Realizability algebras and new models of ZF

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Introduction : classical realizability

- It is a method to get programs from mathematical proofs

by extending the proof-program correspondence up to classical set theory.
The transition from intuitionistic to classical logic is due to Griffin's discovery
that a *control instruction* is typed with the law of Peirce (1990).

- It is also a new technique to build models of ZF

and to obtain relative consistency results.

Until now, only two such methods are known (thus, a third one is welcome)

  - Inner models (particularly the model of *constructible sets*) :
    the model is a *subclass* of the ground model.
  - Forcing : the model is an *extension* of the ground model ;
    the axiom of choice is maintained.

In both cases, ordinals are not changed.

Introduction

A classical realizability model is neither a subclass nor an extension of the ground model. The ordinals and even *the integers* are changed. The axiom of choice *is not* preserved, only dependent choice may be.

The main tools are :

- Realizability algebra

a three-sorted variant of the well known combinatory algebra.

- ZF_ε set theory

a conservative extension of ZF ;

ε is a strong membership relation which lacks extensionality.

Introduction

We prove relative consistency results not obtained by previous methods :

ZF + DC (dependent choice) +

- there exists a sequence of infinite subsets of $\mathbb{R}$ with strictly decreasing cardinals ;
- there exists a sequence $X_n (n \geq 2)$ of infinite subsets of $\mathbb{R}$ with strictly increasing cardinals such that $X_m \times X_n$ is equipotent with X_{mn} ;

Each proposition implies (trivially) that $\mathbb{R}$ is not well ordered.

Remarks.

- It is the *simplest possible realizability model* which has such a strange $\mathbb{R}$.
- A new proof of the independence of the well-ordering axiom.

Classical realizability : an extension of forcing

More precisely, forcing is a *degenerate case* of classical realizability.

The generalization is about *the set of conditions*

which is always a first order structure with a binary operation :

- In the case of forcing, it is a *commutative idempotent monoid*

with an identity $\mathbf{I}$; in other words, a meet-semilattice with a greatest element.

The axioms are : $xy = yx$; $x \bullet yz = xy \bullet z$; $xx = x$; $\mathbf{I}x = x$.

Moreover, we have an *ideal* (initial segment) which is the set of *false conditions*.

Usually, these false conditions are removed.

Then, we get a practically *arbitrary ordered set*

(any ordered set in which two compatible elements have a g.l.b.).

An extension of forcing and combinatory algebra

• In the general case of realizability, we have again a first order structure but with three types ; I call it a *realizability algebra*.

The commutative idempotent monoids of forcing are a simple particular case which is in no way representative (far too degenerate).

Another well known interesting case is the *combinatory algebra* of Curry.

It is only an approximation of a realizability algebra, but is much more representative.

A binary operation with two constants K and S, called *combinators*.

The axioms are : $\mathsf{K}x \bullet y = x$; $\mathsf{S}xy \bullet z = xz \bullet yz$.

Combinatory algebra is very interesting because of its close connection with *λ-calculus* and therefore with *intuitionistic propositional logic*, by the proof-program (a.k.a. *Curry-Howard*) correspondence.

Realizability, forcing and combinatory algebra

We want to extend *intuitionistic propositional logic (IPL)* up to *classical set theory* !
To do this, we need to add some *axioms* to IPL, and therefore,
by the proof-program correspondence, some *constants* to the algebra.

- If the algebra is commutative, the only possible constant is **I**.

Then, there is no problem, we can add all the axioms we need at one go
without changing the algebra ; it is the case of *forcing*.

- In the general case, for some axioms, we need to add new constants,

and even new sorts, to the first order structure.
These problematic axioms are the *excluded middle* and the *dependent choice*.

The *general axiom of choice* is much more difficult to handle
than dependent choice ; it will not be considered in these talks.

The excluded middle

It is, far and away, the toughest axiom.

The solution was not (it could not be !) found by a logician,
but by a computer scientist, *Timothy Griffin*, in 1990.

The constant associated with the law of Peirce is a sophisticated instruction
which can *save and restore the context* (or environment).

This is a *major discovery*, of the same importance, at least,
as the Gödel incompleteness theorem.

We now need a first order language with two sorts
in order to speak about *programs* and *environments*.

We also need to consider the dynamics (execution)
hence a third sort for *processes*.

Realizability algebra

It is a first order structure, which consists of :

- *Three sets* :

  Λ the set of *terms*, Π the set of *stacks* and $\Lambda \star \Pi$ the set of *processes*.
- *Six distinguished terms* : B, C, I, K, W, cc (*elementary combinators*) ;

  they are not necessarily distinct.
- *Four operations* :

  Application : $\Lambda \times \Lambda \to \Lambda$ denoted $(\xi)\eta$ (or often $\xi\eta$)

  Push : $\Lambda \times \Pi \to \Pi$ denoted $\xi . \pi$

  Continuation : $\Pi \to \Lambda$ denoted k_π

  Process : $\Lambda \times \Pi \to \Lambda \star \Pi$ denoted $\xi \star \pi$

  (ξ, η are arbitrary terms and π is an arbitrary stack)
- *A preorder on processes*, denoted $\succ$ (*execution*)
- *A distinguished subset* $\perp\!\!\!\perp$ of $\Lambda \star \Pi$

Axioms of realizability algebra

- The preorder $\succ$ is such that :

  $(\xi)\eta \star \pi \succ \xi \star \eta . \pi$

  $\mathsf{I} \star \xi . \pi \succ \xi \star \pi$

  $\mathsf{K} \star \xi . \eta . \pi \succ \xi \star \pi$

  $\mathsf{W} \star \xi . \eta . \pi \succ \xi \star \eta . \eta . \pi$

  $\mathsf{C} \star \xi . \eta . \zeta . \pi \succ \xi \star \zeta . \eta . \pi$

  $\mathsf{B} \star \xi . \eta . \zeta . \pi \succ \xi \star (\eta)\zeta \star \pi$

  $\mathsf{cc} \star \xi . \pi \succ \xi \star \mathsf{k}_\pi . \pi$

  $\mathsf{k}_\pi \star \xi . \varpi \succ \xi \star \pi$

- The set $\perp\!\!\!\perp$ of processes is a *terminal segment* of $\Lambda \times \Pi$ i.e. :

  $\xi \star \pi \in \perp\!\!\!\perp, \xi' \star \pi' \succ \xi \star \pi \Rightarrow \xi' \star \pi' \in \perp\!\!\!\perp.$

If $\perp\!\!\!\perp = \emptyset$, the realizability algebra is called *trivial*.

c-terms and λ-terms

A *c-term* is a term of the language of realizability algebras
built with variables $x, y, \ldots,$ elementary combinators and application.
A closed c-term is called *proof-like*. It has a value in Λ.
Examples : *integers* in combinatory logic.
$\sigma = (\mathsf{BW})(\mathsf{B})\mathsf{B}$ (the *successor*) ; $\underline{0} = \mathsf{KI}$; $\underline{n+1} = (\sigma)\underline{n}$
Let t be a c-term and x a variable ; define inductively a c-term written $\mathit{\Lambda} x\, t$:

- $\mathit{\Lambda} x\, t = (\mathsf{K})\, t$ if x is not in t
- $\mathit{\Lambda} x\, x = \mathsf{I}$
- $\mathit{\Lambda} x\, tu = (\mathsf{C}\mathit{\Lambda} x\, t)\, u$ if x is in t but not in u
- $\mathit{\Lambda} x\, tx = t$ if x is not in t
- $\mathit{\Lambda} x\, tx = (\mathsf{W})\mathit{\Lambda} x\, t$ if x is in t
- $\mathit{\Lambda} x(t)(u)\, v = \mathit{\Lambda} x(\mathsf{B})\, tuv$ if x is in uv

We now define our translation of λ-calculus, by setting : $\lambda x\, t = \mathit{\Lambda} x(\mathsf{I})\, t$.
We use λ-calculus only as a convenient way of writing c-terms.

c-terms and λ-terms

The rewriting of $\Lambda x\, t$ is finite because :

- no combinator is introduced inside t, but only in front of it ;
- the only changes in t are : moving parentheses, erasing occurrences of x ;
- each rule decreases the part of t which is under Λx ;
- except for the last rule, this decrease is *strict* ;
- the last rule can be applied consecutively only finitely many times.

Theorem. Let $t[x_1,\ldots,x_n]$ be a c-term and $\xi_1,\ldots,\xi_n \in \Lambda$. Then $\lambda x_1 \ldots \lambda x_n\, t \star \xi_1 . \ldots . \xi_n . \pi \succ t[\xi_1/x_1,\ldots,\xi_n/x_n] \star \pi$.

Easily proved, by induction on the length of the rewriting of t.

The usual KS -translation does not satisfy the theorem. For instance :

$\lambda x(x)xx \star \xi . \pi \equiv ((\mathsf{S})(\mathsf{S})\mathsf{I}\,\mathsf{I})\,\mathsf{I} \star \xi . \pi \succ \mathsf{S}\,\mathsf{I}\,\mathsf{I} \star \xi . \mathsf{I}\xi . \pi \succ \xi \star \mathsf{I}\xi . \mathsf{I}\xi . \pi$ instead of $(\xi)\xi\xi \star \pi$.

The above Curry-style translation gives:

$\lambda x(x)xx \star \xi . \pi \equiv (\mathsf{W})(\mathsf{W})(\mathsf{B})(\mathsf{B})\mathsf{I} \star \xi . \pi \succ \mathsf{B} \star \mathsf{BI} . \xi . \xi . \xi . \pi \succ (\xi)\xi\xi \star \pi$

The formal system for ZF_ε

We use first order logic with the only connectives $\top, \bot, \to, \forall$, some function symbols, three binary relation symbols $\not\varepsilon, \notin, \subseteq$ and the usual rules of natural deduction :

- $x_1{:}A_1, \ldots, x_n{:}A_n \vdash x_i{:}A_i$
- $x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}A \to B, \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash u{:}A \quad \Rightarrow \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash (t)u{:}B$
- $x_1{:}A_1, \ldots, x_n{:}A_n, x{:}A \vdash t{:}B \quad \Rightarrow \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash \lambda x\, t{:}A \to B$
- $x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}A \quad \Rightarrow \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}\forall x\, A$ (x is not in $A_1, \ldots, A_n$)
- $x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}\forall x\, A \quad \Rightarrow \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}A[\tau/x]$
  (τ is a *ℓ-term* of ZF_ε, i.e. a term built with variables and function symbols)
- $x_1{:}A_1, \ldots, x_n{:}A_n \vdash \mathsf{cc}{:}((A \to B) \to A) \to A$ (law of Peirce)
- $x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}\bot \quad \Rightarrow \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}A$

Notation. We write $F_1, \ldots, F_k \to F$ for $F_1 \to (F_2 \to \cdots \to (F_k \to F) \cdots)$
and $\exists x\{F_1, \ldots, F_k\}$ for $\forall x(F_1, \ldots, F_k \to \bot) \to \bot$.

Axioms of ZF_ε set theory

The axioms of ZF_ε essentially say that ε is a well founded relation and that its extensional collapse $\in$ satisfies ZF.

- Foundation scheme. $\forall\vec{z}\,\big(\forall x\,\big((\forall y\,\varepsilon\,x)F[y,\vec{z}] \to F[x,\vec{z}]\big) \to \forall a\,F[a,\vec{z}]\big)$
  for every formula $F[x,\vec{z}]$.
- Collapse. $\forall x\forall y\,\big(x \in y \leftrightarrow (\exists z\,\varepsilon\,y)\{x \subseteq z, z \subseteq x\}\big)$; $\forall x\forall y\,\big(x \subseteq y \leftrightarrow (\forall z\,\varepsilon\,x)z \in y\big)$
- Comprehension scheme. $\forall\vec{z}\forall a\exists b\forall x(x\,\varepsilon\,b \leftrightarrow (x\,\varepsilon\,a \wedge F[x,\vec{z}]))$
- Pairing. $\forall a\forall b\exists x\{a\,\varepsilon\,x, b\,\varepsilon\,x\}$
- Union. $\forall a\exists b(\forall x\,\varepsilon\,a)(\forall y\,\varepsilon\,x)\,y\,\varepsilon\,b$
- Power set. $\forall a\exists b\forall x(\exists y\,\varepsilon\,b)\forall z(z\,\varepsilon\,y \leftrightarrow (z\,\varepsilon\,a \wedge z\,\varepsilon\,x))$
- Collection scheme. $\forall\vec{z}\forall a\exists b(\forall x\,\varepsilon\,a)\,\big(\exists y\,F[x,y,\vec{z}] \to (\exists y\,\varepsilon\,b)F[x,y,\vec{z}]\big)$
- Infinity scheme. $\forall\vec{z}\forall a\exists b\,\big\{a\,\varepsilon\,b, (\forall x\,\varepsilon\,b)\,\big(\exists y\,F[x,y,\vec{z}] \to (\exists y\,\varepsilon\,b)F[x,y,\vec{z}]\big)\big\}$

A conservative extension of ZF.

Realizability models of ZF_ε

The *ground* or *standard model* $\mathcal{M}$ is an ordinary model of ZFC.

Its elements are called *individuals*.

The formulas of ZF (i.e. without $\not\varepsilon$) are interpreted in $\mathcal{M}$ (*true or false*).

The *realizability model* $\mathcal{N}$ has the *same domain* as $\mathcal{M}$.

The function symbols have the same interpretation as in $\mathcal{M}$.

The formulas of ZF_ε are interpreted in $\mathcal{N}$, but *with truth values in* $\mathcal{P}(\Pi)$.

Although $\mathcal{M}$ and $\mathcal{N}$ have the same domain (which means that
the quantifier $\forall x$ describes the same domain for both)
$\mathcal{N}$ has *more individuals* than $\mathcal{M}$ because some of them are *not named*.

For instance, in the "thread model" below, there are necessarily
non standard integers in $\mathcal{N}$, i.e. integers which are not named in $\mathcal{M}$.

Therefore, realizability models *are not* forcing models.

Realizability models of ZF_ε

For each closed formula F of ZF_ε with parameters $a_1, \ldots, a_n$ in $\mathscr{M}$
we define its *truth value* $|F| \subset \Lambda$ and its *falsity value* $\|F\| \subset \Pi$.
$\xi \in |F|$ is read *ξ realizes F* and is written $\xi \Vdash F$.
These values are connected by the relation : $\xi \in |F| \Leftrightarrow (\forall \pi \in \|F\|)(\xi \star \pi \in \perp\!\!\!\perp)$
so that we only need to define the falsity value $\|F\|$, by induction :

- F is atomic ;

$\|\top\| = \emptyset$; $\|\bot\| = \Pi$; $\|a \not\varepsilon b\| = \{\pi \in \Pi; (a, \pi) \in b\}$

$\|a \subseteq b\|, \|a \notin b\|$ are defined by induction on the ranks of a, b :

$\|a \subseteq b\| = \bigcup_c \{\xi \centerdot \pi;\ \xi \in \Lambda,\ \pi \in \Pi,\ (c, \pi) \in a,\ \xi \Vdash c \notin b\}$;

$\|a \notin b\| = \bigcup_c \{\xi \centerdot \xi' \centerdot \pi;\ \xi, \xi' \in \Lambda,\ \pi \in \Pi,\ (c, \pi) \in b,\ \xi \Vdash a \subseteq c,\ \xi' \Vdash c \subseteq a\}$.

- $F \equiv A \to B$; then $\|F\| = \{\xi \centerdot \pi\ ;\ \xi \Vdash A,\ \pi \in \|B\|\}$
- $F \equiv \forall x\, A$; then $\|F\| = \bigcup_a \|A[a/x]\|$

Realizability models of ZF_ε

The following *adequacy lemma* is an essential tool.

Theorem. If $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ and $\xi_1 \Vdash A_1, \ldots, \xi_n \Vdash A_n$ then $t[\xi_1/x_1, \ldots, \xi_n/x_n] \Vdash A$. In particular, if $\vdash t : A$, then $t \Vdash A$.

We say that *the model $\mathcal{N}$ realizes F* if there is a proof-like term $\xi \Vdash F$.

Notation : $\mathcal{N} \Vdash F$ or even $\Vdash F$.

By adequacy, the class of realized formulas is closed by classical deduction.

Theorem. The axioms of ZF_ε , and thus also the axioms of ZF , are realized.

Therefore, the realizability model may give us relative consistency results if it is *coherent*, i.e. $\perp$ *is not realized*. This means :

For every proof-like term ξ, there is a stack π such that $\xi \star \pi \notin \perp\!\!\!\perp$

For instance, $\perp\!\!\!\perp = \Lambda \star \Pi$ (the whole set of processes) gives an incoherent model.

Equality

In the realizability model we have two notions of *equality* :

- The *strong* or *Leibniz* equality $x = y$ which is $\forall z(x \notin z \to y \notin z)$.

We have $\Vdash \forall x \forall y(x = y, F[x] \to F[y])$ for *every* formula F.

- The *extensional* equality $x \simeq y$, which is $x \subseteq y, y \subseteq x$.

We have $\Vdash \forall x \forall y(x \simeq y, F[x] \to F[y])$ for every formula F *of ZF* (i.e. without the symbol $\notin$).

Each function symbol f on $\mathscr{M}$ extends immediately to $\mathscr{N}$, with the same values on *named* individuals. ZF_ε remains satisfied with the extended language.

On the other hand, to satisfy ZF, we must check that f is *compatible with* $\simeq$:

$$\Vdash \forall x \forall y(x \simeq y \to f\,x \simeq f\,y)$$

or else $$\Vdash \forall x \forall y(x \subseteq y, y \subseteq x \to f\,x \subseteq f\,y)$$

Equality

In order to compute more easily with Leibniz equality, we introduce the symbol $\neq$:

$\|a \neq b\| = \Pi = \|\bot\|$ if $a = b$; $\|a \neq b\| = \emptyset = \|\top\|$ if $a \neq b$.

Then $x = y$ is defined as $x \neq y \to \bot$. It is equivalent with Leibniz equality ; indeed :

Theorem.

i) $\mathsf{I} \Vdash \forall z(a \notin z \to b \notin z), a \neq b \to \bot$;

ii) $\lambda x \lambda y(\mathsf{cc})\lambda k(x)(k)y \Vdash (a \neq b \to \bot) \to \forall z(a \notin z \to b \notin z)$.

i) Let $\xi \Vdash \forall z(a \notin z \to b \notin z), \eta \Vdash a \neq b$ and $\pi \in \Pi$. We must show $\xi \star \eta . \pi \in \perp\!\!\!\perp$.

If $a \neq b$, then $\|\forall z(a \notin z \to b \notin z)\| = \|\top \to \bot\|$ (take $z = \{b\} \times \Pi$).

Therefore $\xi \Vdash \top \to \bot$ and we are done.

If $a = b$, then $\eta \Vdash \bot$, thus $\eta \Vdash a \notin z$;

take $z = \{(b, \pi)\}$, then $\pi \in \|b \notin z\|$ and $\eta . \pi \in \|a \notin z \to b \notin z\|$. Thus $\xi \star \eta . \pi \in \perp\!\!\!\perp$.

Equality

ii) Let $\xi \Vdash a \neq b \to \bot,\ \ \eta \Vdash a \notin z$ and $\pi \in \|b \notin z\|$.

We must show $(\mathrm{cc})\lambda k(\xi)(k)\eta \star \pi \in \perp\!\!\!\perp$, i.e. $\xi \star \mathrm{k}_\pi \eta \,\centerdot\, \pi \in \perp\!\!\!\perp$.

If $a \neq b$, then $\xi \Vdash \top \to \bot$ and we are done.

If $a = b$, then $\eta \star \pi \in \perp\!\!\!\perp$, and therefore $\mathrm{k}_\pi \eta \Vdash \bot$. Thus $\mathrm{k}_\pi \eta \,\centerdot\, \pi \in \|\bot \to \bot\|$.

But $\xi \Vdash \bot \to \bot$, hence $\xi \star \mathrm{k}_\pi \eta \,\centerdot\, \pi \in \perp\!\!\!\perp$.

Q.E.D.

The axioms of ZF_ε are realized

Foundation. $\mathsf{Y} \Vdash \forall x\big(\forall y(F[y] \to y \not\varepsilon\, x), F[x] \to \bot\big) \to \forall x(F[x] \to \bot)$
with $\mathsf{Y} = \mathsf{AA}$ and $\mathsf{A} = \lambda x \lambda f(f)(x)xf$ (Turing fixed point combinator).

Let $\xi \Vdash \forall x(\forall y(F[y] \to y \not\varepsilon\, x), F[x] \to \bot)$, $\eta \Vdash F[a]$ and $\pi \in \Pi$.

We show $\mathsf{Y} \star \xi . \eta . \pi \in \perp\!\!\!\perp$ by induction on the rank of a.

Since $\mathsf{Y} \star \xi . \eta . \pi \succ \xi \star \mathsf{Y}\xi . \eta . \pi$, it suffices to show $\xi \star \mathsf{Y}\xi . \eta . \pi \in \perp\!\!\!\perp$.

Now, $\xi \Vdash \forall y(F[y] \to y \not\varepsilon\, a), F[a] \to \bot$, so that it suffices to show
$\mathsf{Y}\xi \Vdash \forall y(F[y] \to y \not\varepsilon\, a)$, in other words $\mathsf{Y}\xi \Vdash F[b] \to b \not\varepsilon\, a$ for every b.

Let $\zeta \Vdash F[b]$ and $\varpi \in \|b \not\varepsilon\, a\|$. Thus, we have $(b, \varpi) \in a$, therefore $\mathrm{rk}(b) < \mathrm{rk}(a)$
and $\mathsf{Y} \star \xi . \zeta . \varpi \in \perp\!\!\!\perp$ by induction hypothesis.

It follows that $\mathsf{Y}\xi \star \zeta . \varpi \in \perp\!\!\!\perp$, which is the desired result.

Q.E.D.

The axioms of ZF_ε are realized

Collapse. $\Vdash \forall x \forall y[x \subseteq y \leftrightarrow (\forall z \varepsilon x) z \in y]$; $\Vdash \forall x \forall y[x \in y \leftrightarrow (\exists z \varepsilon y)\{x \subseteq z, z \subseteq x\}]$

Indeed, we have :

$\|a \subseteq b\| = \|\forall z(z \notin b \to z \not\varepsilon a)\|$ and $\|a \notin b\| = \|\forall z(a \subseteq z, z \subseteq a \to z \not\varepsilon b)\|$

This follows immediately from the definition of $\|a \subseteq b\|$ and $\|a \notin b\|$:

$\|a \subseteq b\| = \bigcup_c \{\xi \centerdot \pi;\ \xi \in \Lambda,\ \pi \in \Pi,\ (c,\pi) \in a,\ \xi \Vdash c \notin b\}$;

$\|a \notin b\| = \bigcup_c \{\xi \centerdot \xi' \centerdot \pi;\ \xi, \xi' \in \Lambda,\ \pi \in \Pi,\ (c,\pi) \in b,\ \xi \Vdash a \subseteq c,\ \xi' \Vdash c \subseteq a\}$.

Pairing. If $c = \{a, b\} \times \Pi$, then $\|a \not\varepsilon c\| = \|b \not\varepsilon c\| = \|\bot\|$; thus $I \Vdash a \varepsilon c,\ I \Vdash b \varepsilon c$.

Warning. In $\mathcal{N}$, c may have many other ε-elements than a, b.

An instance of a pair $\{a, b\}$ is $c' = \{(a, \mathsf{K} \centerdot \pi);\ \pi \in \Pi\} \cup \{(b, \underline{0} \centerdot \pi);\ \pi \in \Pi\}$. Indeed :

$\lambda x\, x\mathsf{K} \Vdash a \varepsilon c'$; $\lambda x\, x\underline{0} \Vdash b \varepsilon c'$; $\lambda x \lambda y \lambda z\, zxy \Vdash \forall x(x \neq a, x \neq b \to x \not\varepsilon c')$.

The axioms of ZF$_\varepsilon$ are realized

Comprehension.

Given a set a and a formula $F[x]$, define $b = \{(u, \xi \centerdot \pi);\ (u,\pi) \in a,\ \xi \Vdash F[u]\}$;
then $\|u \notin b\| = \|F(u) \to u \notin a\|$ for every set u.
Therefore $\mathsf{I} \Vdash \forall x(x \notin b \to (F(x) \to x \notin a))$ and $\mathsf{I} \Vdash \forall x((F(x) \to x \notin a) \to x \notin b)$.

and so on ...

The axioms of ZF$_\varepsilon$ are much easier to realize than those of ZF.

Type-like sets in $\mathcal{N}$

Define the function symbol $\daleth$ by $\daleth E = E \times \Pi$. Define the quantifier $\forall x^{\daleth E}$ by :
$\|\forall x^{\daleth E} A[x]\| = \bigcup_{a \in E} \|A[a/x]\|$; therefore $|\forall x^{\daleth E} A[x]| = \bigcap_{a \in E} |A[a/x]|$.
Let us see that this quantifier has the intended meaning $\forall x (x \varepsilon \daleth E \to A[x])$:

Theorem.

i) $\lambda x \lambda y y x \Vdash \forall x^{\daleth E} A[x] \to \forall x (\neg A[x] \to x \not\varepsilon \daleth E)$;

ii) cc $\Vdash \forall x (\neg A[x] \to x \not\varepsilon \daleth E) \to \forall x^{\daleth E} A[x]$.

i) Let $\xi \Vdash \forall x^{\daleth E} A[x]$, $\eta \Vdash \neg A[a]$ and $\pi \in \|a \not\varepsilon \daleth E\|$ i.e. $a \in E$.

Then $\xi \Vdash A[a]$; therefore $\lambda x \lambda y y x \star \xi . \eta . \pi \succ \eta \star \xi . \pi \in \perp\!\!\!\perp$.

ii) Let $\xi \Vdash \forall x (\neg A[x] \to x \not\varepsilon \daleth E)$, $a \in E$ and $\pi \in \|A[a]\|$;

then $\xi \Vdash \neg\neg A[a]$, $\mathsf{k}_\pi \Vdash \neg A[a]$; thus $\mathsf{cc} \star \xi . \pi \succ \xi \star \mathsf{k}_\pi . \pi \in \perp\!\!\!\perp$.

Q.E.D.

Type-like sets in $\mathcal{N}$

Let f, g be some terms built with the function symbols in the ground model $\mathcal{M}$.

If $\mathcal{M} \models f : E_1 \times \cdots \times E_k \to E$ then $\mathcal{N} \Vdash f : \beth E_1 \times \cdots \times \beth E_k \to \beth E$

(in fact, $\mathsf{I} \Vdash \forall x_1^{\beth E_1} \cdots \forall x_k^{\beth E_k} [f(x_1, \ldots, x_k) \not\varepsilon \beth E \to \bot]$).

Moreover, if $\mathcal{M} \models (\forall x_1 \in E_1) \cdots (\forall x_k \in E_k)[f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k)]$

then $\mathsf{I} \Vdash \forall x_1^{\beth E_1} \cdots \forall x_k^{\beth E_k} [f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k)]$.

For instance, let $\wedge, \vee, \neg$ be the (trivial) boolean operations on the set $\mathbf{2} = \{0, 1\}$.

They give a structure of boolean algebra on $\beth\mathbf{2}$ in the realizability model $\mathcal{N}$.

This boolean algebra is, in general, non trivial and even infinite ;

but, only two elements of $\beth\mathbf{2}$ are *named* : 0 and 1.

Remarks about $\beth 2$.

- $|\forall x^{\beth\mathbf{2}} F[x]| = |F[1]| \cap |F[0]|$; thus $\forall x^{\beth\mathbf{2}} F[x]$ behaves like an *intersection type*
- Every ε-element of $\beth\mathbf{2}$ except 1 is empty ; indeed $\mathsf{I} \Vdash \forall x^{\beth\mathbf{2}} \forall y (x \neq 1 \to y \not\varepsilon x)$.

Integers

Define the function symbol s in $\mathcal{M}$ by $s(a) = \{a\} \times \Pi = \daleth(\{a\})$ and $0 = \emptyset$.

$s(a)$ represents some singleton of a in the realizability model $\mathcal{N}$;

The following formulas are realized in $\mathcal{N}$:

$\forall x \forall y(sx = sy \to x = y)$; $\forall x(sx \neq 0)$;

$\forall x \forall y(x \simeq y \to sx \simeq sy)$.

Let us define $\widetilde{\mathbb{N}} = \{(s^n 0, \underline{n} \centerdot \pi);\ n \in \mathbb{N}, \pi \in \Pi\}$;

$\widetilde{\mathbb{N}}$ is the set of integers of the realizability model $\mathcal{N}$ (see below).

Since we have $\daleth\mathbb{N} = \{(s^n 0, \pi);\ n \in \mathbb{N}, \pi \in \Pi\}$, it follows that $\mathsf{I} \Vdash \widetilde{\mathbb{N}} \subset \daleth\mathbb{N}$.

In general, this inclusion is strict.

Integers

Define the quantifier $\forall x^{\text{int}}$ by $\|\forall x^{\text{int}} F[x]\| = \bigcup\{\underline{n}\,.\,\pi;\ n \in \mathbb{N}, \pi \in \|F[s^n 0]\|\}$.

Remark. $\xi \Vdash \forall x^{\text{int}} F[x]$ implies $\xi \underline{n} \Vdash F[s^n 0]$ for each $n \in \mathbb{N}$ *(Kleene realizability)*.

We see, as before, that the quantifier $\forall x^{\text{int}}$ has the intended meaning which is $\forall x(x \varepsilon \widetilde{\mathbb{N}} \to F[x])$.

$\widetilde{\mathbb{N}}$ represents the set of integers of the model $\mathcal{N}$. Indeed :

Theorem. $\lambda x\, x\underline{0} \Vdash 0 \varepsilon \widetilde{\mathbb{N}}$; $\lambda f \lambda x (f)(\sigma)x \Vdash \forall x(sx \not\varepsilon \widetilde{\mathbb{N}} \to x \not\varepsilon \widetilde{\mathbb{N}})$;
I $\Vdash \forall x^{\text{int}}(\forall y(F[sy] \to F[y]), F[x] \to F[0])$ for every formula $F[x]$.

The following theorem gives a characteristic property of recursive functions : *the image of an integer is an integer* and not only an element of $\beth\mathbb{N}$.

Theorem. Let $f : \mathbb{N}^k \to \mathbb{N}$ be a recursive function defined in $\mathcal{M}$.
Then $\mathcal{N} \Vdash \forall x_1^{\text{int}} \ldots \forall x_k^{\text{int}}(f(x_1, \ldots, x_k) \varepsilon \widetilde{\mathbb{N}})$.

Standard realizability algebras

We consider now a (very) special case : the *standard* realizability algebras.

The terms and the stacks are *words* composed with the following alphabet :

- the elementary *combinators* B C I K W cc ς (there is a new one)
- the *symbols* k $\bullet$ ( ) [ ]
- a countable set Π_0 of *empty stacks*.

The sets Λ of *terms* and Π of *stacks* are defined as follows :

- each elementary combinator is a term ; each empty stack is a stack ;
- if ξ, η are terms, then $(\xi)\eta$ is a term (*application*, written also $\xi\eta$) ;
- if ξ is a term and π a stack, then $\xi \bullet \pi$ is a stack (*push*) ;
- if π is a stack, then $\mathrm{k}[\pi]$ is a term (*continuation*, written k_π).

A *process* is an ordered pair (ξ, π) with $\xi \in \Lambda, \pi \in \Pi$; it is written $\xi \star \pi$.

The four operations of *application, push, continuation, process*
are defined in the obvious way.

Execution of processes

Define the preorder $\succ$ on processes (*execution*) by the following rules :

$(\xi)\eta \star \pi \succ \xi \star \eta . \pi$

$\mathsf{I} \star \xi . \pi \succ \xi \star \pi$

$\mathsf{K} \star \xi . \eta . \pi \succ \xi \star \pi$

$\mathsf{W} \star \xi . \eta . \pi \succ \xi \star \eta . \eta . \pi$

$\mathsf{C} \star \xi . \eta . \zeta . \pi \succ \xi \star \zeta . \eta . \pi$

$\mathsf{B} \star \xi . \eta . \zeta . \pi \succ (\xi)(\eta)\zeta \star \pi$

$\mathsf{cc} \star \xi . \pi \succ \xi \star \mathsf{k}_\pi . \pi$

$\mathsf{k}_\pi \star \xi . \varpi \succ \xi \star \pi$

$\varsigma \star \xi . \eta . \pi \succ \xi \star \underline{\mathsf{n}}_\eta . \pi$

where $\eta \longmapsto \mathsf{n}_\eta$ is a fixed (not necessarily recursive) numerotation of terms.

$\perp\!\!\!\perp$ is any set of processes such that $\xi \star \pi \in \perp\!\!\!\perp, \xi' \star \pi' \succ \xi \star \pi \Rightarrow \xi' \star \pi' \in \perp\!\!\!\perp$.

The *proof-like terms* are generated with the *seven combinators* B, C, I, K, W, cc, ς.

Non extensional and dependent choice

Theorem. For each formula $F[x,y]$, we can define a function symbol f such that :
$\lambda x(\varsigma)xx \Vdash \forall x(\forall k^{\text{int}} F[x, f(k,x)] \to \forall y\, F[x,y])$.

Now, let $\phi(x) = f(k,x)$ for the first k s.t. $\neg F[x, f(k,x)]$ if there is one ; else 0. Then

$$\mathcal{N} \Vdash \forall x(F[x,\phi(x)] \to \forall y\, F[x,y])$$

This gives the axiom of choice in the realizability model $\mathcal{N}$ for ZF$_\varepsilon$, *but not for ZF*, because we cannot find a symbol f which is *compatible with* $\simeq$.

This axiom is much weaker than choice, we call it *non extensional choice (NEC)*.

As we shall see below, it does not even imply the well ordering of $\mathbb{R}$.

Nevertheless, *it implies the axiom of dependent choice (DC)*. The proof is easy :
from $\forall x \exists y\, F[x,y]$, using NEC, we get a function ϕ such that $\forall x\, F[x, \phi x]$;
then, given a_0, we have the sequence $a_k = \phi^k(a_0)$ such that $F[a_k, a_{k+1}]$.

The Boolean algebra $\gimel 2$

The Boolean algebra $\gimel 2$ is essential in order to understand the structure of the realizability model $\mathcal{N}$. It is rather difficult to handle because, in general, it is infinite (even atomless) but only its obvious elements 0 and 1 are named.

It has the remarkable property of having *a countable dense subset.*

Theorem. There exists a function $\Delta : \mathbb{N} \to \mathbf{2}$ such that

$\lambda x \lambda y(\varsigma) y x x \Vdash \forall x^{\gimel 2}(x \neq 0 \to \exists n^{\text{int}}\{\Delta(n) \neq 0, (\Delta(n) \vee x) = x\})$.

Δ is defined as follows in $\mathcal{M}$: let $n \longmapsto \xi_n$ be the inverse of the given recursive enumeration of Λ which is $\xi \longmapsto \mathsf{n}_\xi$

(recall : the execution rule of the instruction ς is $\varsigma \star \xi . \eta . \pi \succ \xi \star \underline{\mathsf{n}}_\eta . \pi$). Then

$$\Delta(n) = 0 \Leftrightarrow \xi_n \Vdash \bot.$$

In $\mathcal{N}$, we have $\Delta : \gimel\mathbb{N} \to \gimel 2$ and therefore $\Delta : \widetilde{\mathbb{N}} \to \gimel 2$.

The theorem says that every element $\neq 0$ of $\gimel 2$ has a lower bound $\Delta(n) \neq 0$ with $n \,\varepsilon\, \widetilde{\mathbb{N}}$.

The pseudo integers $\beth\mathbf{n}$

In the ground model $\mathscr{M}$, we put, for each integer n :

$$\mathbf{n} = \{0, 1, \ldots, n-1\} = \{0, s0, \ldots, s^{n-1}0\}.$$

The functions $n \mapsto \mathbf{n}$ and $n \mapsto \beth\mathbf{n}$ are defined in the realizability model $\mathscr{N}$ with domain $\beth\mathbb{N}$.

We define the function $(m{<}n)$ from $(\beth\mathbb{N})^2$ into $\beth\mathbf{2}$, by putting, in $\mathscr{M}$, for $m, n \in \mathbb{N}$:

$(m{<}n) = 1$ if $m < n$ else $(m{<}n) = 0$.

The relation $(m{<}n) = 1$ is a strict (well founded, partial) order on $\beth\mathbb{N}$ which is the usual order on the set $\widetilde{\mathbb{N}}$ of integers in $\mathscr{N}$.

The following formulas are realized :

$\forall x^{\beth\mathbb{N}} \forall m^{\beth\mathbb{N}} ((x{<}m) = 1 \leftrightarrow x \,\varepsilon\, \beth\mathbf{m})$

$\forall m^{\beth\mathbb{N}} \forall n^{\beth\mathbb{N}} ((m{<}n) = 1 \to \beth\mathbf{m} \subset \beth\mathbf{n})$

$\forall m^{\beth\mathbb{N}} \forall n^{\beth\mathbb{N}}$(the application $(x, y) \longmapsto my + x$ is a bijection from $\beth\mathbf{m} \times \beth\mathbf{n}$ onto $\beth(\mathbf{mn})$).

Injection of $\daleth\mathbf{n}$ into $\mathbb{R}$

The application $x \longmapsto \{n\,\varepsilon\,\widetilde{\mathbb{N}};\ \Delta(n) \le x\}$ is, in $\mathcal{N}$, an injection of $\daleth\mathbf{2}$ into $\mathscr{P}(\widetilde{\mathbb{N}})$ (the real line of the model $\mathcal{N}$). Therefore :

$\mathcal{N} \Vdash (\forall n^{\text{int}})(\exists f : (\daleth\mathbf{2})^n \to \mathbb{R})(f \text{ is injective})$.

By recurrence on n, we see that $(\daleth\mathbf{2})^n$ is equipotent with $\daleth(\mathbf{2}^{\mathbf{n}})$.

Now, for each integer n, we have $n < 2^n$ and therefore $\daleth\mathbf{n} \subset \daleth(\mathbf{2}^{\mathbf{n}})$. Thus :

$\mathcal{N} \Vdash (\forall n^{\text{int}})(\exists f : \daleth\mathbf{n} \to \mathbb{R})(f \text{ is injective})$.

We will now choose the set $\perp\!\!\!\perp$ such that, in the realizability model $\mathcal{N}$, $\daleth\mathbf{2}$ is infinite and the “cardinals” of $\daleth\mathbf{n}$ form a *strictly increasing sequence* (which means that there is no surjection of $\daleth\mathbf{n}$ onto $\daleth(\mathbf{n}+\mathbf{1})$).

Since $\daleth\mathbf{m} \times \daleth\mathbf{n}$ is equipotent with $\daleth(\mathbf{mn})$, it follows that *neither $\daleth\mathbf{2}$ nor $\mathbb{R}$ are well ordered in $\mathcal{N}$.*

The model of threads

Remark. If $\gimel 2$ is non trivial, then there are non standard integers in the model $\mathcal{N}$.
Indeed, let $a \varepsilon \gimel 2, a \neq 0, 1$; there is an integer n such that $\Delta(n) \neq 0$ and $\Delta(n) \leq a$.
Thus $\Delta(n) \neq 0, 1$; n is non-standard because $\Delta(m) = 0$ or 1 for each standard m.
Thus, the realizability model $\mathcal{N}$ we are looking for, has non-standard integers.
It cannot be a forcing model or an inner model.
We define now the simplest non trivial *coherent* realizability model. Let :
$n \longmapsto \pi_n$ be an enumeration of the *empty stacks*
$n \longmapsto \theta_n$ be a recursive enumeration of the *proof-like terms*
The *thread with number n* is the set of processes $\xi \star \pi$ such that $\theta_n \star \pi_n \succ \xi \star \pi$.
The only empty stack which appears in the terms of the n-th thread is π_n.

The model of threads

The simplest way to ensure a *coherent model* is to decide that $\theta_n \star \pi_n \in \perp\!\!\!\perp^c$ ($\perp\!\!\!\perp^c$ is the complement of $\perp\!\!\!\perp$). Then, every thread must be in $\perp\!\!\!\perp^c$. Thus, we decide :

$\perp\!\!\!\perp^c$ is the union of all threads

Therefore $\xi \star \pi \in \perp\!\!\!\perp$ iff $\xi \star \pi$ never appears in any thread.

$\xi \Vdash \perp$ iff ξ never appears in head position in any thread.

Theorem. The following are satisfied in the model of threads :

i) There is a proof-like ω such that $\omega \mathsf{k}_\pi \xi \Vdash \perp$ or $\omega \mathsf{k}_\pi \xi' \Vdash \perp$ for any π, ξ, ξ' with $\xi \neq \xi'$.

ii) If $\zeta_0, \zeta_1, \zeta_2$ are distinct, then $\mathsf{k}_\pi \alpha \zeta_0 \Vdash \perp$ or $\mathsf{k}_\pi \alpha \zeta_1 \Vdash \perp$ or $\mathsf{k}_\pi \alpha \zeta_2 \Vdash \perp$ for any α, π.

i) Take $\omega = (\lambda x\, xx)\lambda x\, xx$ or (WI)(W)I.

ii) If the process $\alpha \star \pi$ appears twice in a thread, then the execution enters in a loop, and there will be no third appearance.

Q.E.D.

Consequences of (i)

We now consider any realizability model which satisfies properties (i) or (ii) (or both).

Theorem.

If a realizability model $\mathcal{N}$ satisfies property (i), then it realizes the formulas :

- $\beth\mathbf{2}$ is not countable.
- $\forall m^{\text{int}} \forall n^{\text{int}} ((m<n) = 1 \rightarrow$ there is no surjection from $\beth\mathbf{m}$ onto $\beth\mathbf{n}$).

Since there is an injection of $\beth\mathbf{n}$ into $\mathbb{R}$, it follows that :

there exists a sequence $X_n (n \geq 2)$ of infinite subsets of $\mathbb{R}$ such that

their “cardinals” are strictly increasing and $X_m \times X_n$ is equipotent with X_{mn}.

Dependent choice is true, but $\mathbb{R}$ *is badly not well orderable.*

The behaviour of cardinals is far from the usual one :

compare $\text{card}(X_2)$ with $\text{card}(X_2 \times X_2)$ which is $\text{card}(X_4)$

or worse, $\text{card}(X_5) < \text{card}(X_6) < \text{card}(X_7)$ and $\text{card}(X_5 \times X_7) < \text{card}(X_6 \times X_6)$.

This relative consistency result is *not obtainable with forcing.*

Consequences of (ii)

Theorem.

If a realizability model $\mathcal{N}$ satisfies property (ii), then it realizes the formulas :

- $\gimel\mathbf{2}$ is an atomless Boolean algebra.
- $\forall a^{\gimel 2} \forall b^{\gimel 2}(a \wedge b = 0, b \neq 0 \rightarrow$ there is no surjection from $a\gimel\mathbf{2}$ onto $b\gimel\mathbf{2}$).
- $\forall a^{\gimel 2} \forall b^{\gimel 2}(a < b \rightarrow$ there is no surjection from $a\gimel\mathbf{2}$ onto $b\gimel\mathbf{2}$).

$a\gimel\mathbf{2}$ is the ideal $\{x \,\varepsilon\, \gimel\mathbf{2};\; x \leq a\}$ of the boolean algebra $\gimel\mathbf{2}$.

We have an atomless Boolean algebra $\mathcal{B}$ of infinite subsets of $\mathbb{R}$ such that :

$X, Y \in \mathcal{B}, X \cap Y = \emptyset \Rightarrow$ card(X) and card(Y) are not comparable.

$X, Y \in \mathcal{B}, X \subset Y, X \neq Y \Rightarrow$ card$(X) <$ card(Y).

Thus, there is a family $(X_r)_{r \in \mathbb{R}}$ of subsets of $\mathbb{R}$ such that

$r < s \Rightarrow$ card$(X_r) <$ card(X_s).

Very far from the continuum hypothesis and the well ordering of $\mathbb{R}$.

Appendix

Some proofs

Non extensional choice

Theorem. For each formula $F[x,y]$, there is a function symbol f such that :
$\lambda x(\varsigma)xx \Vdash \forall x \forall y(\forall k^{\text{int}} F[x, f(k,x)] \to F[x,y])$.

For each $j \in \mathbb{N}$, let $P_j = \{\pi \in \Pi; \xi_j \star \underline{j} \centerdot \pi \notin \perp\!\!\!\perp\}$; ξ_j is the term η such that $\mathsf{n}_\eta = j$.

For each individual a, we have $\|\forall y\, F[a,y]\| = \bigcup_b \|F[a,b]\|$.

Thus, there exists a function f such that, given $j \in \mathbb{N}$ and a such that
$P_j \cap \|\forall y\, F[a,y]\| \neq \emptyset$, we have $P_j \cap \|F[a, f(j,a)]\| \neq \emptyset$ (by axiom of choice in $\mathscr{M}$).

Now, we want to show $\lambda x(\varsigma)xx \Vdash \forall k^{\text{int}} F[a, f(k,a)] \to F[a,b]$, for every a, b.

If this is false, we have $\varsigma \star \eta \centerdot \eta \centerdot \pi \notin \perp\!\!\!\perp$, for some $\eta \Vdash \forall k^{\text{int}} F[a, f(k,a)]$ and $\pi \in \|F[a,b]\|$.

Therefore $\eta \star \underline{j} \centerdot \pi \notin \perp\!\!\!\perp$ with $j = \mathsf{n}_\eta$ and it follows that $\pi \in P_j \cap \|F[a,b]\|$.

Thus, there exists $\varpi \in P_j \cap \|F[a, f(j,a)]\|$; then $\underline{j} \centerdot \varpi \in \|\forall k^{\text{int}} F[a, f(k,a)]\|$.

Therefore, by hypothesis on η, we have $\eta \star \underline{j} \centerdot \varpi \in \perp\!\!\!\perp$. Contradiction with $\varpi \in P_j$.

Q.E.D.

$\daleth 2$ has a countable dense subset

Define $\Delta : \mathbb{N} \to 2$ as follows in $\mathscr{M}$: $\Delta(j) = 0 \Leftrightarrow \xi_j \Vdash \bot$

(ξ_j is the term η such that $\mathsf{n}_\eta = j$).

In $\mathscr{N}$, we have $\Delta : \daleth\mathbb{N} \to \daleth 2$ and therefore $\Delta : \widetilde{\mathbb{N}} \to \daleth 2$.

Theorem. $\lambda x \lambda y (\varsigma) y x x \Vdash \forall x^{\daleth 2}(x \neq 0, \forall n^{\text{int}}(\Delta(n) \neq 0 \to x \neq \Delta(n) \vee x) \to \bot)$.

Let $a \in \{0,1\}$, $\xi \Vdash a \neq 0$, $\eta \Vdash \forall n^{\text{int}}(\Delta(n) \neq 0 \to a \neq \Delta(n) \vee a)$ and $\pi \in \Pi$.

We must show $\varsigma \star \eta . \xi . \xi . \pi \in \perp\!\!\!\perp$ that is $\eta \star \underline{\mathsf{n}}_\xi . \xi . \pi \in \perp\!\!\!\perp$.

By hypothesis on η, it suffices to show $\underline{\mathsf{n}}_\xi . \xi . \pi \in \|\forall n^{\text{int}}(\Delta(n) \neq 0 \to a \neq \Delta(n) \vee a)\|$

i.e. by definition of the quantifier $\forall n^{\text{int}}$: $\xi . \pi \in \|\Delta(\mathsf{n}_\xi) \neq 0 \to a \neq \Delta(\mathsf{n}_\xi) \vee a\|$

This amounts to show $\xi \Vdash \Delta(\mathsf{n}_\xi) \neq 0$ and $a = \Delta(\mathsf{n}_\xi) \vee a$.

- Proof of $\xi \Vdash \Delta(\mathsf{n}_\xi) \neq 0$: trivial if $\Delta(\mathsf{n}_\xi) = 1$ because $\|\Delta(\mathsf{n}_\xi) \neq 0\| = \emptyset$; if $\Delta(\mathsf{n}_\xi) = 0$, then $\xi \Vdash \bot$, by definition of Δ.
- Proof of $a = \Delta(\mathsf{n}_\xi) \vee a$: obvious if $a = 1$; if $a = 0$, then $\xi \Vdash \bot$ (hypothesis on ξ) ; thus $\Delta(\mathsf{n}_\xi) = 0$ by definition of Δ, hence the result. Q.E.D.

$\beth 2$ is not equipotent with $\beth 4$

This is the key property to prove that $\mathbb{R}$ is not well ordered.

Theorem. Suppose there is a proof-like ω such that $\xi \neq \xi' \Rightarrow \omega \mathsf{k}_\pi \xi \Vdash \bot$ or $\omega \mathsf{k}_\pi \xi' \Vdash \bot$.

Then $\lambda x \lambda x'(\mathsf{cc})\lambda k(x')\lambda z(xzz)(\omega)kz \Vdash$

$\forall z[(\forall x \forall y \forall y'(F(x,y,z), F(x,y',z), y \neq y' \to \bot), \forall y^{\beth 4} \exists x^{\beth 2} F(x,y,z) \to \bot)]$.

The formula F being arbitrary, this tells us that there is no surjection from $\beth 2$ onto $\beth 4$.

A similar proof will show that there is no surjection from $\widetilde{\mathbb{N}}$ onto $\beth 2$.

Since $\beth 4$ is equipotent with $(\beth 2)^2$ it follows that $\beth 2$ is not well ordered.

Proof. If this is false, there exist $\xi, \xi' \in \Lambda, \pi \in \Pi$ and an individual c such that :

$\lambda x \lambda x'(\mathsf{cc})\lambda k(x')\lambda z(xzz)(\omega)kz \star \xi \,.\, \xi' \,.\, \pi \notin \perp\!\!\!\perp$;

$\xi \Vdash \forall x \forall y \forall y'[F(x,y,c), F(x,y',c), y \neq y' \to \bot]$;

$\xi' \Vdash \forall y^{\beth 4} \neg \forall x^{\beth 2} \neg F(x,y,c)$.

$\beth 2$ is not equipotent with $\beth 4$

Therefore, we have $\xi' \star \eta \centerdot \pi \notin \perp\!\!\!\perp$ with $\eta = \lambda z(\xi z z)(\omega)\mathsf{k}_\pi z$.

By hypothesis on ξ', we have $\eta \not\Vdash \forall x^{\beth 2} \neg F(x, i, c)$ for $i < 4$.

Thus, there exists $\delta_i \in \{0, 1\}$ such that $\eta \not\Vdash \neg F(\delta_i, i, c)$.

Then, there exist $\xi_i \in \Lambda$ and $\pi_i \in \Pi$ such that $\xi_i \Vdash F(\delta_i, i, c)$ and $\eta \star \xi_i \centerdot \pi_i \notin \perp\!\!\!\perp$.

By definition of η, we get $\xi \star \xi_i \centerdot \xi_i \centerdot \omega\mathsf{k}_\pi \xi_i \centerdot \pi_i \notin \perp\!\!\!\perp$.

By hypothesis on ξ, we have $\omega\mathsf{k}_\pi \xi_i \not\Vdash i \neq i$, i.e. $\omega\mathsf{k}_\pi \xi_i \not\Vdash \perp$ for every $i < 4$.

Now, the hypothesis of the theorem gives $\xi_i = \xi_j$ for every $i, j < 4$.

But, since $\delta_i < 2$, there exist $i, j < 4$, $i \neq j$ such that $\delta_i = \delta_j = \delta$.

Then, $\xi_i = \xi_j \Vdash F(\delta, i, c), F(\delta, j, c)$ and $\omega\mathsf{k}_\pi \xi_i \Vdash i \neq j$ since $\| i \neq j \| = \emptyset$.

Thus, by hypothesis on ξ, we have $\xi \star \xi_i \centerdot \xi_i \centerdot \omega\mathsf{k}_\pi \xi_i \centerdot \pi_i \in \perp\!\!\!\perp$, which is a contradiction.

Q.E.D.

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