{S}$ are said to strongly commute iff

$$\begin{array}{ccc} s & \rightarrow_{\mathcal{R}} & t \\ \downarrow_{\mathcal{S}} & & \downarrow_{*\mathcal{S}} \\ u & \rightarrow^=_{\mathcal{R}} & v \end{array}$$

Strong Commutation

Theorem

If $\mathcal{R}$ and $\mathcal{S}$ strongly commute, then they commute.

Proof.

Let $s \to_{\mathcal{R}}^{n} t$ and $s \to_{\mathcal{S}}^{m} u$. We reason by induction on $\langle n, m \rangle$ using the lexicographic order.

- If $n = 0$ or $m = 0$, then the property is trivial.
- If $n > 0$ and $m > 0$, then $s \to_{\mathcal{R}}^{1} t' \to_{\mathcal{R}}^{n-1} t$. $s \to_{\mathcal{S}}^{1} u' \to_{\mathcal{S}}^{m-1} u$.
  By the hypothesis $u' \to_{\mathcal{R}}^{=} v$ and $t' \to_{\mathcal{S}}^{*} v$.
  Now, we consider $u' \to_{\mathcal{R}}^{=} v$ and $u' \to_{\mathcal{S}}^{m-1} u$, we remark that $\langle 0/1, m-1 \rangle <_{lex} \langle n, m \rangle$, so by the i.h. $u \to_{\mathcal{R}}^{*} v'$ and $v \to_{\mathcal{S}}^{*} v'$.
  Similarly, $t' \to_{\mathcal{R}}^{n-1} t$ and $t' \to_{\mathcal{S}}^{*} v'$, we remark that $\langle n-1, k \rangle <_{lex} \langle n, m \rangle$ for any k, and then the diagram closes by the i.h.

□

Theorem

If $\mathcal{R}$ and $\mathcal{S}$ are both confluent and commute, then $\mathcal{R} \cup \mathcal{S}$ is confluent.

Proof.

Let $s \to^*_{\mathcal{R}\cup\mathcal{S}} t$ and $s \to^*_{\mathcal{R}\cup\mathcal{S}} u$. We decompose both sequences as $s \to^*_{\mathcal{R}} \to^*_{\mathcal{S}} \ldots \to^*_{\mathcal{R}} \to^*_{\mathcal{S}} t$ and $s \to^*_{\mathcal{R}} \to^*_{\mathcal{S}} \ldots \to^*_{\mathcal{R}} \to^*_{\mathcal{S}} u$.
Intuitively, the steps with $\mathcal{R}$ are closed by confluence of $\mathcal{R}$, and the steps with $\mathcal{S}$ are closed by confluence of $\mathcal{S}$. The other ones by commutation.
Formally,

- We show $\to_{\mathcal{R}\cup\mathcal{S}} \subseteq \to^*_{\mathcal{R}} \to^*_{\mathcal{S}} \subseteq \to^*_{\mathcal{R}\cup\mathcal{S}}$
- We remark that $\to^*_{\mathcal{R}} \to^*_{\mathcal{S}}$ is strongly confluent.
- We apply confluence by equivalence.

□

(Famous) Example: confluence of $\beta\eta$

Example

$$\begin{array}{lll} (\lambda x.t)u & \mapsto_\beta & t\{x\backslash u\} \\ \lambda x.t\ x & \mapsto_\eta & t \qquad \text{If } x \notin \mathtt{fv}(t) \end{array}$$

Let $\mathcal{R} = \beta$ and $\mathcal{S} = \eta$. Now,

- Show that β is confluent (done).
- Show that η is confluent.
- Show that β and η strongly commute.
- Conclude that $\beta \cup \eta$ is confluent.

Confluence by interpretation

Theorem

Let $\mathcal{R}$ and $\mathcal{S}$ be two relations s.t. $\mathcal{R}$ is confluent and terminating. If there is a relation $\mathcal{T}$ on the set of $\mathcal{R}$-normal forms s.t.

1. $\rightarrow_{\mathcal{T}}^{*} \subseteq \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*}$ *and*
2. $a \rightarrow_{\mathcal{S}} b$ *implies* $\mathcal{R}(a) \rightarrow_{\mathcal{T}}^{*} \mathcal{R}(b)$
3. $\mathcal{T}$ *is confluent*

Then $\mathcal{R} \cup \mathcal{S}$ is confluent.

Proof.

Let $s \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*} t$ and $s \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*} u$.
Since $\mathcal{R}$ is confluent and terminating we can compute $\mathcal{R}$-normal forms $s \rightarrow_{\mathcal{R}}^{*} \mathcal{R}(s)$, $t \rightarrow_{\mathcal{R}}^{*} \mathcal{R}(t)$, and $u \rightarrow_{\mathcal{R}}^{*} \mathcal{R}(u)$.
We can then simulate the first two $\rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*}$-sequences by $\mathcal{R}(s) \rightarrow_{\mathcal{T}}^{*} \mathcal{R}(t)$ and $\mathcal{R}(s) \rightarrow_{\mathcal{T}}^{*} \mathcal{R}(u)$ using point 2.
We can close this divergence by $\mathcal{R}(t) \rightarrow_{\mathcal{T}}^{*} v$ and $\mathcal{R}(u) \rightarrow_{\mathcal{T}}^{*} v$ using point 3.
We obtain $\mathcal{R}(t) \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*} v$ and $\mathcal{R}(u) \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*} v$ using point 1.
We thus conclude since $t \rightarrow_{\mathcal{R}}^{*} \mathcal{R}(t) \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*} v$ and $u \rightarrow_{\mathcal{R}}^{*} \mathcal{R}(u) \rightarrow_{\mathcal{R}\cup\mathcal{S}}^{*} v$. □

(Famous) Example: confluence of $\lambda\mathtt{x}$

$$
\begin{array}{llll}
(\lambda x.t)\ u & \mapsto_B & t[x/u] & \\
(t\ u)[x/v] & \mapsto_{\mathtt{x}} & (t[x/v]\ u[x/v]) & \\
(\lambda y.t)[x/v] & \mapsto_{\mathtt{x}} & \lambda y.t[x/v] & \text{if } x \neq y \,\&\, y \notin \mathtt{fv}(v) \\
y[x/v] & \mapsto_{\mathtt{x}} & y & \text{if } x \neq y \\
x[x/v] & \mapsto_{\mathtt{x}} & v &
\end{array}
$$

Let $\mathcal{R} = \mathtt{x}$ and $\mathcal{S} = B$ and $\mathcal{T} = \beta$, Now,

- Show that $\mathtt{x}$ is confluent and terminating.
- Show that $\beta \subseteq (\mathtt{x} \cup B)^*$.
- Show that $a \to_B b$ implies $\mathtt{x}(a) \to_\beta^* \mathtt{x}(b)$.
- Since β is confluent $\lambda\mathtt{x} = \mathtt{x} \cup B$ is confluent.

Confluence by critical pairs

Lemma

(Newmann) *Let $\mathcal{R}$ be a SN system. Then $\mathcal{R}$ is locally confluent iff $\mathcal{R}$ is confluent.*

Proof.

(By Huet) By well-founded induction on $s \in SN$.

$$
\begin{array}{ccccc}
s & \to & t' & \to^* & t \\
\downarrow & & \downarrow_* & & \downarrow_* \\
u' & \to^* & v & \text{i.h. since } t' < s & \downarrow_* \\
\downarrow_* & \text{i.h. since } u' < s & \downarrow_* & & \downarrow_* \\
u & \to^* & p & \to^* & p'
\end{array}
$$

□

Important remark

The following (infinite) system on natural numbers:

$$\mathcal{R} = \begin{cases} 2.n & \mapsto & 2.n+1 \\ 2.n & \mapsto & a \\ 2.m+1 & \mapsto & 2.m+2 \\ 2.m+1 & \mapsto & b \end{cases}$$

is locally confluent but not confluent: $a \leftarrow 0 \rightarrow^* b$
In fact it is not SN

$$0 \rightarrow \; 1 \rightarrow \; 2 \rightarrow \; 3 \rightarrow \; \ldots$$

Towards local confluence: critical pairs

A critical pair between two variable disjoint rules $l \mapsto r$ and $g \mapsto d$ of $\mathcal{R}$ (not necessarily distinct rules) is a pair $\langle \sigma(r), \sigma(l)[\sigma(d)]_p \rangle$ s.t.

1. $p \in Pos(l)$ and $l|_p$ is not a variable.
2. σ is a principal unifier of $l|_p$ and g.
3. If $l \mapsto r$ and $g \mapsto d$ are the same rule, then $p \neq \Lambda$.

Observe that

$$\begin{aligned} \sigma(r) \leftarrow \sigma(l) \quad &= \sigma(l)[\sigma(l)|_p]_p \\ &= \sigma(l)[\sigma(l|_p)]_p \\ &= \sigma(l)[\sigma(g)]_p \rightarrow \sigma(l)[\sigma(d)]_p \end{aligned}$$

Example

$$\mathcal{R} = \begin{cases} f(g(x), g(y), a) & \mapsto_{r1} \quad j(x, x, a) \\ g(b) & \mapsto_{r2} \quad b \\ h(b) & \mapsto_{r3} \quad b \end{cases}$$

The critical pairs are in red:

$$j(b, b, a) \;{}_{r1}\!\leftarrow \underline{f(\underline{g(b)}, g(y), a)} \rightarrow_{r2} f(b, g(y), a) \qquad (r1, r2, pos = 1)$$

$$j(x, x, a) \;{}_{r1}\!\leftarrow \underline{f(g(x), \underline{g(b)}, a)} \rightarrow_{r2} f(g(x), b, a) \qquad (r2, r2, pos = 2)$$

Putting the information in a table:

	$r1$	$r2$	$r3$
$r1$	X	$positions = 1, 2$	X
$r2$		X	X
$r3$			X

Example

$$\mathcal{R} = \begin{cases} f(f(x)) & \mapsto_{r1} & g(x) \\ f(b) & \mapsto_{r2} & c \\ b & \mapsto_{r3} & a \end{cases}$$

The critical pairs are in red:

$$g(f(x)) \quad {}_{r1}\!\leftarrow \quad \underline{f(\underline{f(f(x))})} \quad \rightarrow_{r1} \quad f(g(x)) \quad (r1, r1, pos = 1)$$

$$g(b) \quad {}_{r1}\!\leftarrow \quad \underline{f(\underline{f(b)})} \quad \rightarrow_{r2} \quad f(c) \quad (r1, r2, pos = 1)$$

$$c \quad {}_{r2}\!\leftarrow \quad \underline{f(\underline{b})} \quad \rightarrow_{r3} \quad f(a) \quad (r2, r3, pos = 1)$$

Putting the information in a table:

	$r1$	$r2$	$r3$
$r1$	$positions = 1$	$positions = 1$	X
$r2$		X	$positions = 1$
$r3$			X

Local confluence by critical pairs

Theorem

Let $\mathcal{R}$ be a rewrite system. Then $\mathcal{R}$ is locally confluent iff every critical pair of $\mathcal{R}$ is joinable.

Proof.

The **only if** implication is trivial.
For the **if** implication, let us take any case of the form

$$v \leftarrow t \rightarrow u$$

Three cases are possible:

- Disjoint reductions:

$$\begin{array}{ccccc} f(b,h(b),a) & \leftarrow & f(\underline{g(b)},\underline{h(b)},a) & \rightarrow & f(g(b),b,a) \\ f(b,\underline{h(b)},a) & \rightarrow & f(b,b,a) & \leftarrow & f(\underline{g(b)},b,a) \end{array}$$

- Not disjoint and not critical:

$$\begin{array}{ccccc} j(\underline{h(b)},h(b),a) & \leftarrow & \underline{f(g(\underline{h(b)}),g(b),a)} & \rightarrow & f(\underline{g(b)},g(b),a) \\ \downarrow & & & & \downarrow \\ j(b,\underline{h(b)},a) & \rightarrow & & & j(b,b,a) \end{array}$$

- Not disjoint and critical: we close the diagram by the hypothesis.

Decidable case of confluence

Theorem

Let $\mathcal{R}$ be a finite and SN rewrite system. Then confluence of $\mathcal{R}$ is decidable.

Proof.

The algorithm:

1. Generate all the critical pairs.
2. For each critical pair $\langle u, v \rangle$, if there is some normal form $\widehat{u}$ of u and some normal form $\widehat{v}$ of v such that $\widehat{u} \neq \widehat{v}$, then fail.
3. Otherwise (no fail for some critical pair), succeed.

□

Remark that

- If the algorithm fails, then there is a critical pair which is not joinable, so $\mathcal{R}$ is not confluent by the previous theorem.
- If the algorithm succeeds, then every critical pair is joinable, so that $\mathcal{R}$ is locally confluent by the previous theorem. To obtain confluence, apply Newmann's Lemma.

Orthogonal Systems

A system is orthogonal iff it is left linear (no duplication of variables on the left of rules) and has no critical pairs.

Example

$$\begin{array}{lcl} 0 + y & \mapsto & y \\ s(x) + y & \mapsto & s(x + y) \\ 0 * y & \mapsto & 0 \\ s(x) * y & \mapsto & (x * y) + y \end{array}$$

Confluence by orthogonality

Theorem

If $\mathcal{R}$ is orthogonal, then it is confluent.

Proof.

We define a notion of parallel reduction associated to $\mathcal{R}$ for first-order terms:

$$\frac{}{s \gg_{\mathcal{R}} s}\ (R) \qquad \frac{l \mapsto r \in \mathcal{R} \text{ and } \sigma \text{ a subst.}}{\sigma(l) \gg_{\mathcal{R}} \sigma(r)}\ (S) \qquad \frac{s_1 \gg_{\mathcal{R}} t_1 \ \ldots\ldots\ s_n \gg_{\mathcal{R}} t_n}{f(s_1, \ldots, s_n) \gg_{\mathcal{R}} f(t_1, \ldots, t_n)}\ (C)$$

Example:

For $\mathcal{R} = \{f(x, y) \mapsto h(y, y), a \mapsto b\}$, we have $f(f(a, c), a, f(a, c)) \gg_{\mathcal{R}} f(h(c, c), b, f(b, c))$

Now use the confluence by equivalence technique:

1. Observe that $\rightarrow_{\mathcal{R}} \ \subseteq \ \gg_{\mathcal{R}} \ \subseteq \ \rightarrow^*_{\mathcal{R}}$.
2. Show that $\gg_{\mathcal{R}}$ has the diamond property.
3. Conclude that $\rightarrow_{\mathcal{R}}$ is confluent.

□

Obs: from now on, we simply write $\gg$ insted of $\gg_{\mathcal{R}}$ is $\mathcal{R}$ is clear from the context.

The relation $\gg$ has the diamond property

Define $\sigma \gg \delta$ for two substitutions iff $\mathtt{dom}(\sigma) = \mathtt{dom}(\delta)$ and $\sigma x \gg \delta x$ for every $x \in \mathtt{dom}(\sigma)$.

Lemma (A)

If $\sigma \gg \delta$, then $\sigma t \gg \delta t$ for every term t.

Proof.

By induction on t. □

Lemma (B)

Let $\mathcal{R}$ be an orthogonal rewriting system. Let σ be a substitution. Let s be a strict subterm of l, where $l \mapsto r \in \mathcal{R}$. If $\sigma s \gg t$, then there is δ s.t. $t = \delta s$ and $\sigma \gg \delta$.

The proof is by induction on s.

- If $s = x$, then define $\delta x = t$ and $\delta y = \sigma y$ for $y \neq x$.
- If $s = f(s_1, \ldots, s_n)$, we distinguish three cases according to the definition of $\sigma s \gg t$.
  - If $\sigma s \gg t$ holds by (R), then $t = \sigma s$ and thus we simply let $\delta := \sigma$.
  - If $\sigma s \gg t$ holds by (S), then there is some rule $g \mapsto d \in \mathcal{R}$ and some substitution τ such that $\sigma s = \tau g$ and $t = \tau d$. We can assume $l \mapsto r$ and $g \mapsto d$ do not share variables and thus $(\sigma \cup \tau)s = \sigma s = \tau g = (\sigma \cup \tau)g$. Then $\sigma \cup \tau$ unifies s (a strict subterm of l) and g, thus contradicting absence of critical pairs, and thus orthogonality of $\mathcal{R}$.
  - If $\sigma s \gg t$ holds by (C), then $t = f(t_1, \ldots, t_n)$ and $\sigma s_i \gg t_i$. Since l (and thus s) are linear, we can write σ as $\bigcup \sigma_i$, where each σ_i is σ restricted to the variables of s_i. We then have $\sigma_i s_i \gg t_i$ and s_i is a strict subterm of l. The i.h. gives $t_i = \delta_i s_i$ and $\sigma_i \gg \delta_i$. We take $\delta := \bigcup \delta_i$. It is easy to check $\sigma \gg \delta$ and $t = \delta s$.

Theorem

Let $\mathcal{R}$ be an orthogonal system and $\gg$ its associated parallel reduction relation. The reduction relation $\gg$ has the diamond property.

Proof.

Supose $t \gg t_1$ and $t \gg t_2$. We reason by cases.

- If one of them uses (R), then we trivially close the diagram.
- If both use (S), then $t = \sigma l$, $t_1 = \sigma r$, $t = \delta g$ and $t_2 = \delta d$ for $l \mapsto r$ and $g \mapsto d$ in $\mathcal{R}$. If the two rules are the same, then $\sigma = \delta$ and we trivially close the diagram. Otherwise, we can assume that the rules do not share variables so that$(\delta \cup \sigma)l = \sigma l = t = \delta g = (\sigma \cup \delta)g$, *i.e.* $(\delta \cup \sigma)$ unifies l and g, which contradicts absence of critical pairs, and thus orthogonality of $\mathcal{R}$.
- If both use (C), the property trivially holds by the i.h.
- Suppose one uses (C) and the other one uses (S). We have $t = f(v_1, \ldots, v_n) \gg f(u_1, \ldots, u_n) = t_1$ where $v_i \gg u_i$ and $t = \sigma l \gg \sigma r = t_2$ for some rule $l \mapsto r \in \mathcal{R}$ and some substitution σ. The left-hand side l is necessarily of the form $f(l_1, \ldots, l_n)$ so that $t = f(v_1, \ldots, v_n) = f(\sigma l_1, \ldots, \sigma l_n)$ and $\sigma l_i \gg u_i$. But l is linear so that we can write $\sigma = \bigcup_1^n \sigma_i$ where $\sigma_i l_i \gg u_i$. Each term l_i is a strict subterm of l so that by Lemma (B) there are substitutions δ_i such that $u_i = \delta_i l_i$ and $\sigma_i \gg \delta_i$. We let $\delta := \bigcup_1^n \delta_i$. Then $t_1 = f(\delta_1 l_1, \ldots, \delta_n l_n) = \delta l$ and also $\sigma \gg \delta$. We can now close the diagram with $t_3 = \delta r$ as follows
  - $t_1 = \delta l \gg \delta r = t_3$ holds using the (S) rule.
  - $t_2 = \sigma r \gg \delta r = t_3$ holds by Lemma (A) using the fact that $\sigma \gg \delta$.

□

Important remark (I)

Left linearity alone is not sufficient for confluence.

Example

$$
\begin{array}{ccc} a \mapsto b & b \mapsto a & c \mapsto a \\ a \mapsto c & b \mapsto d & c \mapsto e \end{array}
$$

Two not joinable terms:

$$e \; {}^{*}\!\!\leftarrow a \rightarrow^{*} \; d$$

Important remark (II)

Absence of critical pairs is not sufficient for confluence.

Example

$$\begin{array}{lcl} f(x,x) & \mapsto & a \\ f(x,g(x)) & \mapsto & b \\ c & \mapsto & g(c) \end{array}$$

Two not joinable terms:

$$b \ {}^{*}\!\leftarrow f(c,c) \rightarrow^{*} a$$

Relaxing orthogonality

$$\begin{array}{lcl} \texttt{por}(\texttt{t}, x) & \mapsto & \texttt{t} \\ \texttt{por}(x, \texttt{t}) & \mapsto & \texttt{t} \end{array}$$

This system is not orthogonal but the critical pair is trivial.

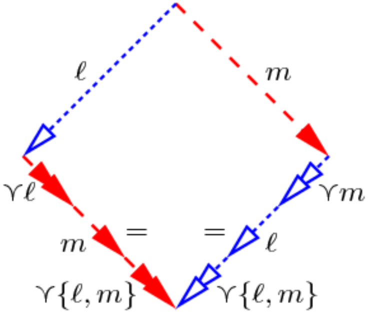
A system $\mathcal{R}$ is parallel closed iff for every critical pair $\langle u, v \rangle$ of $\mathcal{R}$ we have $v \gg u$.

Theorem

If $\mathcal{R}$ is left-linear and parallel closed then it is confluent.

Confluence by decreasing diagrams

Vincent van Oostrom (1994)
Idea: every reduction step is labelled



where $t \to_{\curlyvee \mathcal{S}} u$ means $t \to_s u$ for some $s < \mathcal{S}$, *i.e.* $\exists i \in \mathcal{S}$ such that $s < i$.
As a consequence, $t \to^*_{\curlyvee l} u$ means that every step from t to u is labelled with some $s' < l$, and $t \to^*_{\curlyvee \{l,m\}} u$ means that every step from t to u is labelled with some $s' < \mathtt{max}(l, m)$ (if l and m are comparable).